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QUASI-ADDITIVITY AND SETS OF FINITE L^p-CAPACITY

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The Bessel L^p-capacity of order $\alpha>0$, $B_{\alpha,p}$, and the Riesz L^p-capacity of order α, $R_{\alpha,p}$, are shown to have the same sets of finite capacity in Euclidean R^n, $\alpha p<n$. However, they have markedly different behavior as countably "almost" additive (quasi-additive) set functions - i.e., as applied to sets that are partitioned by increasing concentric rings.

There are several useful versions of an L^p-capacity (defined on subsets of R^n) in the literature. They are all more or less direct generalizations of the classical notion of capacity based on Laplace's equation in two and three dimensions (i.e., the capacity used by N. Wiener et al.) which corresponds to the case $p = 2$, $\alpha = 1$ below. We will be interested here in two important canonical examples that have attracted some attention of late: the L^p-Bessel capacity, $B_{\alpha,p}$, and the L^p-Riesz capacity, $R_{\alpha,p}$. These set functions are quite useful in the function theory for the Sobolev spaces and in the theory of partial differential equations. Most of these applications require either a detailed knowledge of the nature of the exceptional sets (the sets of capacity zero) or estimates on the rate at which the capacity of a sequence of bounded sets tends to zero (e.g., in the Wiener criteria). In either case, it is local information that is being sought, either about the capacities themselves or about the potentials used to define them.

But these capacities give global information as well, especially with regard to the existence of certain Sobolev functions and solutions to certain elliptic partial differential equations. For example, a subset $A\subset R^n$ has finite Bessel capacity $B_{\alpha,p}$ iff there is a Sobolev function $u\in W^{\alpha,p}(R^n)$ (i.e., $u\in L^p(R^n)$ and $D^\alpha u\in L^p(R^n)$, D^α denoting all derivatives of order α) such that $u\equiv 1$ q.e. (quasi-everywhere) on A. Or if Ω is an open set of R^n with $R_{1,p}(\tilde{\Omega}) < \infty$, $\tilde{\Omega}$ = complement of Ω, $1 < p < n$, then there is a solution to the Euler equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω which is equal to one q.e. on $\tilde{\Omega}$ and with $\int_\Omega |\nabla u|^p dx < \infty$ — the "equilibrium potential." These examples are extended to more general boundary values in [1].

In this note, however, we are interested in two special global properties of the capacities R and B: (a) the sets in R^n of finite capacity and (b) quasi-additivity.

(a_1) Our first result shows that both $R_{\alpha,p}$ and $B_{\alpha,p}$ have the same sets of finite capacity, $\alpha p < n$. Indeed, the inequality $R_{\alpha,p}(A) \leqq Q\cdot B_{\alpha,p}(A)$, with Q independent of A, is elementary, but the reverse inequality is false. We show that if $\alpha p < n$, $1 < p < \infty$, then there is a constant Q independent of A such that

$$Q \cdot B_{\alpha,p}(A) \leqq R_{\alpha,p}(A) + R_{\alpha,p}(A)^{n/(n-\alpha p)} \tag{1}$$

for all $A \subset R^n$. Previously it was only known that $B_{\alpha,p}(A) \leqq Q(A) \cdot R_{\alpha,p}(A)$, with $Q(A)$ depending on the diameter of A (see [2]). This result is rather interesting in view of the fact that the Bessel kernel defining $B_{\alpha,p}$ has exponential decay at infinity while the Riesz kernel defining $R_{\alpha,p}$ decays only algebraically.

(a_2) For $\alpha p > n$, we show that $B_{\alpha,p}(A) < \infty$ iff A is bounded. Furthermore, there is a constant Q independent of A such that

$$Q^{-1} \cdot B_{\beta,q}(A) \leqq B_{\alpha,p}(A) \leqq Q \cdot B_{\beta,q}(A) \tag{2}$$

whenever $\beta q > n$ and $\alpha p > n$.

(b_1) It is quite natural to think of the L^p-capacities as "refinements" of the classical Hausdorff measures on R^n. They are refinements in the sense that they can be used to distinguish between sets of Hausdorff measure zero. In obtaining such a refinement, however, we are forced to give up a useful property that all measures enjoy, namely additivity. Indeed, the classical capacity of the closed unit ball in three space is the same as that of its boundary or its interior. Thus the question becomes: how non-additive are B and R? Since they are of course countably subadditive, we seek estimates of the form $\Sigma B(A_j) \leqq Q \cdot B(A)$, where $A_j = A(q_j) = A \cap [q_{j-1} \leqq |x| < q_j]$, $j = 1, 2, \cdots, q_j \uparrow +\infty$, Q independent of A. When this is the case, we say that B is quasi-additive (qa) with respect to the sequence $\{q_j\}$. If $A_j = A(q_j) = A \cap [q_j \leqq |x| < q_{j-1}]$, $j = 1, 2, \cdots$, with now $q_j \downarrow 0$, we say that B is locally quasi-additive (lqa) with respect to $\{q_j\}$.

(b_2) A related question is: when is $\Sigma B(A_j) < \infty$ given $B(A) < \infty$?

The surprising thing is that R is qa with respect to $q_j = \lambda^j$, for all $\lambda > 1$ (and lqa for $q_j = \lambda^j$, $\lambda < 1$), but not qa for any algebraically increasing sequence $\{q_j\}$. However, B is qa with respect to $q_j = j$. Nevertheless, it is still true that with $q_j = j$, $\Sigma R_{\alpha,p}(A_j) < \infty$ iff $\Sigma B_{\alpha,p}(A_j) < \infty$ iff $R_{\alpha,p}(A) < \infty$.

Now if g_α denotes the usual Bessel kernel of order α, $\alpha > 0$, i.e., the nonnegative L^1 function on R^n whose Fourier transform is $(1 + |\xi|^2)^{-\alpha/2}$, and $h_\alpha(x) = |x|^{\alpha - n}$, $0 < \alpha < n$, is the Riesz kernel, then $g_\alpha(x) \leqq Q h_\alpha(x)$, for all $x \in R^n$. And for $A \subset R^n$, $B_{\alpha,p}(A) = \inf \int f^p dx$, $1 < p < \infty$, where the infimum is over all $f \in L_+^p(R^n)$ such that $g_\alpha * f(x) \geqq 1$ on A. For $R_{\alpha,p}(A)$, we just replace g_α by h_α. The reader is referred to [10], [2], [3], and [1] for a detailed account of these capacities and various of their properties that we will use freely throughout this article.

The letter Q will be used to denote various constants, and the symbol $\sim$ ("is comparable to") to mean that the ratio of two functions is bounded above and below by finite positive constants independent of the variables in question.

1. **Sets of finite capacity.** We begin by showing that (1) and (2) hold.

THEOREM 1. (a) *For $\alpha p<n$, $1<p<\infty$, $\alpha>0$, there is a constant $Q = Q(\alpha, p, n)$ such that* (1) *holds for all $A\subset R^n$.*

(b) *There is no a priori inequality of the form*

$$R_{\alpha,p}(A)^t \leqq Q\cdot B_{\alpha,p}(A) \tag{3}$$

for any $\alpha p < n$ when $t > 1$ and Q independent of A.

It should be noted that (1) is asymptoticly sharp in the sense that for large balls $B_R(0) = [|x| < R]$, $B_{\alpha,p}(B_R(0)) \sim R^n$, as $R \to \infty$, whereas $R_{\alpha,p}(B_R(0)) \sim R^{n-\alpha p}$ for all $R > 0$.

The proof of (a) is a modification of the basic argument given in [4] which we now outline. It depends on several preliminary lemmas.

Let $I_\alpha f$ be the operator (defined initially on smooth functions f) corresponding to the Fourier multiplier $|\xi|^{-\alpha}$, $\alpha\in R$, and $J_\alpha f$ the operator corresponding to the Fourier multiplier $(1 + |\xi|^2)^{-\alpha/2}$, $\alpha\in R$. We then set $||u||_{\alpha,p} = ||J_{-\alpha}u||_p = L^p$ norm over R^n of $J_{-\alpha}u$. D^s denotes differentiation of order $|s| = s_1 + s_2 + \cdots + s_n$, in the usual way. $\mathscr{L}^{\alpha,p}$ denotes $\{u = J_\alpha f : f\in L^p(R^n)\}$, with norm $||\cdot||_{\alpha,p}$.

LEMMA 1. (a) *For $\alpha > 0$, $1 < p < \infty$, and $u\in\mathscr{L}^{\alpha,p}$,*

$$||u||_{\alpha,p} \sim ||u||_p + ||I_{-\alpha}u||_p\ ,$$

(b) $||I_{-\alpha}u||_p \sim \Sigma_{|s|=\alpha}||D^s u||_p$, *for α = positive integer.*

This lemma follows in a standard way from the theory of Fourier multipliers, see in particular [12] p. 96.

LEMMA 2. *If $\varphi = I_\alpha f$, $f\in L^p_+$, $\alpha p < n$, then*

$$||I_{\alpha_1}f||_{\alpha p/\alpha_2} \leqq Q\,||\varphi||_\infty^{\alpha_1/\alpha}||f||_p^{\alpha_2/\alpha}$$

where $\alpha = \alpha_1 + \alpha_2$, $\alpha_1, \alpha_2 > 0$.

Lemma 2 is an immediate consequence of the pointwise estimates for Riesz potentials in terms of the Hardy-Littlewood maximal function due to Hedberg [6], see in particular Theorem 3 there.

Finally, we need the fractional differentiation operators of Strichartz and Polking; see [13] and [11] for the next two lemmas. For $0 < \sigma < 1$, $1 \leqq q < \infty$, set

$$\mathscr{D}_q^\sigma(u)(x) = \left(\int_0^\infty \left(\int_{|y|\leqq 1} |u(x+\rho y) - u(x)|^q dy\right)^2 \frac{d\rho}{\rho^{1+2\alpha q}}\right)^{1/2q} .$$

LEMMA 3. (a) $\|\mathscr{D}_q^\sigma(I_\sigma f)\|_p \leqq Q\|f\|_p$, *for* $0 < \sigma < 1$, $1 \leqq q < \infty$, $\max(1, nq/(n+\sigma q)) < p < \infty$, (b) $\|\mathscr{D}_1^\sigma(I_\sigma f)\|_p \sim \|f\|_p$, $0 < \sigma < 1$, $1 < p < \infty$.

LEMMA 4. *For* $0 < \mu, \lambda < 1$,

$$\mathscr{D}_q^\sigma(uv) \leqq |u|\mathscr{D}_q^\sigma(v) + |v|\mathscr{D}_q^\sigma(u) + \mathscr{D}_{q/\lambda}^{\sigma\mu}(u)\cdot\mathscr{D}_{q/1-\lambda}^{\sigma(1-\mu)}(v) .$$

Now let $H(t)$ be a C^∞-function on the line for which $H(t) = 0$, $|t| \leqq 1/2$ and $H(t) = \operatorname{sgn}(t)$, $|t| \geqq 1$.

LEMMA 5. *If* $\varphi = I_\alpha f$, $f \in L^p$, *then*

$$\|H(\varphi)\|_p \leqq Q\|f\|_p^{p^*/p} ,$$

where $p^* = np/(n-\alpha p)$, $\alpha p < n$.

Proof. $\int |H(\varphi)|^p dx \leqq \int_{|\varphi| \geqq 1/2} dx \leqq 2^{p^*}\int |\varphi|^{p^*} dx \leqq Q\|f\|_p^{p^*/p}$ by the Sobolev inequality.

The claim now is that Lemmas 1-5 imply that there is a constant Q independent of $f \in L_+^p$ such that

$$\|H(I_\alpha f)\|_{\alpha,p} \leqq Q\{\|f\|_p \sum_{j<\alpha} \|I_\alpha f\|_\infty^j + \|f\|_p^{p^*/p}\} \tag{4}$$

and that (4) in turn implies (1). The latter fact follows by taking f to be the $R_{\alpha,p}$-capacitary distribution for A (i.e., the L_+^p function that minimizes in the definition) and recalling that its Riesz potential of order α is bounded on R^n, [10] and [2]. The first fact is obtained by mimicing the procedure in [4]. Of course, when $\alpha =$ positive integer, we need only Lemmas 1, 2, and 5. We content ourselves with an outline of the proof in a representative case for fractional α, namely $2 < \alpha < 3$. Writing $\alpha = 2 + \sigma$, $0 < \sigma < 1$, we clearly need only concentrate on estimating $\|\mathscr{D}_1^\sigma(D^2 H(\varphi))\|_p$, where D^2 now represents any second order derivative and $\varphi = I_\alpha f$. But that quantity is dominated by

$$\|\mathscr{D}_1^\sigma(D^2\varphi)\|_p + \|D^2\varphi\cdot\mathscr{D}_1^\sigma(H'(\varphi))\|_p + \|\mathscr{D}_1^\sigma(D\varphi\cdot D\varphi)\|_p + \|D\varphi\cdot D\varphi\cdot\mathscr{D}_1^\sigma(H''(\varphi))\|_p$$

where now D denotes a first order derivative in D^2. In the first term, we write $D^2\varphi = I_\sigma(D^2I_2f)$ and use Lemma 3 together with the Calderón-Zygmund estimates for singular integrals. Thus 1st term $\leqq Q\|f\|_p$. The second term is dominated by $Q\|I_\sigma f\|_{p(1+\sigma/2)}\cdot\|\mathscr{D}_1^\sigma(\varphi)\|_{p(1+2/\sigma)}$ and we just use Lemmas 2 and 3. The result is: 2nd term $\leqq Q\|f\|_p\|\varphi\|_\infty$. The fourth term is handled in a similar way leading to: 4th term $\leqq Q\|f\|_p\|\varphi\|_\infty^2$. Finally, for the third term, we use Lemma 4 (the choice $\lambda = \mu = 1/2$ works here), and then an application of Lemmas 2 and 3 finishes off the proof by giving: 3rd term $\leqq Q\|f\|_p\|\varphi\|_\infty$.

We defer the proof of Theorem 1(b) to § 2.

THEOREM 2. *For $\alpha, \beta > 0$, $1 < p, q < \infty$, and $\alpha p, \beta q > n$, there is a constant $Q = Q(\alpha, \beta, p, q, n)$ such that* (2) *holds for all $A \subset R^n$.*

Note that from [2]: if $\beta q < \alpha p$ or if $\beta q = \alpha p$ and $\beta < \alpha$, then there is a Q such that $B_{\beta,q}(A) \leqq QB_{\alpha,p}(A)$, for all A. For Theorem 2, however, we need only the following lemma from [2] - see the proof of Theorem 2.1 there.

LEMMA 6. *Suppose $\beta q < \alpha p$, then for all γ such that $\beta q < 2\gamma < \alpha p$, there is a Q such that*

$$Q^{-1}B_{\beta,q}(A) \leqq B_{\gamma,2}(A) \leqq QB_{\alpha,p}(A)$$

for all A.

LEMMA 7. *Let $n < 2\alpha < 2\beta$, then there is a Q such that*

$$Q^{-1}B_{\alpha,2}(A) \leqq B_{\beta,2}(A) \leqq QB_{\alpha,2}(A)$$

for all A.

Proof of Lemma 7. The first inequality follows from Lemma 6. For the second, we note that $g_{2\alpha}(x) \sim 1$ for $|x| \leqq 1$, and $g_{2\alpha}(x) \sim |x|^{\alpha-(n-1)/2}e^{-|x|}$ for $|x| \geqq 1$, see [10]. Hence $g_{2\alpha}(x) \leqq Qg_{2\beta}(x)$ and then for any Borel measure μ (supported in K, a compact set), $g_{2\alpha}*\mu(x) \leqq Qg_{2\beta}*\mu(x)$. Thus $b_{2\beta,1}(K) \leqq Qb_{2\alpha,1}(K)$ or $B_{\beta,2}(K) \leqq QB_{\alpha,2}(K)$. The general inequality follows by capacitability - again see [10] and [2].

To prove Theorem 2, we may assume without loss of generality that $\beta q \leqq \alpha p$. Choose γ and δ so that $n < 2\gamma < \beta q \leqq \alpha p < 2\delta$. Then by Lemmas 6 and 7

$$B_{\beta,q}(A) \leqq QB_{\delta,2}(A) \leqq QB_{\gamma,2}(A) \leqq QB_{\alpha,p}(A)$$
$$B_{\alpha,p}(A) \leqq QB_{\delta,2}(A) \leqq QB_{\gamma,2}(A) \leqq QB_{\beta,q}(A) .$$

2. Quasi-additivity. For our next result we will need the following lemmas. They are contained in Theorems 3.3 and, 3.4 and Corollary 4.1 of [3].

LEMMA 8. *Given* $M > 1$, *let*

$$R_{\alpha,p}(A; M) = \inf \|\mu\|_1$$

where the infimum is over all Borel measures μ *on* R^n *(of finite total variation* $\|\mu\|_1$*) such that* $\cup^\mu(x) \geqq 1$ *on* A *and* $\cup^\mu(x) \leqq M$ *on* R^n. *Here* $\cup^\mu(x) \equiv h_\alpha*(h_\alpha*\mu)^{1/(p-1)}(x)$. *Then there is a constant* $M_0 = M_0(\alpha, p, n)$ *such that*

$$R_{\alpha,p}(A; M_0) \leqq R_{\alpha,p}(A) \leqq M_0 \cdot R_{\alpha,p}(A; M_0) ,$$

$1 < p < \infty$, $\alpha p < n$.

LEMMA 9. *For all* $x \in R^n$,

(i) $\cup^\mu(x) \leqq Q \int_0^\infty [r^{\alpha p - n}\mu(B_r(x))]^{1/(p-1)} dr/r$, $p > 2 - \alpha/n$, $\alpha p < n$;

(ii) *the exponent* $1/(p-1)$ *in* (i) *may be taken to be* $(n-\alpha)/(n-\alpha p)$ *when* $1 < p < 2 - \alpha/n$, *and* $[(n-\alpha)/(n-\alpha p)] - \varepsilon$, $\varepsilon > 0$, *when* $p = 2 - \alpha/n$.

Now suppose that q_j is sequence which tends to infinity with j and that $R_{\alpha,p}(A) < \infty$. Furthermore, let μ be such that $\cup^\mu \geqq 1$ on A, $\cup^\mu \leqq M_0$ on R^n, and $\|\mu\|_1 < R_{\alpha,p}(A; M_0) + \varepsilon$, some $\varepsilon > 0$. Next set $\nu_j = \mu|[|z| \geqq q_{j+1}]$, $\lambda_j = \mu|[|z| < q_{j-2}]$, and $\mu_j = \mu - \nu_j - \lambda_j$. Then

$$\cup^{\mu_j} \geqq 1 - Q[\cup^{\nu_j} + \cup^{\lambda_j}] ,$$

and (for $p > 2 - \alpha/n$) taking $x \in A(q_j)$,

$$\begin{aligned} \cup^{\nu_j} &\leqq Q\int_{q_{j+1}-q_j}^\infty [r^{\alpha p - n}\mu(B_r(x))]^{1/(p-1)}\frac{dr}{r} \\ &\leqq Q\|\mu\|_1^{1/(p-1)}(q_{j+1} - q_j)^{(\alpha p - n)/(p-1)} . \end{aligned}$$

Similarly,

$$\cup^{\lambda_j} \leqq Q\|\mu\|_1^{1/(p-1)}(q_{j-1} - q_{j-2})^{(\alpha p - n)/(p-1)} .$$

Thus we have

THEOREM 3. *Suppose* $q_j = q_j(\sigma)$, $\sigma > 0$, *and that* $q_{j+1} - q_j \to \infty$ *as* $\sigma \to \infty$ *uniformly in* j, *then for any set* A *with* $R_{\alpha,p}(A) < \infty$, *there is a* $\sigma = \sigma(n, p, \alpha, R_{\alpha,p}(A))$ *such that*

$$\sum_j R_{\alpha,p}(A(q_j)) \leqq Q\cdot R_{\alpha,p}(A)\ , \tag{5}$$

for Q independent of A.

The same basic argument also gives

COROLLARY 1. *If $q_{j+1} - q_j \to \infty$ as $j \to \infty$, then $\Sigma R_{\alpha,p}(A(q_j)) < \infty$ provided $R_{\alpha,p}(A) < \infty$.*

COROLLARY 2. *If $q_j = j^\sigma$, $\sigma > 1$, then*

$$\sum R_{\alpha,p}(A(j^\sigma)) \leqq Q\cdot R_{\alpha,p}(A)^{1+1/(n-\alpha p)(\sigma-1)}\ ,$$

Q independent of A.

REMARKS. (i) Corollaries 1 and 2 extend a previous result of Landkof's in this direction for the case $p = 2, 0 < \alpha < n$; see [8] p. 304.

(ii) With minor modifications in Lemmas 8 and 9, we could also prove Theorem 3 and its corollaries for $B_{\alpha,p}$, $1 < p < \infty$, $\alpha > 0$. This then easily gives

$$B_{\alpha,p}(A) < \infty,\ \alpha p > n \text{ implies } A \text{ is bounded.} \tag{6}$$

Indeed $B_{\alpha,p}(A) \geqq \|g_\alpha\|_{p'}^{-p}$ provided $A \neq \varnothing$; see [10]. Also, (6) can be more directly deduced by an approximation argument since (6) is equivalent to $f \in L^p$, $\alpha p > n$ implies $g_\alpha * f(x) \to 0$ as $|x| \to \infty$. When $p = 2$, this is just the Riemann-Lebesgue lemma.

(iii) In [1], we showed that for $q_j = \lambda^j$, $\lambda > 0$, then

$$\sum_{j=-\infty}^{\infty} R_{\alpha,p}(A(\lambda^j)) \leqq Q\cdot R_{\alpha,p}(A) \tag{7}$$

with Q independent of A and λ, $\alpha p < n$, and α = positive integer $\in (0, n)$. The proof relied on the capacitary strong type inequality developed there. (7) is now known to be valid for all $\alpha \in (0, n)$, $\alpha p < n$, due to the capacitary strong type estimates recently obtained by Dahlberg [5]. For an earlier result of this type ($\alpha = 1$) see [7]. Note that (7) not only gives a quasi-additivity result at infinity, but a local such result as well.

Proof of Theorem 1(b). Set $E_N = \cup_1^N B_1(x_j)$, a disjoint union of balls centered at x_j chosen so that $B_1(x_j) \subset A(j^\sigma)$, for some sufficiently large σ. Then by Theorem 3, $R_{\alpha,p}(E_N) \sim N$ and likewise $B_{\alpha,p}(E_N) \sim N$, as $N \to \infty$.

For the Bessel capacities, we can do somewhat better than (7).

THEOREM 4. *For $\alpha > 0, 1 < p < \infty$, there is a constant $Q = Q(n, \alpha, p)$ such that*

$$\sum B_{\alpha,p}(A(j)) \leqq Q \cdot B_{\alpha,p}(A) \tag{8}$$

for all $A \subset R^n$.

Proof. This is basicly nothing more than a restatement of the "uniform localization theorem" of Strichartz [12] p. 1041. Just choose $\varphi(t) \in C_0^\infty(R)$ satisfying: $0 \leqq \varphi \leqq 1$, $\varphi(t) = 1$ for $1 \leqq t \leqq 2$, and $\varphi(t) = 0$ for $t \leqq 0$ and $t \geqq 3$. Then setting $\eta_j(x) = \varphi(|x| - j)$, we have

$$\sum_j ||\eta_j u||_{\alpha,p}^p \leqq Q||u||_{\alpha,p}^p$$

by a simple modification of Strichartz's proof. This completes the argument since $B_{\alpha,p}(A) = \inf ||u||_{\alpha,p}^p$, where $u \in \mathscr{L}^{\alpha,p}$ such that $u \geqq 1$ on A quasi-every where.

For a related result see [9].

REMARKS. (iv) Theorems 1 and 4 imply that Corollary 1 holds for $q_j = j$ and furthermore,

$$\sum R_{\alpha,p}(A(j)) \leqq Q[R_{\alpha,p}(A) + R_{\alpha,p}(A)^{n/(n-\alpha p)}] .$$

(v) We have seen that (5) holds for $q_j = j^\sigma$ provided σ is chosen sufficiently large (depending on A). But in fact it is impossible to choose a $\sigma > 0$ to be independent of A. Furthermore, (8) does not hold with $A(j)$ replaced by $A(j^\sigma)$ for any $\sigma \in (0, 1)$. To see these claims, we need only note that (at least in the case $\alpha p > 1$) $B_{\alpha,p}(A_{ab}) \sim (a + b)^{n-1}$ and $R_{\alpha,p}(A_{ab}) \sim (a + b)^{n-\alpha p}$, where $A_{ab} = [a \leqq |x| < b]$, $a > 1$. (For these estimates, see Theorem 5.2 of [2]. Similar estimates hold for the cases $\alpha p \leqq 1$.) Now just take $A = B_{N^\sigma}(0)$ and let $N \to \infty$.

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