PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

FUNCTIONS WHICH OPERATE ON THE REAL PART OF A UNIFORM ALGEPRA

S. J. SIDNEY

Three theorems are proved to the effect that a nonaffine function h on an interval cannot operate by composition on the real part of a uniform algebra on X unless the algebra is $C(X)$. The additional hypotheses necessary are, respectively, that h be continuously differentiable, that h be "highly" nonaffine in a suitable sense, and that h operate in a rather weakly bounded manner. These results contain and extend work of J. Wermer and of A. Bernard.

1. Introduction. Given a space of functions, its symbolic calculus is a standard object of study, particularly if the space is associated with a Banach algebra. This paper is concerned with the space $\mathrm{Re}A = \{\mathrm{Re}(f): f \in A\}$ where A is a uniform algebra on a compact Hausdorff space X. It is an old conjecture that, unless $A = C(X)$, the symbolic calculus of $\mathrm{Re}A$ is trivial in that the only functions which operate by composition on $\mathrm{Re}A$ are the affine functions $t \rightarrow at + b$, which obviously operate.

Precisely, suppose A is a uniformly closed subalgebra of $C(X)$ which contains the constant functions and separates the points of X, I is an interval, and $h: I \rightarrow \boldsymbol{R}$ is not the restriction of an affine function. The conjecture is that under these cooditions, if h operates by composition on $\mathrm{Re}A$ in the sense that $h \circ u \in \mathrm{Re}A$ whenever $u \in \mathrm{Re}A$ has range in I, then it follows that $A = C(X)$. We shall prove three theorems along these lines.

The history of this problem probably begins with J. Wermer's paper [6], whose conclusion that $\mathrm{Re}A$ cannot be closed under products is equivalent to the conjecture for $h(t) = t^2$ (and, by induction on degree, implies the conjecture for any polynomial of degree at least 2) on any interval. Some time later, A. Bernard [2] proved the conjecture for $h(t) = |t|$ on $I = \boldsymbol{R}$. Our first two results, like Wermer's and Bernard's, place restrictions on h. Either contains Wermer's theorem, but not Bernard's.

THEOREM 1. *Suppose I is an open interval and $h: I \rightarrow \boldsymbol{R}$ is not affine but is continuously differentiable. If h operates by composition on* $\mathrm{Re}A$, *then $A = C(X)$.*

THEOREM 2. *Suppose that $h: I \rightarrow \boldsymbol{R}$ and that I contains a nondegenerate subinterval J such that h is not affine on any nonde-*

generate subinterval of J. If h operates by composition on $\mathrm{Re}\, A$, *then* $A = C(X)$.

Any antiderivative of the standard Cantor function satisfies the hypotheses of Theorem 1 but not those of Theorem 2. What sort of function remains uncovered by any known theorem? The Cantor function is an example, which is in fact more or less typical. For suppose $h: I \to \boldsymbol{R}$ is continuous and not affine, but still operates on $\mathrm{Re}A$. Let P denote the set of all points of I in no neighborhood of which h is affine. If P has an isolated point interior to I, then the graph of h has a corner there, hence $|t|$ operates on $\mathrm{Re}A$, so $A = C(X)$ by Bernard's theorem. If P contains a nondegenerate interval, then $A = C(X)$ by Theorem 2. Thus if $A \neq C(X)$, then P, which is closed (in I), must be nowhere dense but dense-itself, so (essentially) a Cantor set.

It is also possible to obtain results by placing conditions on the manner in which h operates on $\mathrm{Re}A$. In Bernard's paper, the conjecture is proved if h operates boundedly in a suitable sense. Our third theorem is of this type, with a very weak boundedness hypothesis. $\mathrm{Re}\, A$ is a Banach space with the usual quotient norm $N(u) = \inf\{\|f\|_X : f \in A, \mathrm{Re}(f) = u\}$.

THEOREM 3. *Suppose that I is an open interval and that $h: I \to \boldsymbol{R}$ is not affine but operates by composition on* $\mathrm{Re}\, A$ *in such a fashion that the following boundedness condition holds: Whenever $u \in \mathrm{Re}A$ has range in I, there are positive numbers $\sigma(u)$, $M(u)$ and a dense subset $S(u)$ of $(-\sigma(u), \sigma(u))$ such that $N(h \circ (u + t)) < M(u)$ for all $t \in S(u)$. Then $A = C(X)$.*

Thus we postulate a week local boundedness condition only in the direction of the constants. This condition holds if, for instance, the mapping $u \to h \circ u$ is either continuous or locally bounded on its domain in $\mathrm{Re}\, A$. Theorem 3 contains Bernard's boundedness results.

A third approach to obtaining results is to place restrictions on the uniform algebra A. While we shall present no results of this type here, it is worth noting two instances in the literature. First, in his cited paper Bernard proves the conjecture if $\mathrm{Re}A$ is "regular", and he shows that the real part of the disc algebra on the circle is regular. Second, Bernard and A. Dufresnoy [3] prove the conjecture when h operates in a suitably bounded manner on certain restrictions of boundary value algebras for analytic functions.

The remainder of this paper will be devoted to proving the three theorems, taking considerable advantage of Bernard's machinery. Perhaps our methods can be adapted to prove the general conjecture.

In §2 we shall present the common broad outline of the three proofs. §3 is devoted to the details of the proof of Theorem 2. Section 4 provides some preliminary information necessary to prove Theorems 1 and 3, and §§5 and 6 consist of the proofs of these theorems.

2. Outline of the proofs. If E is a (real or complex) Banach space of continuous functions on X whose norm N dominates the supremum norm $\|\cdot\|_X$, $l^\infty(\boldsymbol{N}, E)$ consists of all sequences $(f_n)_{n\in\boldsymbol{N}}$ where $f_n \in E$ and $\tilde{N}((f_n)) \equiv \sup\{N(f_n)\colon n \in \boldsymbol{N}\} < \infty$. Endowed with $\tilde{N}$ as norm, $l^\infty(\boldsymbol{N}, E)$ is a Banach space which embeds continuously in $C(\tilde{X})$ where $\tilde{X} = \beta(\boldsymbol{N}\times X)$, the Stone-Čech compactification of $\boldsymbol{N}\times X$; the embedding, which we will often interpret as an identification, associates to $(f_n) \in l^\infty(\boldsymbol{N}, E)$ that unique $\tilde{f} \in C(\tilde{X})$ for which $\tilde{f}(n, x) = f_n(x)$ for every $(n, x) \in \boldsymbol{N}\times X$. If $f \in C(X)$, $\tilde{f}$ denotes the element $(f, f, f, \cdots)$ of $C(\tilde{X})$. The point of passing to $\tilde{E}$, the image of $l^\infty(\boldsymbol{N}, E)$ under this embedding, is that it enables one to use Bernard's lemma [2, Proposition 1]: If E consists of real-valued functions and $\tilde{E}$ is uniformly dense in $C_R(\tilde{X})$, then $E = C_R(X)$.

This lemma will be applied to $E = \mathrm{Re}A$. Let V denote the (uniform) closure in $C_R(\tilde{X})$ of $(\mathrm{Re}\,A)^\sim = \mathrm{Re}(\tilde{A})$, and let $B = \{\tilde{u} \in C_R(\tilde{X})\colon \tilde{u}\tilde{v} \in V\ \forall \tilde{v} \in (\mathrm{Re}\,A)^\sim\} = \{\tilde{u} \in C_R(\tilde{X})\colon \tilde{u}\tilde{v} \in V\, \forall \tilde{v} \in V\}$. Clearly B is a closed subalgebra of $C_R(\tilde{X})$ which contains the constants and (because V contains the constants) is contained in V. Our objective will be to verify that B separates the points of $\tilde{X}$. Then the Stone-Weierstrass theorem will imply that $B = C_R(\tilde{X})$, hence $V = C_R(\tilde{X})$, so $(\mathrm{Re}A)^\sim$ will be uniformly dense in $C_R(\tilde{X})$ and by Bernard's lemma $\mathrm{Re}A = C_R(X)$, whence $A = C(X)$ by, for instance, a well-known theorem of K. Hoffman and Wermer [5; cf. also 2].

Suppose now that $h\colon I \to \boldsymbol{R}$ operates on $\mathrm{Re}A$. It is well-known that h must be continuous. If h is not affine, applying a theorem of K. de Leeuw and Y. Katznelson [4; cf. also 2, appendice] to the uniform closure of $\mathrm{Re}A$, it follows readily that $\mathrm{Re}A$ is uniformly dense in $C_R(X)$, hence [2, Corollaire 3] $\tilde{A}$ and $(\mathrm{Re}\,A)^\sim$ separate points on $\tilde{X}$.

It is worth noting that this conclusion—that $(\mathrm{Re}A)^\sim$ separates points—depends on the fact that A is a uniform algebra; no further portion of the proof that $\mathrm{Re}A = C_R(X)$ in Theorem 2 involves $\mathrm{Re}\,A$ being the real part of an algebra of functions. Thus we may conclude that $E = C_R(X)$ whenever a continuous h as in Theorem 2 operates on E, a Banach space in $C_R(X)$ which contains the constant functions and is "ultraseparating" in that $\tilde{E}$ separates the points of $\tilde{X}$. Similar comments via-à-vis the other theorems will be made later. Ultraseparating Banach spaces and Banach algebras are discussed in [2] and [3]; for an "intrinsic" description of when a semisimple commutative Banach algebra is ultraseparating, see the paper of B.T. Batikjan and E.A. Gorin [1].

Continuing with our outline, we use a category argument to find roughly (precisely, in the case of Theorem 2) a ball in $(\mathrm{Re}A)^\sim$ which is carried by h into V. A device from the proof of the de Leeuw-Katznelson theorem is then used to replace h by a suitably nonaffine continuously differentiable function ϕ; this step is not needed for Theorem 1. It is then shown that ϕ' carries the ball into B, that is, that $\phi' \circ \tilde{u} \in B$ for each $\tilde{u}$ in the ball. Careful choices of $\tilde{u}$ yield the point-separating property. In the proofs of Theorems 1 and 3, there are pairs of points which we cannot separate in this manner, but for them alternative separating schemes exist.

3. Proof of Theorem 2. Now assume h is as in Theorem 2. Choose $a < b$ so that $[a, b] \subset J$. Let $D = \{u \in \mathrm{Re}A: a \leqq u \leqq b\}$, and for each $n \in \boldsymbol{N}$ let $D_n = \{u \in D: N(h \circ u) < n\}$. D is closed in $\mathrm{Re}A$, so complete and $D = \cup D_n$, so by the Baire category theroem the closure in $\mathrm{Re}A$ of some D_n has nonempty interior in D. Thus there are $u^0 \in D$, $\eta > 0$, and $r \in \boldsymbol{N}$ such that $U \cap D_r$ is dense in $U \cap D$ where $U = \{u \in \mathrm{Re}A: N(u - u^0) < 3\eta\}$. If necessary we may replace u^0 by $su^0 + t$ for appropriate numbers $s \in (0, 1)$ and t, and shrink η somewhat, to arrange that $U \subset D$ and that $U \cap D_r$ be dense in U.

Let $(v_n) = \tilde{v} \in (\mathrm{Re}\,A)^\sim$ satisfy $\tilde{N}(\tilde{v}) < 3\eta$. For each $(n, k) \in \boldsymbol{N} \times \boldsymbol{N}$ choose $u_{nk} \in U \cap D_r$ such that $N(u_{nk} - (u^0 + v_n)) < 1/k$. For each $k \in \boldsymbol{N}$ let $\tilde{u}_k = (u_{nk})_{n \in N} \in (\mathrm{Re}A)^\sim$. Because $u_{nk} \in D_r$, $N(h \circ u_{nk}) < r$, so for each k, $h \circ \tilde{u}_k = (h \circ u_{nk})_{n \in N} \in (\mathrm{Re}A)^\sim$. As $k \to \infty$, $h \circ \tilde{u}_k$ converges uniformly on $\tilde{X}$ to $h \circ ((u^0)^- + \tilde{v})$, hence $h \circ ((u^0)^- + \tilde{v}) \in V$. Thus if for each $\varepsilon > 0$ we let W_ε denote the open ball in $(\mathrm{Re}A)^\sim$ centered at $(u^0)^-$ with radius ε, we have just shown that $h \circ \tilde{u} \in V$ whenever $\tilde{u} \in W_{3\eta}$.

We shall now to some extent imitate the proof of the de Leeuw-Katznelson theorem. For $0 < \delta < \eta$ let λ_δ be a nonnegative continuously differentiable function on $\boldsymbol{R}$ supported in $(-\delta, \delta)$ and with integral 1. Let ϕ_δ denote the convolution

$$\phi_\delta(x) = \int_{-\delta}^{\delta} h(x - t)\lambda_\delta(t)dt = \int_a^b h(t)\lambda_\delta(x - t)dt .$$

ϕ_δ is continuously differentiable on a neighborhood of $[a + \eta, b - \eta]$, and as $\delta \to 0$, ϕ_δ converges to h uniformly on $[a + \eta, b - \eta]$. If $\tilde{u} \in W_{2\eta}$ then $\tilde{u} - t \in W_{3\eta}$ and so $h \circ (\tilde{u} - t) \in V$ whenever $t \in [-\delta, \delta]$, hence $\phi_\delta \circ \tilde{u} \in V$ since $\phi_\delta \circ \tilde{u} = \int (h \circ (\tilde{u} - t)) \cdot \lambda_\delta(t) dt$ and V is uniformly closed; if also $\tilde{v} \in (\mathrm{Re} A)^\sim$ then for small nonzero t, $\tilde{u} + t\tilde{v} \in W_{2\eta}$, hence $(\phi_\delta \circ (\tilde{u} + t\tilde{v}) - \phi_\delta \circ \tilde{u})/t \in V$ and, letting $t \to 0$, $(\phi_\delta' \circ \tilde{u})\tilde{v} \in V$. Thus $\phi_\delta' \circ \tilde{u} \in B$ whenever $\tilde{u} \in W_{2\eta}$. We shall complete the proof of Theorem 2 by showing that the family of such $\phi_\delta' \circ \tilde{u}$ separates the points of $\tilde{X}$.

Let $p, q \in \tilde{X}$, $p \neq q$. Choose $\tilde{w} \in W_\eta$ such that $\tilde{w}(p) \neq \tilde{w}(q)$, possible because $(\mathrm{Re} A)^\sim$ separates points on $\tilde{X}$. Choose $\varepsilon > 0$ so that $\varepsilon < \eta |\tilde{w}(p) - \tilde{w}(q)|/(2\tilde{N}(\tilde{w})) \leqq \eta$. Choose numbers t_1, t_2, t_3 in $(\tilde{w}(p) - \varepsilon, \tilde{w}(p) + \varepsilon)$ so that the three points $(t_j, h(t_j))$ are not collinear, possible because this interval is contained in J. Then choose a positive $\delta < \eta$ small enough so that the points $(t_j, \phi_\delta(t_j))$ are close enough to the points $(t_j, h(t_j))$ to prevent them from being collinear; thus ϕ_δ' is not constant on $(\tilde{w}(p) - \varepsilon, \tilde{w}(p) + \varepsilon)$. If $\phi_\delta'(\tilde{w}(p)) \neq \phi_\delta'(\tilde{w}(q))$, let $\tilde{u} = \tilde{w}$. If $\phi_\delta'(\tilde{w}(p)) = \phi_\delta'(\tilde{w}(q))$, choose $s \in (\tilde{w}(p) - \varepsilon, \tilde{w}(p) + \varepsilon)$ for which $\phi_\delta'(s) \neq \phi_\delta'(\tilde{w}(q))$ and let $\tilde{u} = \tilde{w} + (\tilde{w} - \tilde{w}(q))(s - \tilde{w}(p))/(\tilde{w}(p) - \tilde{w}(q))$, so $\tilde{N}(\tilde{u} - \tilde{w}) < \eta$, $\tilde{u}(q) = \tilde{w}(q)$ and $\tilde{u}(p) = s$. In either case, $\tilde{u} \in W_{2\eta}$ and $\phi_\delta'(\tilde{u}(p)) \neq \phi_\delta'(\tilde{u}(q))$, that is, $\phi_\delta' \circ \tilde{u}$ is an element of B which separates p and q. Theorem 2 is proved.

4. Reduction of other proofs. For each $p \in \tilde{X}$, the functional $u \to \tilde{u}(p)$ on $C_R(X)$ is linear and multiplicative, so there is a unique $x_p \in X$ such that $\tilde{u}(p) = u(x_p)$ for all $u \in C_R(X)$.

For the remainder of the paper, p and q will be fixed distinct points in $\tilde{X}$, and we must find an element of B which separates them. In this section, we show that we may suppose either that (1) $x_p \neq x_q$, or that (2) $x_p = x_q$ and there is $\tilde{w} \in (\mathrm{Re} A)^\sim$ which vanishes on $N \times \{x_p\}$ and at q, but is 1 at p.

Indeed, suppose that (1) fails, so $x_p = x_q$. Suppose further that $\tilde{f}(p) = \tilde{f}(q)$ whenever $\tilde{f} \in \tilde{A}$ vanishes identically on $N \times \{x_p\}$. Then there is a linear functional L on l^∞ such that $\tilde{f}(p) - \tilde{f}(q) = L((\tilde{f}(n, x_p)))$ for all $\tilde{f} \in \tilde{A}$; L is not identically zero because $\tilde{A}$ separates p and q. Choose a real sequence $(c_n) \in l^\infty$ for which $L((c_n)) \neq 0$. The function $\tilde{u} \in C_R(\tilde{X})$ which is identically equal to c_n on $\{n\} \times X$ belongs to $\tilde{A} \cap C_R(\tilde{X})$ and so to B, and $\tilde{u}(p) - \tilde{u}(q) = L((c_n)) \neq 0$, so $\tilde{u}$ separates p and q.

Thus we may suppose that there is $\tilde{f} \in \tilde{A}$ which vanishes on $N \times \{x_p\}$ but for which $\tilde{f}(p) \neq \tilde{f}(q)$. If $\tilde{f}(p) = 0$, interchange p and q. Then, by replacing $\tilde{f}$ by $\alpha\tilde{f} + \beta\tilde{f}^2$ with $\alpha = -\tilde{f}(q)/[\tilde{f}(p)(\tilde{f}(p) - \tilde{f}(q))]$ and $\beta = 1/[\tilde{f}(p)(\tilde{f}(p) - \tilde{f}(q))]$ if necessary, we may arrange that $\tilde{f}(p) = 1$ and $\tilde{f}(q) = 0$. Now $\tilde{w} = \mathrm{Re}\tilde{f}$ will do in case (2).

This is the last point in the proof of Theorem 3 at which the algebraic structure of A is used. Thus Theorem 3 remains true if

A is any ultraseparating Banach algebra in $C(X)$ which contains the constant functions, and h is assumed to be continuous.

5. **Proof of Theorem 1.** Now assume h is as in Theorem 1, and let p, q be distinct points of $\tilde{X}$. We may assume either case (1) or case (2) of § 4 is in effect.

First suppose case (1) holds, so $x_p \neq x_q$. Choose numbers $a < b < c < d$ so that $[a, d] \subset I$ and $h'([a, b]) \cap h'([c, d]) = \emptyset$, possible by the hypothesis on h. Let $D = \{u \in \mathrm{Re}A: a \leqq u \leqq d, a \leqq u(x_p) \leqq b, c \leqq u(x_q) \leqq d\}$, and for each $u \in N$ let $D_n = \{u \in D: N(h \circ u) < n\}$. D is nonempty (since $\mathrm{Re}A$ is dense in $C_R(X)$) and closed in $\mathrm{Re}A$, so complete, and $D = \cup D_r$, so the closure in $\mathrm{Re}A$ of some D_n has nonempty interior in D. Thus there are $u^0 \in D$, $\eta > 0$, and $r \in N$ such that $U \cap D_r$ is dense in $U \cap D$ where $U = \{u \in \mathrm{Re}A: N(u - u^0) < \eta\}$. We may choose $w \in \mathrm{Re}A$ so that $a < w < d$, $a < w(x_p) < b$, $c < w(x_q) < d$; replacing u^0 by $(1 - t)u^0 + tw$ for small $t > 0$ and shrinking η, we may arrange that $U \subset D$ and that $U \cap D_r$ be dense in U. Arguing as in § 3, $h \circ \tilde{u} \in V$ and then $h' \circ \tilde{u} \in B$ whenever $\tilde{u} \in W_\eta$, the open ball in $(\mathrm{Re}A)^\sim$ centered at $(u^0)^-$ with radius η. In particular, $\tilde{v} = h' \circ (u^0)^- \in B$. But $\tilde{v}(p) = h'(u^0(x_p)) \in h'([a, b])$ and $\tilde{v}(q) = h'(u^0(x_q)) \in h'([c, d])$, so $\tilde{v}$ separates p and q.

Now suppose case (2) holds, so $x_p = x_q$ and there is $\tilde{w} \in (\mathrm{Re}A)^\sim$ which vanishes on $N \times \{x_p\}$ and at q but is 1 at p. Choose $s \in I$ such that h is affine—equivalently, h' is constant—on no neighborhood of s. Choose numbers a, b so that $a < s < b$ and $[a, b] \subset I$, let $D = \{u \in \mathrm{Re}A: a \leqq u \leqq b, u(x_p) = s\}$, and for each $n \in N$ let $D_n = \{u \in D: N(h \circ u) < n\}$. As usual there are $u^0 \in D$, $\eta > 0$, and $r \in N$ such that $U \cap D_r$ is dense in $U \cap D$ where $U = \{u \in \mathrm{Re}A: u(x_p) = s$ and $N(u - u^0) < \eta\}$, and on replacing u^0 by $(1 - t)u^0 + ts$ we may arrange that $U \subset D$ and that $U \cap D_r$ be dense in U. Arguing as in § 3, we find that $h \circ \tilde{u} \in V$ whenever $\tilde{u} \in W_\eta$, where $W_\eta = \{\tilde{u} \in (\mathrm{Re}A): \tilde{u}(n, x_p) = s \ \forall n \in N$ and $\tilde{N}(\tilde{u} - (u^0)^-) < \eta\}$. We wish now to show that $h' \circ \tilde{u} \in B$ whenever $\tilde{u} \in W_\eta$.

Let $T = \{\tilde{v} \in (\mathrm{Re}A)^\sim: \tilde{v}(n, x_p) = 0 \ \forall n \in N\}$, a closed subspace of $(\mathrm{Re}A)^\sim$, so W_η is the open η-ball in T translated by $(u^0)^-$. Arguing again with quotients $(h \circ (\tilde{u} + t\tilde{v}) - h \circ \tilde{u})/t$, we find that $(h' \circ \tilde{u})\tilde{v} \in V$ whenever $\tilde{u} \in W_\eta$ and $\tilde{v} \in T$. Since $(\mathrm{Re}A)^\sim$ is spanned by T and the functions which are constant on each $\{n\} \times X$, we need only show that if $\tilde{u} \in W_\eta$, if (c_n) is any real sequence in l^∞, and if $\tilde{v} \in C_R(\tilde{X})$ is identically equal to c_n on $\{n\} \times X$, then $(h' \circ \tilde{u})\tilde{v} \in V$. Replacing h by $t \to h(t) - h'(s)t$ if necessary, we may suppose that $h'(s) = 0$, hence $h' \circ \tilde{u}$ vanishes on $N \times \{x_p\}$. Given $\varepsilon > 0$, choose positive numbers α and β so that $\|h' \circ \tilde{u}\|_{\tilde{X}} \cdot \|(c_n)\|_{l^\infty} \cdot \alpha < \varepsilon$ and $\beta \cdot \|(c_n)\|_{l^\infty} \cdot e < \varepsilon$. For each $n \in N$ let $K_n = \{x \in X: |(h' \circ \tilde{u})(n, x)| \geqq \beta\}$ and choose $g_n \in A$ such that

$\mathrm{Re}(g_n) < 1$, $g_n(x_p) = 0$, and $\mathrm{Re}(g_n) < \log \alpha$ on K_n. Let $f_n = 1 - e^{g_n} \in A$, so $|f_n - 1| < e$, $f_n(x_p) = 0$, and $|f_n - 1| < \alpha$ on K_n. The sequence $(c_n f_n)$ belongs to $l^\infty(N, A)$, so the sequence $(c_n \mathrm{Re}(f_n))$ defines an element $\tilde{z}$ of T. Thus $(h' \circ \tilde{u})\tilde{z} \in V$, and it is easy to verify that $|(h' \circ \tilde{u})\tilde{z} - (h' \circ \tilde{u})\tilde{v}| \leqq \varepsilon$ on $N \times X$, so on $\tilde{X}$. Since ε is arbitrary and V is uniformly closed, it follows that $(h' \circ \tilde{u})\tilde{v} \in V$. Thus $h' \circ \tilde{u} \in B$, as we wished to show. This argument is the last place in which the function algebras context is used.

Choose $t \neq 0$ so that $h'(s + t) \neq h'(s)$, and so that $|t|\tilde{N}(\tilde{w}) < \eta$. Set $\tilde{u} = (u^0)^{\sim} + t\tilde{w} \in W_\eta$. Then $h' \circ \tilde{u} \in B$ by what we have just proven, and $(h' \circ \tilde{u})(p) = h'(s + t) \neq h'(s) = (h' \circ \tilde{u})(q)$. This completes the proof of Theorem 1.

6. Proof of Theorem 3. Now assume the hypotheses of Theorem 3, and let p, q be distinct points of $\tilde{X}$. We may again assume that either case (1) or case (2) of § 4 holds. Choose $s \in I$ so that h is not affine on any neighborhood of s.

Suppose case (1) holds, so $x_p \neq x_q$. Take numbers $a < b < c < d$ such that $a < s < d$, $[a, d] \subset I$, and $s \notin [b, c]$. Let $D = \{u \in \mathrm{Re}A$: $a \leqq u \leqq d$, $u(x_p) = s$, $b \leqq u(x_q) \leqq c\}$, and for each $n \in N$ let $D_n = \{u \in D$: $\sigma(u) > 2/n$, $M(u) < n\}$. Then there are $u^0 \in D$, $\tau > 0$ and $r \in N$ such that $U \cap D_r$ is dense in $U \cap D$ where $U = \{u \in \mathrm{Re}\, A$: $u(x_p) = s$, $N(u - u^0) < 4\tau\}$. By choosing $w \in D$ so that $a < w < d$ and $b < w(x_q) < c$, shrinking τ, and replacing u^0 by $(1 - t)u^0 + tw$ for small $t > 0$, we may ensure that $U \subset D$ and that $U \cap D_r$ be dense in U. Let $\eta = \min\{\tau, 1/r\}$. Suppose $v \in \mathrm{Re}A$ and $N(v) < 2\eta$. Then $u^0 + v - v(x_p) \in U$. Given $\varepsilon > 0$, take $u \in U \cap D_r$ so that $N(u - (u^0 + v - v(x_p))) < \varepsilon/2$ and then take $t \in S(u)$ so that $|t - v(x_p)| < \varepsilon/2$, possible because $|v(x_p)| \leqq N(v) < 2\eta \leqq 2/r < \sigma(u)$. Then $N(h \circ (u + t)) < M(u) < r$ while $N((u + t) - (u^0 + v)) \leqq N(u - (u^0 + v - v(x_p))) + |t - v(x_p)| < \varepsilon$. In other words, the open ball in $\mathrm{Re}A$ centered at u^0 with radius 2η has a dense subset consisting of functions u with the property that $N(h \circ u) < r$. As usual, it follows that $h \circ \tilde{u} \in V$ whenever $\tilde{u} \in W_{2\eta}$, where for each $\varepsilon > 0$, W_ε denotes the open ball in $(\mathrm{Re}A)^{\sim}$ centered at $(u^0)^{\sim}$ with radius ε.

For $0 < \delta < \eta$, construct ϕ_δ as in § 3, so $\phi'_\delta \circ \tilde{u} \in B$ whenever $\tilde{u} \in W_\eta$. If $0 < \varepsilon < \eta|s - u^0(x_q)|/(2N(u^0)) \leqq \eta$ then, as in § 3, for small enough δ, ϕ'_δ will not be constant on $(s - \varepsilon, s + \varepsilon)$; choose δ this small. If $\phi'_\delta(s) \neq \phi'_\delta(u^0(x_q))$ take $\tilde{u} = (u^0)^{\sim}$; if $\phi'_\delta(s) = \phi'_\delta(u^0(x_q))$ let $\tilde{u} = (u^0)^{\sim} + ((u^0)^{\sim} - u^0(x_q))(t - s)/(s - u^0(x_q))$ for some $t \in (s - \varepsilon, s + \varepsilon)$ such that $\phi'_\delta(t) \neq \phi'_\delta(u^0(x_q))$. In either event, $u \in W_\eta$ and $\phi'_\delta \circ \tilde{u}$ is an element of B which separates p and q. This completes the argument when case (1) holds.

Finally, suppose that case (2) holds, so $x_p = x_q$ and there is $\tilde{w} \in (\mathrm{Re}A)^{\sim}$ which vanishes on $N \times \{x_p\}$ and at q, but $\tilde{w}(p) = 1$. Choose numbers a, b with $a < s < b$ and $[a, b] \subset I$. Let $D = \{u \in \mathrm{Re}A\colon a \leqq u \leqq b, u(x_p) = s\}$, and let u^0, τ, r, U, η and W_ε be as in the above argument for case (1). Exactly as in that argument, $h \circ \tilde{u} \in V$ whenever $\tilde{u} \in W_{2\eta}$; construct the ϕ_δ again, and choose δ small enough so that ϕ'_δ is not constant on $(s - \varepsilon, s + \varepsilon)$, where ε is chosen as in case (1). Choose t, $0 < |t| < \varepsilon$, small enough that $|t|\,\tilde{N}(\tilde{w}) < \eta$, and so that $\phi'_\delta(s + t) \neq \phi'_\delta(s)$. Then $\tilde{u} = (u^0)^{-} + t\tilde{w} \in W_\eta$, so $\phi'_\delta \circ \tilde{u} \in B$, and $(\phi'_\delta \circ \tilde{u})(p) = \phi'_\delta(s + t) \neq \phi'_\delta(s) = (\phi'_\delta \circ \tilde{u})(q)$. Thus $\phi'_\delta \circ \tilde{u}$ is an element of B which separates p and q. This completes the argument for case (2), as well as the proof of Theorem 3.

REFERENCES

1. B. T. Batikjan and E. A. Gorin, *Ultraseparating algebras for continuous functions*, Vestnik Moskov. Univ. Ser. I Mat. Meh., **31** (1976), 15–20; English transl. Moscow Univ. Math. Bull., **31** (1976), 71–75.

2. A. Bernard, *Espaces des parties réelles des éléments d'une algèbre de Banach de fonctions*, J. Functional Analysis, **10** (1972), 387–409.

3. A. Bernard and A. Dufresnoy, *Calcul symbolique sur la frontière de Šilov de certaines algèbres de fonctions holomorphes*, Ann. Inst. Fourier, Grenoble, **25** 2(1975), 33–43.

4. K. de Leeuw and Y. Katznelson, *Functions that operate on non-selfadjoint algebras*, J. Analyse Math., **11** (1963), 207–219.

5. K. Hoffman and J. Wermer, *A characterization of $C(X)$*, Pacific J. Math., **12** (1962), 941–944.

6. J. Wermer, *The space of real parts of a function algebra*, Pacific J. Math., **13** (1963), 1423–1426.

Received January 15, 1978. Presented to the American Mathematical Society at the 84th Annual Meeting in Atlanta, Georgia, on Wednesday, January 4, 1978.

THE UNIVERSITY OF CONNECTICUT
STORRS, CT 06268