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APPROXIMATION BY RATIONAL MODULES ON NOWHERE DENSE SETS

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Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space of all functions of the form

$$r_0(z) + r_1(z)\bar{z} + \cdots + r_m\bar{z}^m$$

where each r_i is a rational function with poles off X. We prove that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dese in $L^p(X)$ for all $1 \leqq p < \infty$ and $\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in $\mathscr{C}(X)$ if X has no interior point. As corollaries, we also prove that $\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in lip (α, X) for all $0<\alpha<1$ and $\mathscr{R}(X)\bar{\mathscr{P}}_3$ is dense in $D^1(X)$ for the same X.

1. **Introductions.** Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space $\mathscr{R} + \mathscr{R}\bar{z} + \cdots + \mathscr{R}\bar{z}^m$

$$= \{r_0(z) + r_1(z)\bar{z} + \cdots + r_m(z)\bar{z}^m\},$$

where each r_i is a rational function with poles off X. In [3, 4], O'Farrell has studied the relation of the problems of approximation by rational modules in different Lipschitz norms, and in the uniform norm, etc., to one another.

In this note, we investigate the problem of determining the set X so that $\mathscr{R}(X)\bar{\mathscr{P}}_m$ is uniformly dense in $\mathscr{C}(X)$ for each m.

Vitushkin [8] has given a necessary and sufficient condition in terms of analytic capacities for the case $m = 0$. In [3], O'Farrell has given an example of an X such that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is uniformly dense in $\mathscr{C}(X)$ whereas $\mathscr{R}(X)$ fails to be dense in $\mathscr{C}(X)$.

It is apparent that if X has interior, then $\mathscr{R}(X)\bar{\mathscr{P}}_m$ can not be dense in $\mathscr{C}(X)$. Thus we restrict our attention to a compact set X without interior throughout this note. Let $L^p(X) = L^p(\chi_X dm)$, where dm denotes the 2-dimensional Lebesgue measure. We prove the following theorem:

THEOREM. *Let X be a compact set with no interior. Then*

I. *$\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(X)$ for all $1 \leqq p < \infty$, and*

II. *$\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in $\mathscr{C}(X)$.*

2. *Proof of theorem.* Let μ be a (finite Borel) measure on X. The Cauchy transform $\hat{\mu}$ is defined by

$$\hat{\mu}(z) = \int_X \frac{d\mu(\zeta)}{\zeta - z}.$$

It is well known that $\hat{\mu}$ is absolutely convergent for almost all z (dm) and $\hat{\mu}$ belongs to L^p_{loc} for $1 \leqq p < 2$ (see [2], p. 37). If g is a function on X, we write $\hat{g}$ for $\widehat{gdm}$. Sinanjan [6] was the first to prove that $\mathscr{R}(X)$ is dense in $L^p(X)$ when $1 \leqq p < 2$. However, it is Brennan's [1] proof that leads to the results we obtain here.

We use the symbol $\bar{\partial}$ for the operator $\partial/\partial x + i(\partial/\partial y)$ and write $g \perp \mathscr{R}(X)\bar{\mathscr{P}}_m$ if $\int fgdm = 0$ for all $f \in \mathscr{R}(X)\bar{\mathscr{P}}_m$. The following lemma is a special case of the key lemma in [3].

LEMMA. *Let μ be a measure on X. Then $\mu \perp \mathscr{R}(X)\bar{\mathscr{P}}_{m+1}$ if and only if $\hat{\mu} \perp \mathscr{R}(X)\bar{\mathscr{P}}_m$.*

Proof. Because $\int fd\mu = \pi^{-1}\int(\bar{\partial}f)\hat{\mu}dm$ for all $f \in \mathscr{R}(X)\bar{\mathscr{P}}_{m+1}$ (see [2]).

Proof of theorem. Of course, the case $1 \leqq p < 2$ is contained in Sinanjan's theorem.

Let $g \in L^p(X)$, $q>1$, $p^{-1}+q^{-1}=1$, such that $g \perp \mathscr{R}(X)\bar{\mathscr{P}}_1$. Lemma implies $\hat{g} \perp \mathscr{R}(X)$. In particular, $\hat{\hat{g}} = 0$ if $z \notin X$. Being the convolution of g and $\zeta^{-1} \in L^r_{\mathrm{loc}}(1 \leqq r < 2)$, $\hat{g}$ belongs to $L^s(X)$ for some $s > 2$, by the classical Young's inequality (see [7], p. 271). And being the convolution of $\hat{g}$ and ζ^{-1}, $\hat{\hat{g}}$ is continuous everywhere in the plane. It follows that $\hat{\hat{g}} \equiv 0$ since X has no interior. It is well known that this implies $\hat{g} = 0$ in $L^s(X)$, which in turn implies $g = 0$ in $L^q(X)$ and hence $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(X)$. Part I is proved.

Let μ be a measure on X such that $\mu \perp \mathscr{R}(X)\bar{\mathscr{P}}_2$. Lemma implies $\hat{\mu} \perp \mathscr{R}(X)\bar{\mathscr{P}}_1$ and Part I implies $\hat{\mu} = 0$ and hence $\mu = 0$. Part II is proved.

3. Some remarks. Let $\mathscr{E}$ and $\mathscr{D}$ be the usual linear topological space of complex-valued $\mathscr{C}^\infty$ functions on the complex plane. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathscr{D}$ and $\mathscr{E}$ respectively. $\mathscr{R}(X)\bar{\mathscr{P}}_m$ can be regarded as a subspace of $\mathscr{D}|_X$ and $\mathscr{E}|_X$. We denote the closure (i.e., the completion) of $\mathscr{R}(X)\bar{\mathscr{P}}_m$ with respect to the norm $\|\cdot\|_j$ (restricted to X) by $[\mathscr{R}(X)\bar{\mathscr{P}}_m]_j$, for $j = 1, 2$. O'Farrell's theorems in [3] can be roughly stated in the following general form:

If $\phi \rightarrow \hat{\phi}$ is continuous from $(\mathscr{D}, \|\cdot\|_1)$ to $(\mathscr{E}, \|\cdot\|_2)$, then

$[\mathscr{R}(X)\bar{\mathscr{P}}_m]_1 = [\mathscr{D}]_1$ *implies* $[\mathscr{R}(X)\bar{\mathscr{P}}_{m+1}]_2 = [\mathscr{E}]_2$.

As corollaries, we have the following:

COROLLARY. *Let X be a compact set with no interior. Then*

I. $\mathscr{R}(X)\bar{\mathscr{P}}_2$ *is dense in* $\text{lip}\,(\alpha, X)$ *for all* $0 < \alpha < 1$, *and*

II. $\mathscr{R}(X)\bar{\mathscr{P}}_3$ *is dense in* $D^1(X)$.

For the definitions of $\text{lip}\,(\alpha, X)$ and $D^1(X)$, we refer the reader to [3].

It is not known whether $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $\mathscr{C}(X)$ for all X without interior. The annular Swiss Cheese of A. Roth [5] has the property that the continuous function $|z|^2$ is not in the uniform closure of $\mathscr{R}(X)$. However, it is not clear whether or not the function $|z|$ is in the uniform closure of $\mathscr{R}(X)\bar{\mathscr{P}}_1$.

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