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A CHARACTERIZATION OF R^2 BY THE CONCEPT OF MILD CONVEXITY

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Let S be an open, connected set in a locally convex, Hausdorff topological vector space L. If the boundary of S contains exactly one point not a mild convexity point of S and this point is not isolated in $\operatorname{bd} S$, then $\dim L = 2$.

NOTATION. $[S]$ denotes the convex hull of S. $\langle S \rangle$ denotes the interior of $[S]$ relative to the affine closure, aff S, of S. int S, cl S, and bd S represent the interior, closure and boundary of S, respectively, while ext S and exp S denote the sets of extreme and exposed points of S. codim S denotes the codimension of aff S.

DEFINITION. Let S be a set in a topological vector space L. A point x is called a mild convexity point of S if there do not exist two points y and z such that $x \in \langle y, z \rangle$ and $[y, z] \sim \{x\} \subseteqq \operatorname{int} S$. [1].

The proof of Theorem 2 proceeds through some lemmas. Easy proofs are omitted.

LEMMA 1. *A topological vector space over* $\boldsymbol{R}$ *induces a locally convex, relative topology on every finite-dimensional linear subspace. Hence the relative topology on every finite-dimensional subspace is coarser than the standard Hausdorff topology on the subspace.*

Proof. Suppose the subspace M of L has finite dimension m and U is an arbitrary 0-neighborhood of L. Choose a balanced 0-neighborhood V such that

$$\sum_{1}^{m+1} V \subseteqq U .$$

Then by Caratheodory's theorem [1]

$$V \cap M \subseteqq [V \cap M] \subseteqq U \cap M .$$

LEMMA 2. *Let S be an open set in a topological vector spaces. Suppose $[x, y] \cup [y, z] \subseteqq S$ and $[x, y, z] \cap \operatorname{bd} S$ contains mild convexity points of S only. Then $\langle x, y, z \rangle \subseteqq S$.*

Proof. If x, y, z are collinear then there is nothing to prove; otherwise S intersects aff $\{x, y, z\}$ in a set which is open relative to the standard Hausdorff topology by Lemma 1. Therefore

$[x, y, z] \sim S$ is compact relative to this topology and so is its convex hull C. It is known that $C = [\text{ext}\, C]$. If $\text{ext}\, C \nsubseteq [x, z]$ then the inclusion $\text{ext}\, C \subseteq \text{cl}\,\text{exp}\, C$ demonstrates the existence of a point $e \in \text{exp}\, C \cap \langle x, y, z \rangle$. Since $\text{exp}\, C \subseteq \text{ext}\, C \subseteq [x, y, z] \sim S$ this point belongs to $\text{bd}\, S$ and is not a mild convexity point of S. This contradiction implies $\text{ext}\, C \subseteq [x, z]$ and the conclusion follows.

LEMMA 3. *If the nondegenerate interval $[x, y]$ does not intersect an affine subspace M of a vectorspace, then there is a point x' such that $x \in \langle x', y \rangle$ and $[x', y] \cap M = \emptyset$.*

LEMMA 4. *If $[x, y] \cup [y, z] \cup [z, w]$ is contained in an open set belonging to a topological vector space over $\boldsymbol{R}$ and u is an arbitrary vector, then y, z may be moved somewhat in the direction of u to the points y', z' so that $[x, y'] \cup [y', z'] \cup [z', w]$ still belongs to the same open set.*

LEMMA 5. *If S is an open, connected set in a topological vector space over $\boldsymbol{R}$ and T is a subset of the same space with codim $T \geqq 2$, the $S \sim T$ is polygonally connected.*

Proof. S is polygonally connected. If an interval $[y, z]$ intersecting $\text{aff}\, T$ belongs to a polygonal path, then by Lemma 4, y and z may be replaced by y' and z' so that the new path is in $S \sim \text{aff}\, T$.

LEMMA 6. *Let S be an open, connected set in a topological vector space over $\boldsymbol{R}$. Suppose that the set N of points in* bd S *which are not mild convexity points of S is empty or has codimension at least* 3. *Then if $x, y \in S$ and $[x, y] \cap \text{aff}\, N = \emptyset$ we have $[x, y] \subseteq S$.*

Proof. By Lemma 5 there is a polygonal path in S from x to y which does not intersect $\text{aff}\,(N \cup x) \sim x$. If $[x, x_1]$, $[x_1, x_2]$ are the first intervals in this path, then by application of the Lemmas 3 and 2 (in that order), $[x, x_2]$ lies in S and clearly does not intersect $\text{aff}\,(N \cup x) \sim x$. Proceeding in this manner we eventually obtain $[x, y] \subseteq S$. A digression is given here.

THEOREM 1. *Suppose S is an open, connected set in a topological vector space over $\boldsymbol{R}$, and suppose* bd S *contains only mild convexity points. Then S is convex.*

REMARK. This theorem which follows immediately from Lemma 6 is established in [1] with the additional assumption that the space is Hausdorff.

LEMMA 7. *Let S be an open, connected set in a locally convex Hausdorff space over* $\boldsymbol{R}$. *Suppose the set N of points in* $\operatorname{bd} S$ *which are not mild convexity points has the property* $\operatorname{codim} \operatorname{cl} \operatorname{aff} N \geqq 3$. *Then for every $x \in N$ there exists a closed hyperplane H and an x-neighborhood U such that $U \sim H \subseteqq S$.*

Proof. Choose two points x_1, x_2 both different from x such that $x \in [x_1, x_2] \subseteqq S \cup x$. The set $(\operatorname{cl} \operatorname{aff} N) \cup x_1$ is contained in a hyperplane H. Call the corresponding open halfspaces H^+ and H^- respectively. Choose an x_i-neighborhood $V_i \subseteqq S$. Then the union of $U^+ = [(V_1 \cup V_2) \cap H^+]$, U^- (defined similarly) and H gives the required U by Lemma 6.

The announced result may be stated forthwith.

THEOREM 2. *Let S be an open connected set in a locally convex, Hausdorff space over* $\boldsymbol{R}$. *If* $\operatorname{bd} S$ *contains exactly one point which is not a mild convexity point of S and this point is not isolated in* $\operatorname{bd} S$, *then the dimension of the space is 2.*

It is trivial to exhibit such a set in R^2, and it is easy to show that the set is starshaped.

REFERENCE

1. F. A. Valentine, *Convex Sets*, McGraw Hill (1964).

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