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SOME PROPERTIES OF THE CHEBYSHEV METHOD

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Several properties of the Chebyshev method of summability, defined by G. G. Bilodeau, are investigated. Specifically, it is shown that the Chebyshev method is translative and is a Gronwall method. It is shown that the de Vallee Poussin method is stronger than the Chebyshev method, and that the Chebyshev method is not stronger than the $(C, 1)$ method. The final result shows that the Chebyshev method exhibits the Gibbs phenomenon.

Let $\Sigma(-1)^i u_i$ be an alternating series with partial sums $s_n = \sum_{i=0}^{n}(-1)^i u_i$. Define a sequence of polynomials $\{P_n(t)\}$ by $P_n(t) = \sum_{k=0}^{n} a_{nk}t^k$, $P_n(1) = 1$, $n = 0, 1, 2, \cdots$. The series $\Sigma(-1)^i u_i$ will be called summable (P_n) to the value s if $\lim \sigma(P_n) = s$, where $\sigma(P_n) = \sum_{k=0}^{n} a_{nk}s_k$. Bilodeau [1] considered the following question. What are sufficient conditions on P_n for $\sigma(P_n)$ to speed up the rate of convergence of a convergent sequence $\{s_n\}$? For sequences $\{u_n\}$ which are moment sequences, i.e., u_n has the representation $u_n = \int_0^1 t_n d\alpha(t)$, where $\alpha(t) \in BV[0, 1]$, he obtains the estimate $|\sigma(P_n) - s|/|r_n| \leqq (\mu_n/|r_n|)\int_0^1 t(1+t)^{-1}|d\alpha(t)|$, where $s = \sum_{i=0}^{\infty}(-1)^i u_i$, $r_n = s_n - s$, and $\mu_n = \max_{0\leqq t\leqq 1}|P_n(-t)|$. Adopting μ_n as a measure of the value of the method $\sigma(P_n)$, the most desirable sequence of polynomials will be those for which μ_n is a minimum, subject to the constraint $P_n(1) = 1$ for each n. The Chebyshev polynomials, defined by $T_n(x) = \cos nx$, $n = 0, 1, 2, \cdots, x = \cos\theta$, form the best approximation to the zero function over the interval $[-1, 1]$. When translated to $[0, 1]$ they give $P_n(t) = T_n(1+2t)/T_n(3)$ as the best polynomials to minimize μ_n, where

$$T_n(x) = [(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n]/2\,, \tag{1}$$

and

$$T_n(3) = (\alpha^n + \alpha^{-n})/2,\ \alpha = 3+\sqrt{8} \approx 5.828\,.$$

The infinite matrix $A = (a_{nk})$, associated with these polynomials, has entries

$$a_{nk} = \begin{cases} 1/T_n(3), & k = 0 \\ \dfrac{2^{2k-1}}{T_n^{(3)}}\left[2\binom{n+k}{n} - \binom{n+k-1}{n-k}\right], & 0 < k \leqq n \\ 0, & k > n\,. \end{cases} \tag{2}$$

Bilodeau calls the associated summability method the Chebyshev or σ-method.

We begin by establishing some properties of the maximal entry in each row of σ.

LEMMA 1. *For each positive integer* $n > 2$, *there exists an integer* p *such that*

$$a_{nk} < a_{n,k+1} \quad \text{for } 0 \leqq k < p$$
$$a_{nk} \geqq a_{n,k+1} \quad \text{for } p \geqq k < n .$$

Proof. For $0 < k \leqq n$ we may write

$$a_{nk} = \frac{2^{2k-1}n}{kT_n(3)}\binom{n+k-1}{n-k}, \tag{3}$$

so that $a_{nk}/a_{n,k+1} = (k+1)(2k+1)/2(n^2-k^2)$. Treating k as a continuous variable and differentiating with respect to k, it follows that $a_{nk}/a_{n,k+1}$ is increasing in k. The proof is completed by noting that $a_{n0} < a_{n1} < a_{n2}$ and $a_{n,n-1} > a_{nn}$ for each $n > 2$.

LEMMA 2. *For each* n, $p = [x_0]$ *where* $x_0 = (-3 + (32n^2 - 7)^{1/2})/8$.

Proof. Since $a_{n1} < a_{n2}$ and $a_{n,n-1} > a_{nn}$, there exists a real positive number x_0 such that $a_{nx_0} = a_{n,x_0+1}$ which implies $2x_0^2 + 3x_0 + 1 = 2n^2 - 2x_0^2$. Since x_0 is positive, $x_0 = (-3 + (32n^2 - 7)^{1/2})/8$.

LEMMA 3. *For each* $n > 6$, $p = [x_0] > n/2$.

It is sufficient to show that $x_0 - 1 \geqq n/2$; i.e., $8(2n^2 - 11n - 16) \geqq 0$, for $n > 6$. With $g(n) = 2n^2 - 11n - 16$ we have $g'(n) > 0$ for $n > 11/4$, hence g is increasing for $n > 11/4$, and g is positive for $n > 6$ and n an integer.

LEMMA 4. *With* p *and* a_{np} *as defined in Lemmas* 2 *and* 3, $\lim_n a_{np} = 0$.

From (3), and Stirling's formula,

$$a_{np} = \frac{n2^{2p-1}\Gamma(n+p)}{PT_n(3)\Gamma(n-p+1)\Gamma(2p)}$$

$$\sim \frac{n2^{2p-1}(n+p-1)^{n+p-1}e^{-(n+p-1)}(2\pi(n+p-1))^{1/2}}{p\alpha^n(n-p)^{n-p}e^{-(n-p)}(2\pi(n-p))^{1/2}(2p-1)^{2p-1}e^{-(2p-1)}(2\pi(2p-1))^{1/2}} \tag{4}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{\left(p-\frac{1}{2}\right)^{1/2}}{p}\frac{n}{((n+p-1)(n-p))^{1/2}}\left(\frac{n+p-1}{\alpha(n-p)}\right)^{n-p}\left(\frac{n+p-1}{\sqrt{\alpha}\left(p-\frac{1}{2}\right)}\right)^{2p}.$$

Both $((n+p-1)/\alpha(n-p)^{n-p})$ and $((n+p-1)/\sqrt{\alpha}(p-1/2)^{2p})$ are bounded above. Therefore $\lim_n a_{np} = 0$.

Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all bounded sequences $\{z_n\}$ is that the matrix A satisfies $\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0$.

THEOREM 1. *The σ-method is absolutely translative for all bounded sequences.*

Proof. Bilodeau [1, p. 296] has shown that the σ-method is regular. From Lemma 1,

$$\sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}|$$
$$= \sum_{k=1}^{p-1} (a_{n,k+1} - a_{nk}) + \sum_{k=p}^{n} (a_{nk} - a_{n,k+1})$$
$$= 2a_{np} - a_{n0}.$$

The regularity of A implies that $\lim_n a_{n0} = 0$, and the result follows from Lemma 4.

For unbounded sequences, we consider the class of sequences $\{z_n\}$ satisfying $|z_k| \leqq \theta_k$ (θ_k real, positive, and increasing), where $\sum_{k=0}^{\infty} a_{nk}\theta_{k+1}$, $\sum_{k=0}^{\infty} a_{n,k+1}\theta_{k+1}$, and $\rho_n = \sum_{k=0}^{\infty} |(a_{nk} - a_{n,k+1})\theta_{k+1}|$ exist for each n. Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all (unbounded) $\{z_n\}$ satisfying $|z_k| \leqq \theta_k$ together with conditions stated above, is that $\lim_n \rho_n = 0$.

THEOREM 2. *The σ-method is absolutely translative for all (unbounded) sequences $\{z_n\}$ such that $z_k = o(\sqrt{k})$. This result is best possible.*

With $|z_n| = \theta_n$, and using Lemma 1,

$$\rho_n = \sum_{k=0}^{p-1} (a_{n,k+1} - a_{nk})\theta_{k+1} + \sum_{k=p}^{n} (a_{nk} - a_{n,k+1})\theta_{k+1}$$
$$(5) \qquad \leqq \theta_{p-1} \sum_{k=1}^{p-1} (a_{n,k+1} - a_{nk}) + \theta_n \sum_{k=p}^{n} (a_{nk} - a_{n,k+1})$$
$$\leqq \theta_n(a_{np} - a_{n0} + a_{np} - 0) = 0(\sqrt{n})(2a_{np} - a_{n0}) .$$

It will be sufficient to show that $\overline{\lim}_n 2\sqrt{n}a_{np}$ is finite. But this follows immediately from (4), since $\lim_n (n(p-1/2))^{1/2}/p = 2^{1/4}$, and the remaining limits have already been shown to be finite.

To show that the result is best possible we shall replace $o(\sqrt{k})$

by $\sqrt{k}$ and verify that ρ_n does not tend to zero.

From (5), $\rho_n \geqq \sqrt{p} \sum_{k=p}^{n} (a_{nk} - a_{n,k+1}) = \sqrt{p} a_{np}$, which does not tend to zero.

Direct calculations verify that σ is not a weighted mean, Nörlund, Hausdorff, or generalized Hausdorff method.

Gronwall [4, p. 102] defined a general class of summability methods, each member of which involves a pair of analytic functions f and g. Specifically, the (f, g)-transform of a given series $\sum_{k=0}^{\infty} u_k$ is the sequence $\{U_n\}$ defined implicitly by the formal power series identity

$$(6) \qquad g(w) \sum_{n=0}^{\infty} u_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n \,,$$

where f and g satisfy the following properties. Let $\varDelta = \{w \mid |w| < 1\}$. The function $z = f(w)$ is analytic in $\bar{\varDelta} - \{1\}$, continuous and $1 - 1$ in $\bar{\varDelta}$, with $f(0) = 0$, $f(1) = 1$, and $|f(w)| < 1$ for $w \in \varDelta$. Moreover, $w = f^{-1}(z)$ has the representation $w = 1 - (1 - z)^{\lambda}[a + a_1(1 - z) + \cdots]$, where $\lambda \geqq 1$, $a > 0$, and the quantity in brackets is a power series in $1 - z$ with a positive radius of convergence. The function g satisfies $g(w) \neq 0$ for $w \in \varDelta$ and has the form $g(w) = (1 - w)^{-\delta} + \gamma(w)$ for some $\delta > 0$, where $\gamma(w)$ is analytic in $\bar{\varDelta}$. Also $g(w) = \sum_{n=0}^{\infty} b_n w^n$, with $b_n \neq 0$ for each n. The series $\sum_{k=0}^{\infty} u_k$ is said to be (f, g)-summable to s if $\lim U_n = s$.

Examples of (f, g)-methods are the Cesàro methods of order k, (C, k), for k real and greater than -1; (E, β) (Euler-Knopp) for $0 < \beta \leqq 1$; de la Vallée Poussin summability (V); a generalized (V)-summability (Vk), introduced by Gronwall; and a method of summation of Obrechkoff. We will now show that the Chebyshev method is also a Gronwall method.

Writing $s_n = \sum_{k=0}^{n} u_k$, the (f, g)-method can be expressed as a sequence to sequence method by rewriting (6) in the form

$$(7) \qquad g(w)[1 - f(w)] \sum_{n=0}^{\infty} s_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n \,.$$

Using (7), (f, g) can be expressed as a triangular matrix transformation of the form $U_n = \sum_{k=0}^{n} a_{nk} s_k$, with $a_{nk} = \gamma_{nk}/b_n$, where γ_{nk} is defined by

$$(8) \qquad [1 - f(w)] g(w) [f(w)]^k = \sum_{n=k}^{\infty} \gamma_{nk} w^k \,.$$

(See, for example, the discussion on page 40 of [2], where the roles of γ_{nk} and a_{nk} have been interchanged.) From (8) it follows that

$$a_{nn} = [f'(0)]^n/b_n\,, \quad n \geqq 0\,. \tag{9}$$

THEOREM 3. *The Chebyshev method is a Gronwall method with* $f(w) = w(\alpha - 1)^2/(\alpha - w)^2$, $g(w) = (1 - w)^{-1} + \gamma(w)$, *and* $\gamma(w) = w/(\alpha^2 - w)$, *where* $\alpha = 3 + \sqrt{8}$.

Proof. If (6) is a Gronwall method, then, from (8) with $k = 0$ and (2),

$$[1 - f(w)]g(w) = \sum_{n=0}^{\infty} b_n a_{n0} w^n = \sum_{n=0}^{\infty} b_n w^n / T_n(3)\,.$$

Thus

$$\begin{aligned} f(w) &= 1 - [g(w)]^{-1} \sum_{n=0}^{\infty} b_n w^n / T_n(3)\,, \\ f'(w) &= [g'(w)/g^2(w)] \sum_{n=0}^{\infty} b_n w^n / T_n(3) - [g(w)]^{-1} \sum_{n=1}^{\infty} n b_n w^{n-1} / T_n(3) \end{aligned} \tag{10}$$

and $f'(0) = [g'(0)/g^2(0)](b_0/T_0(3)) - b_1/g(0)T_1(3) = 2b_1/3b_0$, since $T_0(3) = 1$ and $T_1(3) = 3$.

From (9) and (3),

$$b_n = (2b_1/3b_0)^n T_n(3)/2^{2n-1} = (b_1/6b_0)^n(\alpha^n + \alpha^{-n})\,. \tag{11}$$

In particular, $b_1 = b_1/b_0$, which implies $b_0 = 1$, since each $b_n \neq 0$. One can also deduce that $b_0 = 1$ from (9), since $a_{00} = 1$.

Thus

$$\begin{aligned} g(w) &= 1 + \sum_{n=1}^{\infty} b_n w^n \\ &= 1 + \sum_{n=1}^{\infty} [(b_1 \alpha w/6)^n + (b_1 w/6\alpha)^n] \\ &= 1 + \frac{b_1 \alpha w}{6 - b_1 \alpha w} + \frac{b_1 w}{6\alpha - b_1 w} \\ &= \frac{6}{6 - b_1 \alpha w} + \frac{b_1 w}{6\alpha - b_1 w}\,. \end{aligned}$$

For g to have the required form choose $b_1 = 6/\alpha$.

From (10), and (11), with $b_1 = 6/\alpha$,

$$\begin{aligned} f(w) &= 1 - [g(w)]^{-1}\left[1 + \sum_{n=1}^{\infty} 2(w/\alpha)^n\right] \\ &= 1 - [g(w)]^{-1}\left[1 + \frac{2w}{\alpha - w}\right] \\ &= 1 - \frac{(\alpha + w)}{\alpha - w} \cdot \frac{(1 - w)(\alpha^2 - w)}{(\alpha^2 - w^2)} \\ &= 1 - \frac{(1 - w)(\alpha^2 - w)}{(\alpha - w)^2} = \frac{w(\alpha - 1)^2}{(\alpha - w)^2}\,. \end{aligned}$$

We now show that f is a $1-1$ selfmapping of $\varDelta$. If $f(w_1) = f(w_2)$, i.e.,

$$\frac{w_1(\alpha-1)^2}{(\alpha-w_1)^2} = \frac{w_2(\alpha-2)^2}{(\alpha-w_2)^2},$$

then $(w_1 - w_2)(\alpha^2 - w_1 w_2) = 0$. Since $w_1, w_2 \in \varDelta$, $w_1 w_2 \neq \alpha^2$, so $w_1 = w_2$. By the Maximum Modules Theorem, it is sufficient to show that $|f(w)| \leqq 1$ for $w = e^{i\theta}$. $|f(e^{i\theta})| = (\alpha-1)^2(\alpha^2 - 2\cos\theta + 1) \leqq 1$.

We now verify that $w = f^{-1}(z)$ is regular on $\bar{\varDelta} - \varDelta$, except possibly at $z = 1$, and that $0 \in \varDelta$. f^{-1} is regular except at $z = 0$, so now we must show

$$\min_{0 \leqq \theta < 2\pi} |f(e^{i\theta})| \geqq \delta > 0 .$$

$|f(e^{i\theta})| = (\alpha-1)^2/T(\theta)$, where $T(\theta) = (\alpha+1)^2 - 4\alpha\cos^2\theta/2$. A direct calculation certifies that the maximum of $T(\theta)$ occurs at $\theta = \pi$, and $T(\pi) = [(\alpha-1)/(\alpha+1)]^2 > 0$.

It remains to show that at $z = 1$, $1 - w = (1-z)^\lambda[a + a_1(1-z) + \cdots]$, $\lambda \geqq 1$, $a > 0$. $z = f(w) = (\alpha-1)^2 w/(\alpha-w)^2$. From the equation $z = f(w)$ we obtain $1 - z = (1-w)(\alpha^2 - w)/(\alpha-w)^2$, which when solved for $1-w$ yields

$$1 - w = \frac{-(\alpha-1)(1-2z-\alpha) \pm (\alpha-1)(\alpha+1)\sqrt{1-4\alpha(1-z)/(\alpha+1)^2}}{-2z} .$$

Now divide the numerator and the denominator by -2 and write z in the denominator as $1-(1-z)$.

$$1 - w = \left\{\frac{(\alpha-1)}{2}[2(1-z) - (\alpha+1)] \pm \frac{(\alpha^2-1)}{-2}\left[1 - \frac{4\alpha}{2(\alpha+1)^2}(1-z) + -\frac{1}{8}\frac{16\alpha^2}{(\alpha+1)^4}(1-z)^2 + \cdots\right]\right\} \cdot \sum_{k=0}^{\infty}(1-z)^k .$$

Using the negative branch,

$$1 - w = \left\{(\alpha-1)(1-z) - \frac{\alpha(\alpha^2-1)}{(\alpha+1)^2}(1-z) - \frac{1}{8}\frac{(\alpha^2-1)}{2}\frac{16\alpha^2}{(\alpha+1)^4}(1-z)^2 + \cdots\right\} \cdot \{1 + (1-z) + (1-z)^2 + \cdots\} .$$

$$= (1-z)\left\{(\alpha-1) - \frac{\alpha(\alpha-1)}{\alpha+1} + \sum_{k=1}^{\infty} b_k(1-z)^k\right\}$$

Theorefore $1 - w = (1-z)^\lambda[a + a_1(1-z) + \cdots]$ where $\lambda = 1$ and $a = (\alpha-1)/(\alpha+1) > 0$.

Theorem 3, along with Theorems 1 and 2 of [2] show that the Chebyshev method is neither an $[F, d_n]$ nor a Sonnenschein method.

One of the important properties of (f, g)-summability is the following [5, p. 267]:

Let (f, g), (f_1, g_1) be two Gronwall means which map regions D, D_1 and with exponents λ, λ_1. If $\lambda > \lambda_1$, and D is interior to D_1, then (f, g) is stronger than (f_1, g_1); i.e., $(f, g) \supset (f_1, g_1)$.

The de la Vallee Poussin method (V) [4, p. 103] is a Gronwall method with $\delta = 2^{-1}$, $f(w) = (1 - \sqrt{1-w})/(1 - \sqrt{1-w})$, $g(w) = (1-w)^{-1/2}$ and $\lambda = 2$.

THEOREM 4. $(V) \supset (\sigma)$.

Proof. Since $\lambda_{(V)} = 2$, $\lambda_{(\sigma)} = 1$, it is enough to show that $D(V)$ is interior to $D(\sigma)$, that is,

$$\left|\frac{1-\sqrt{1-w}}{1+\sqrt{1-w}}\right| \leqq \left|\frac{(\alpha-1)^2 w}{(\alpha-w)^2}\right| .$$

It suffices to consider $|w| = 1$; thus we need to show

$$\frac{1}{|(1+\sqrt{1-w^2}|} \leqq \frac{(\alpha-1)^2}{|(\alpha-w)^2|} . \tag{12}$$

Writing $1 - w = \rho e^{i\phi}$, where $-\pi < \phi < \pi$, (12) becomes

$$|\alpha - 1 + \rho e^{i\phi}|^2 \leqq (\alpha-1)^2 |1 + \rho^{1/2} e^{i\phi/2}|^2 ,$$

i.e.,

$$2(\alpha-1)\cos\phi + \rho \leqq 4\alpha(2\rho^{-1/2}\cos\phi/2 + 1) .$$

Since $\cos\phi/2 > 0$, it is sufficient to show that $2(\alpha-1)\cos\phi + \rho \leqq 4\alpha$, which is readily verified.

THEOREM 5. $\sigma \not\supseteqq (C, 1)$.

We shall make use of the well-known result that if A and B are regular summability methods, and B is a triangle, then $(A) \supseteqq (B)$ if and only if AB^{-1} is regular.

Consider $D = AC^{-1}$, where A is the Chebyshev method and C is $(C, 1)$. C^{-1} has entries

$$c_{nk}^{-1} = \begin{cases} -n, \ k = n-1 \\ n+1, \ k = n \\ 0, \ \text{elsewhere} . \end{cases}$$

Then

$$d_{nk} = \begin{cases} (k+1)a_{nk} - (k+1)a_{n,k+1}, & k < n \\ (n+1)a_{nn}, & k = n \\ 0, & \text{elsewhere} . \end{cases}$$

We shall show that D has infinite norm.

$$\sum_{k=0}^{n} |d_{nk}| = \sum_{k=0}^{p-1} (k+1)(a_{n,k+1} - a_{nk}) + \sum_{k=p}^{n-1} (k+1)(a_{nk} - a_{n,k+1}) + a_{nn}(n+1) .$$

Now,

$$\begin{aligned} \sum_{k=0}^{p-1} (k+1)(a_{n,k+1} - a_{nk}) &= \sum_{k=0}^{p-1} (k+1)a_{n,k+1} - \sum_{k=0}^{p-1} ka_{nk} - \sum_{k=0}^{p-1} a_{nk} \\ &= \sum_{j=1}^{p} ja_{nj} - \sum_{k=0}^{p-1} ka_{nk} - \sum_{k=0}^{p-1} a_{nk} \\ &= pa_{np} - \sum_{k=0}^{p-1} a_{nk} . \end{aligned}$$

$$\begin{aligned} \sum_{k=p}^{n-1} (k+1)(a_{nk} - a_{n,k+1}) &= \sum_{k=p}^{n-1} ka_{nk} + \sum_{k=p}^{n-1} a_{nk} - \sum_{k=p}^{n-1} (k+1)a_{n,k+1} \\ &= \sum_{k=p}^{n-1} ka_{nk} + \sum_{k=p}^{n-1} a_{nk} - \sum_{j=p+1}^{n} ja_{nj} \\ &= pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk} . \end{aligned}$$

Therefore,

$$\sum_{k=0}^{n} |d_{nk}| = pa_{np} - \sum_{k=0}^{p-1} a_{nk} + pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk} + a_{nn}(n+1) .$$

Since the Chebyshev method has row sums equal to 1,

$$\sum_{k=p}^{n-1} a_{nk} = 1 - \sum_{k=0}^{p-1} a_{nk} - a_{nn} .$$

Thus

$$\sum_{k=0}^{n} d_{nk} = 2pa_{np} - 2\sum_{k=0}^{p-1} a_{nk} + 1 .$$

But $\sum_{k=0}^{p-1} a_{nk} \leqq 1$, so it is sufficient to show $pa_{np} \to \infty$. This follows immediately from (2), since $\lim \sqrt{n} = \infty$ and the remaining limits have already been shown to be finite and nonzero.

The Fourier series

$$\sum_{k=1}^{\infty} \sin kt/k = (\pi - t)/2, \quad 0 < t \leqq \pi ,$$

converges for all t, and the function has a jump at $t = 0$. Hence the convergence is nonuniform at $t = 0$; that is, the sequence $\{s_n(t_n)\}$, where $\{t_n\}$ is a positive null sequence and

$$s_n(t) = \sum_{k=1}^{n} \sin kt/k \ , \quad s_0 = 0 \ , \tag{13}$$

has several limit points, depending on the manner in which t_n approaches 0.

If $\lim nt_n = \tau \geqq 0$, then $\lim s_n(t_n) = \int_0^\tau (\sin t/t)dt$, and the maximal limit is attained when $\tau = \pi$, in which case

$$\lim s_n(t_n) = \int_0^\pi \frac{\sin t}{t} dt = \frac{\pi}{2} \times 1.17897\cdots \ . \tag{14}$$

On the other hand, $(\pi - t)/2 \to \pi/2$ as $t \downarrow 0$. Thus the limit points of $\{s_n(t_n)\}$ cover an interval which extends beyond $f(0+)$ if $f(0+) \neq 0$. This situation is called the Gibbs phenomenon relative to the partial sums.

We shall now show that the corresponding phenomenon occurs for the Chebyshev means.

THEOREM 6. *The Chebyshev means of* (13) *satisfy*

$$\lim \sigma_n(t_n) = \int_0^{\tau/\sqrt{2}} \frac{\sin y}{y} dy \quad \text{as } nt_n \longrightarrow \tau \text{ and } nt_n^2 \longrightarrow 0 \ , \tag{15}$$

and

$$\limsup \sigma_n(t_n) \leqq \int_0^\pi \frac{\sin t}{t} dt \ .$$

The lim sup inequality is an immediate consequence of (14) *and the well-known fact that, for any totally regular matrix A, and any sequence $x = \{x_n\}$,* $\limsup A_n(x) \leqq \limsup x_n$.

The proof of the theorem is similar to that of [6]. One may write $s_n(t)$ in the form

$$s_n(t) = -t/2 + \int_0^t \frac{\sin (n + 1/2)x}{2 \sin (x/2)} dx \ .$$

Since $\sin (k + 1/2)x = \mathscr{I}(\exp (i(k + 1/2)x))$,

$$\sigma_n(t) = -t/2 + \mathscr{I}\left[\int_0^t \frac{1}{2 \sin (x/2)} \sum_{k=0}^{n} a_{nk} e^{ikx} e^{ix/2} dx\right] .$$

From [1, p. 297], $\sum_{k=0}^{n} a_{nk} e^{ikx} = T_n(1 + 2e^{ix})/T_n(3)$, where $T_n(x)$ is defined by (1).

Define

$$\rho e^{i\beta} = 1 + 2e^{ix} + [(1 + 2e^{ix})^2 - 1]^{1/2}$$
$$= 1 + 2e^{ix} + 2e^{ix/2}e^{ix/4}(2\cos x/2)^{1/2} .$$

Let $a = (2\cos x/2)^{1/2}$. Then $\rho\cos\beta = 1 + 2(\cos x + a\cos(3x/4))$,

(16) $$\rho\sin\beta = 2(\sin x + a\sin(3x/4)) ,$$

and

(17) $$\rho^2 = 5 + 4(\cos x + a\cos(3x/4)) + 8(\cos(x/2) + a\cos(x/4)) .$$

Therefore $1 + 2e^{ix} - [(1 + 2e^{ix})^2 - 1]^{1/2} = \rho^{-1}e^{-i\beta}$, and assume $0 < x \leqq t \leqq \pi/2$.

$$\sigma_n(t) + t/2 = \frac{1}{2T_n(3)}\int_0^t \frac{1}{2\sin(x/2)}[\rho^n \sin(n\beta + x/2)$$
$$- \rho^{-n}\sin(n\beta - x/2)]dx = \frac{1}{4T_n(3)}\left\{\int_0^t \rho^n \cot(x/2)\sin n\beta dx\right.$$
$$\left. + \int_0^t \rho^n \cos n\beta dx + - \int_0^t \rho^{-n}\cot(x/2)\sin n\beta dx + \int_0^t \rho^{-n}\cos n\beta dx\right\} .$$

From (17), ρ is monotone decreasing in x for $0 < x \leqq \pi/2$. Therefore for $0 < x \leqq \pi/2$, $\rho < \alpha$. Thus

$$\left|\frac{1}{2T_n(3)}\int_0^t \rho^n \cos n\beta dx\right| < \int_0^t (\rho/\alpha)^n dx < t ,$$

so that there exists an η satisfying $|\eta| < 1$ such that

$$\frac{1}{2T_n(3)}\int_0^t \rho^n \cos n\beta dx = \eta t .$$

Now assume that $t = t_n$, $nt_n \to \tau$, $0 \leqq \tau \leqq \infty$, and $nt_n^2 \to 0$. Since, from (17), $\rho \geqq \sqrt{5}$,

$$\left|\frac{1}{4T_n(3)}\int_0^t \rho^{-n}\cos n\beta dx\right| < \pi/4(\alpha\sqrt{5})^n = o(1) .$$

(18) $$\left|\frac{1}{4T_n(3)}\int_0^t \rho^{-n}\cot(x/2)\sin n\beta dx)\right| < \frac{1}{2(\alpha\sqrt{5})^n}\int_0^t n\beta\cot(x/2)dx .$$

We wish to show that $\beta < x$. For $0 < x \leqq \pi/2$, from (16), $\rho\sin\beta < 2(1 + a)\sin x$. From (17), if $\cos(3x/4) + 2\cos(x/4) \geqq 2$, then $\rho > 2(a + 1)$. In the interval $[0, \pi/2]$,

$$\cos(3x/4) + 2\cos(x/4) \geqq \cos(3\pi/8) + \cos(\pi/8)$$
$$= \cos(\pi/8)(4\cos^2(\pi/8) - 1) .$$

Since $\cos(\pi/8) = \sqrt{2 + \sqrt{2}}/2$, it is sufficient to show that

$$\frac{\sqrt{2+\sqrt{2}}}{2}\left(\frac{4}{4}(2+\sqrt{2})-1\right) \geqq 2 ,$$

which is easily verified. Therefore $0 < \sin < \beta(\rho/2(1+a)) \sin\beta < \sin x$, and $\beta < x$.

For $0 < x \leqq \pi/2$, $2 \leqq x/\sin(x/2) \leqq \pi/\sqrt{2}$. Substituting in (18) we have

$$\left|\frac{1}{4T_n(3)}\int_0^t \rho^{-n}\cot(x/2)\sin n\beta dx\right| < \frac{n}{2(\alpha\sqrt{5})^n}\int_0^{\pi/2}\cos(x/2)\cdot\frac{x}{\sin(x/2)}dx$$
$$< \frac{n\pi^2}{4\sqrt{2}(\alpha\sqrt{5})^n} = o(1) ,$$

and

$$\sigma_n(t) + (1-\eta)t/2 = \frac{1}{4T_n(3)}\int_0^t \rho^n \cot(x/2)\sin n\beta dx + o(1) .$$

Using (17), and the values of a and α,

$$1-(\rho/\alpha)^2 = [17 + 12\sqrt{2} - 5 - 4(\cos x + a\cos(3x/4))$$
$$- 8(\cos(x/2) + a\cos(x/4))]/\alpha^2$$

$$= \frac{4}{\alpha^2}[1-\cos x + 2(1-\cos(x/2)) + \sqrt{2}(1-\cos(3x/4)\sqrt{\cos(x/2)})$$
$$+ 2\sqrt{2}(1-\cos(x/4)\sqrt{\cos(x/2)})] .$$

Since $0 < \cos(x/2) < 1$,

$$1-\cos(x/4)\sqrt{\cos(x/2)} \leqq 1-\cos(x/4)\cos(x/2)$$
$$= 1-(\cos(3x/4)+\cos(x/4))/2 .$$

Similarly, $1-\cos(3x/4)\sqrt{\cos(x/2)} \leqq 1-(\cos(5x/4)+\cos(x/4))/2$. Therefore,

$$1-(\rho/\alpha)^2 \leqq \frac{4}{\alpha^2}[2\sin^2(x/2) + 4\sin^2(x/4) + \sqrt{2}(2\sin^2(5x/8)$$
$$+ 2\sin^2(x/8))/2 + \sqrt{2}(2\sin^2(3x/8) + 2\sin^2(x/8))]$$
$$\leqq \frac{4}{\alpha^2}[2(x/2)^2 + 4(x/4)^2 + \sqrt{2}((5x/8)^2 + (x/8)^2)$$
$$+ 2\sqrt{2}((3x/8)^2 + (x/8)^2)]$$
$$= \frac{4}{\alpha^2}\left(3/4 + \frac{46\sqrt{2}}{64}\right)x^2 < \frac{x^2}{4} .$$

Since $0 < \rho/\alpha < 1$, $1-\rho/\alpha \leqq 1-(\rho/\alpha)^2$, so that $1-\rho/\alpha < x^2/4$. $0 <$

$1-(\rho/\alpha)^n = (1-\rho/\alpha)\sum_{k=0}^{n-1}(\rho/\alpha)^k < n(1-\rho/\alpha) < nx^2/4$. Therefore $1-(\rho/\alpha)^n = \lambda n x^2$ for some λ satisfying $0 < \lambda < 1/4$, so that we may write

$$\frac{1}{2T_n(3)}\int_0^t \rho^n \cot(x/2)\sin n\beta dx = \frac{\alpha^n}{2T_n(3)}\Big[\int_0^t \cot(x/2)\sin n\beta dx$$
$$- n\int_0^t \lambda x^2 \cot(x/2)\sin n\beta dx\Big]$$
$$\times\ n\left|\int_0^t \lambda x^2 \cot(x/2)\sin n\beta dx\right| < n\int_0^t x^2\cot(x/2)dx$$
$$\leqq \frac{nt\pi}{\sqrt{2}}\int_0^t dx < nt^2 = o(1)\ ,$$

since $\lim nt_n^2 = 0$. Note that $\lim \alpha^n/2T_n(3) = 1$.

Using (17),

$$\frac{\rho\beta}{2} - \frac{2}{\alpha}\Big(1+\frac{3\sqrt{2}}{4}\Big)x = \frac{\rho}{\alpha}(\beta-\sin\beta) - \frac{2}{\alpha}(x-\sin x)$$
$$- \frac{2\sqrt{2}}{\alpha}\Big(\frac{3x}{4} - \sin(3x/4)\sqrt{\cos(x/2)}\Big)\ ,$$

so that

$$|\rho\beta/\alpha - rx| \leqq \frac{\rho}{\alpha}|\beta - \sin\beta| + \frac{2}{\alpha}|x-\sin x|$$
$$+ \frac{2\sqrt{2}}{\alpha}\left|\frac{3x}{4} - \sin(3x/4)\sqrt{\cos(x/2)}\right|\ ,$$

where $r = 2(1+3\sqrt{2}/4)/\alpha = (4+3\sqrt{2})/2\alpha = (4+3\sqrt{2})(3-2\sqrt{2})/2 = 1/\sqrt{2}$.

But $0 \leqq 3x/4 - \sin(3x/4)\sqrt{\cos(x/2)} \leqq 3x/4 - \sin(3x/4)\cos(x/2)$, $\sin(3x/4) \geqq 3x/4 - (3x/4)^3/3!$, and $\cos(x/2) \geqq 1 - x^2/4$, so that

$$|3x/4 - \sin(3x/4)\sqrt{\cos(x/2)}| \leqq 3x/4 - (3x/4 - (3x/4)^3/6)(1-x^2/4)$$
$$= 33x^3/128\ .$$

Since $0 < x - \sin x < x^3$ and $\beta < x$,

$$|\rho\beta/\alpha - x/\sqrt{2}| \leqq (\rho\beta^3 + 2x^3 + 33\sqrt{2}x^3/64)/\alpha < 2x^3\ .$$

Also, $|\beta - x/\sqrt{2}| \leqq |\rho\beta/\alpha - x/\sqrt{2}| + (1-\rho/\alpha)\beta < 2x^3 + x^3 = x^3$, so that $\beta = x/\sqrt{2} + \mu x^3$, where $|\mu| < 3$.

The remainder of the proof of (15) is the same as that of [6], beginning with formula (2.7), and will therefore be omitted.

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