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A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP

GARY SHERMAN

A lower bound is given for the number of conjugacy classes in a finite nilpotent group which reflects the nilpotency class of the group.

The problem of estimating the number of conjugacy classes, k, in a finite group G, has been around since the turn of the century. Probably the earliest version of the problem is the question: Do there exist groups of arbitrarily large finite order with a fixed number of conjugacy classes? In 1903 Landau [4] answered this question in the negative by showing $k(G)$ goes to infinity with $|G|$. By refining Landau's technique, Erdos and Turan [2] proved $k(G) > \log_2 \log_2 |G|$. The known lower bound for $k(G)$ when G is nilpotent is somewhat better, $k(G) > \log_2 |G|$. This follows from a parametric equation for $k(G)$ when G is a p-group given by Poland [5].

In [3] Gustafson posed the problem of finding improved lower bounds for $k(G)$. Recently, Bertram [1] provided a substantial improvement of the $\log_2 \log_2 |G|$ bound which holds for "most" group orders. The purpose of this note is to give a lower bound for $k(G)$ when G is nilpotent which reflects the nilpotency class of G and improves the $\log_2 |G|$ bound.

THEOREM. *If G is a finite nilpotent group of nilpotency class n, then* $k(G) \geqq n|G|^{1/n} - n + 1$.

Proof. We observe that

$$G = Z_0 \cup \left(\bigcup_{i=1}^{n} Z_i - Z_{i-1} \right) \tag{1}$$

where $e = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n = G$ is the upper central series of G. Since Z_i and Z_{i-1} are normal subsets of G, $Z_i - Z_{i-1}$ is a union of conjugacy classes of G. Indeed, for $x \in Z_i - Z_{i-1}$ and $g \in G$ we have $x^{-1}g^{-1}xg \in Z_{i-1}$ because Z_i/Z_{i-1} is the center of G/Z_{i-1}. This implies $g^{-1}xg \in xZ_{i-1}$ and we conclude $\bar{x}$, the conjugacy class of x in G, is contained in xZ_{i-1}. Thus $|\bar{x}| \leqq |xZ_{i-1}| = |Z_{i-1}|$ and therefore $Z_i - Z_{i-1}$ is a union of at least $|Z_i|/|Z_{i-1}| - 1$ conjugacy classes. It follows from (1) that

$$k(G) \geqq 1 + \sum_{i=1}^{n} (|Z_i|/|Z_{i-1}| - 1)$$

$$= \left(\sum_{i=1}^{n} |Z_i|/|Z_{i-1}|\right) - n + 1$$

$$(2) \qquad = \frac{1}{n}\left(\sum_{i=1}^{n} n\,|Z_i|/|Z_{i-1}|\right) - n + 1\,.$$

The arithmetic-geometric means inequality applied to the sum in (2) yields

$$k(G) \geqq \left(\prod_{i=1}^{n} n\,|Z_i|/|Z_{i-1}|\right)^{1/n} - n + 1$$
$$= n|G|^{1/n} - n + 1\,.$$

Let us illustrate how this result can be used to sharpen the $\log_2 |G|$ bound for $k(G)$. Specifically, suppose G is a nilpotent group of order $2^5 5^7 7^4$. We note that $k(G) \geqq 33$ since $\log_2(2^5 5^7 7^4) > 32$.

Can we determine the nilpotency class of G? Not exactly, but the class of a nilpotent group is the maximum of the classes of its p-Sylow subgroups and the class of a p-group of order p^m, $m \geqq 3$, is at most $m - 1$ so the class of G is at most 6. Fortunately $n|G|^{1/n} - n + 1$ is a decreasing function of n and therefore $k(G) \geqq 6(2^5 5^7 7^4)^{1/6} - 5 > 250$. Thus $k(G) \geqq 251$. To improve this bound we make use of the fact that $k(G)$ is multiplicative; i.e., the number of conjugacy classes in a direct product is the product of the number of conjugacy classes in each factor. This implies $k(G) \geqq (4 \cdot 2^{5/4} - 3)(6 \cdot 5^{7/6} - 5)(3 \cdot 7^{4/3} - 2) > 8510$. Thus $k(G) \geqq 8511$.

As a corollary to the theorem and the preceding remarks:

THEOREM. *If G is a finite nilpotent of order $p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ and nilpotency class n, then*

$$k(G) \geqq \prod_{i=1}^{s} (t_i(p_i^{r_i/t_i}) - t_i + 1) \geqq n|G|^{1/n} - n + 1 > \log_2 |G|\,,$$

where the p_i's are distinct primes and $t_i = \max\{1, r_i - 1\}$.

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ROSE-HULMAN INSTITUTE OF TECHNOLOGY
TERRE HAUTE, IN 47803