PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

COMPACT OPERATORS OF THE FORM uC_φ

HERBERT KAMOWITZ

If A is the disc algebra, the uniform algebra of functions analytic on the open unit disc D and continuous on its closure, and if $u, \varphi \in A$ with $||\varphi|| \leqq 1$, then the operator uC_φ is defined on A by $uC_\varphi: f(z) \to u(z)f(\varphi(z))$. In this note we characterize compact operators of this form and determine their spectra.

We recall that a bounded linear operator T from a Banach space B_1 to a Banach space B_2 is *compact* if given a bounded sequence $\{x_n\}$ in B_1, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in B_2.

If $\varphi: \bar{D} \to \bar{D}$, we let φ_n denote n^{th} the iterate of φ, i.e., $\varphi_0(z) = z$ and $\varphi_n(z) = \varphi(\varphi_{n-1}(z))$ for $z \in \bar{D}$ and $n \geqq 1$. Our main result is the following.

THEOREM. *Let $u \in A$, $\varphi \in A$, $||\varphi|| \leqq 1$ and suppose φ is not a constant function.*

I. *The operator uC_φ is compact if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.*

II. *Suppose uC_φ is compact and let $z_0 \in \bar{D}$ be the unique fixed point of φ for which $\varphi_n(z) \to z_0$ for all $z \in D$. If $|z_0| = 1$, then uC_φ is quasinilpotent, while if $|z_0| < 1$, the spectrum $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n$ is a positive integer$\} \cup \{0, u(z_0)\}$.*

1. Characterization of compact uC_φ.

We first consider the easy case in which φ is a constant function.

THEOREM 1.1. *Suppose $u \in A$ and $\varphi(z) = a \in \bar{D}$ for all $z \in \bar{D}$. Then uC_φ is compact.*

Proof. Since $\varphi(z) = a$ for all $z \in \bar{D}$, $(uC_\varphi)f(z) = u(z)f(\varphi(z)) = f(a)u(z)$. Therefore the range of uC_φ is one-dimensional and so uC_φ is compact.

We next give a necessary and sufficient condition that uC_φ be a compact operator for those φ which are not constant functions.

THEOREM 1.2. *Suppose $u \in A$, $\varphi \in A$, $||\varphi|| \leqq 1$ and φ is not a constant function. Then uC_φ is a compact operator on A if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.*

Proof. Since everything holds if $u \equiv 0$, we will assume that u is not identically zero.

1. Suppose uC_φ is a compact operator on A. We must prove that if $z \in \bar{D}$ and $u(z) \neq 0$, then $|\varphi(z)| < 1$. Since φ is not a constant function, the maximum modulus principle implies that $|\varphi(z)| < 1$ whenever $|z| < 1$ and thus it suffices to show that $|\varphi(z)| < 1$ when $u(z) \neq 0$ and z lies on the unit circle. Assume the contrary and let θ satisfy $u(e^{i\theta}) \neq 0$ and $|\varphi(e^{i\theta})| = 1$. Set $\mu = \varphi(e^{i\theta})$ and for each positive integer n, define f_n by $f_n(z) = (\frac{1}{2}(z + \mu))^n$. Then $||f_n|| = 1$. Since uC_φ is assumed to be compact, there exists a subsequence $\{f_{n_k}\}$ and a function F in A with $(uC_\varphi)f_{n_k} \to F$ in A. That is, $u(z)(\frac{1}{2}(\varphi(z)+\mu))^{n_k} \to F(z)$ uniformly for $z \in \bar{D}$. But $(\frac{1}{2}(\varphi(z) + \mu))^{n_k} \to 0$ for $|z| < 1$ and so $F(z) = 0$ on D. However, F is continuous on $\bar{D}$ and therefore $F(z) = 0$ on $\bar{D}$. Hence $(uC_\varphi)f_{n_k} \to 0$ uniformly on $\bar{D}$. In particular, $u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k} \to 0$. But for all k, we have $|u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k}| = |u(e^{i\theta})| \neq 0$. This is a contradiction. Hence if uC_φ is compact and $u(z) \neq 0$, then $|\varphi(z)| < 1$.

2. Conversely, assume $|\varphi(z)| < 1$ whenever $u(z) \neq 0$. To show that uC_φ is compact, assume $f_n \in A$ and $||f_n|| \leqq 1$. Since $\{f_n\}$ is a uniformly bounded sequence of functions on D, it is a normal family in the sense of Montel and so there exists a subsequence $\{f_{n_k}\}$ and a function g analytic on D with $f_{n_k} \to g$ uniformly on compact subsets of the open disc D. We observe that this convergence implies $\sup_{|w|<1} |g(w)| \leqq 1$. Now defined a function G on the closed disc $\bar{D}$ by setting $G(z) = 0$ whenever $|z| = 1$ and $u(z) = 0$, and letting $G(z) = u(z)g(\varphi(z))$ otherwise. We claim that $G \in A$ and $(uC_\varphi)f_{n_k} \to G$ uniformly on $\bar{D}$.

We first show that G is continuous on $\bar{D}$. Indeed, G is continuous on $\{z \mid u(z) \neq 0\}$ since $|\varphi(z)| < 1$ on this set and g is continuous on D. Further, if $|z^*| = 1$ and $u(z^*) = 0$, let $\{z_m\}$ be a sequence in $\bar{D}$ converging to z^*. For each m, $G(z_m) = 0$ or $G(z_m) = u(z_m)g(\varphi(z_m))$. Since $|g(\varphi(z_m))| \leqq 1$ it follows that $\lim_{m\to\infty} G(z_m) = 0 = G(z^*)$ and so G is continuous at each $z \in \bar{D}$. Also G is analytic on D since u and $g \circ \varphi$ are analytic on D. Hence $G \in A$.

To show that $(uC_\varphi)f_{n_k} \to G$ uniformly on $\bar{D}$, let $V = \{e^{i\theta} \mid u(e^{i\theta}) = 0\}$ and suppose $\varepsilon > 0$. Since u is continuous, there exists an open set $U \supset V$ for which $|u(t)| < \varepsilon$ for $t \in U$. Also since $|u(z)| < 1$ for $z \notin U$ and $\bar{D} \backslash U$ is a compact set, there exists r, $0 < r < 1$, such that $|\varphi(z)| \leqq r$ for $z \notin U$. Moreover, since $f_{n_k} \to g$ uniformly on compact subsets of D, $u(z)f_{n_k}(\varphi(z)) \to u(z)g(\varphi(z))$ uniformly for $z \notin U$. That is, there exists an integer N such that $|u(z)f_{n_k}(\varphi(z)) - G(z)| =$

$|u(z)f_{n_k}(\varphi(z)) - u(z)g(\varphi(z))| < \varepsilon$ for $k \geqq N$ and all $z \notin U$. On the other hand, for $z \in U\backslash V$ and for all k,

$$\begin{aligned}|(uC_\varphi)f_{n_k}(z) - G(z)| &= |u(z)f_{n_k}(\varphi(z)) - u(z)g(\varphi(z))| \\ &\leqq \sup_{z \in U \backslash V} [|u(z)|\,|f_{n_k}(\varphi(z)) - g(\varphi(z))|] \leqq \varepsilon[||f_{n_k}|| + ||g||_\infty] = 2\varepsilon\,.\end{aligned}$$

Finally, if $z \in V$, then $(uC_\varphi)f_{n_k}(z) = u(z)f_{n_k}(\varphi(z)) = 0 = G(z)$. Hence given $\varepsilon > 0$, there exists an integer N such that $|(uC_\varphi)f_{n_k}(z) - G(z)| < 2\varepsilon$ for $k \geqq N$ and all $z \in \bar{D}$. That is, $(uC_\varphi)f_{n_k} \to G$ uniformly. Thus if $|\varphi(z)| < 1$ whenever $u(z) = 0$, then the operator uC_φ is compact.

2. Spectra of compact uC_φ. If T is a bounded linear operator from A to A we let $\sigma(T)$ denote the spectrum of T. As before, we first consider the case where φ is a constant function.

THEOREM 2.1. *Suppose $u \in A$ and $\varphi(z) = a \in \bar{D}$ for all $z \in \bar{D}$. Then $\sigma(uC_\varphi) = \{0, u(a)\}$.*

Proof. 0 and $u(a)$ are both eigenvalues uC_φ. For, if $F(z) = z - a$, then $(uC_\varphi)F(z) = u(z)F(\varphi(z)) = u(z)F(a) = 0$, while if $G(z) = u(z)$, then $(uC_\varphi)G(z) = u(z)G(\varphi(z)) = u(a)G(z)$. Thus $\{0, u(a)\} \subset \sigma(uC_\varphi)$.

On the other hand, since the range of uC_φ is one-dimensional, $\sigma(uC_\varphi)$ contains at most two elements and therefore $\sigma(uC_\varphi) = \{0, u(a)\}$.

In determining the spectra of the remaining compact operators of the form uC_φ we will make use of the following theorem of Denjoy and Wolf.

THEOREM A (Denjoy [2], Wolf [6]). *Suppose φ is an analytic function mapping D to D. If φ is not conformally equivalent to a rotation about a fixed point, then there exists a unique $z' \in \bar{D}$ for which $\varphi_n(z) \to z'$ for all $z \in D$. If φ is continuous at z', then $\varphi(z') = z'$.*

Suppose $\varphi \in A$ and $\varphi\colon D \to D$. It is easy to show that if $\varphi \not\equiv z$, then there is at most one fixed point of φ in the open disc D. There may, however, be infinitely many fixed points on the boundary of D. However, if the function φ is not equivalent to a rotation, then Theorem A asserts that there exists a unique fixed point $z_0 \in \bar{D}$, which we call the *Denjoy-Wolf fixed point of φ*, for which $\varphi_n(z) \to z_0$ for all $z \in D$. The spectrum of a compact operator of the form uC_φ will depend on the location of the Denjoy-Wolf fixed point of φ.

THEOREM 2.2. *Suppose $u \in A$, $\varphi \in A$, $||\varphi|| = 1$, φ is not a constant* *function and φ has all its fixed points on the unit circle. If uC_φ is a compact operator, then uC_φ is quasinilpotent.*

Proof. Let z_0 be the Denjoy-Wolf fixed point of φ, which by hypothesis has modulus 1. Since uC_φ is compact, Theorem 1.2 implies $u(z_0) = 0$. Let $V = \{e^{i\theta} \mid u(e^{i\theta}) = 0\}$.

Choose $\varepsilon > 0$. As in Theorem 1.2 there exists an open set U such that $U \supset V$ and $|u(t)| < \varepsilon$ for all $t \in U$. Also, since $\bar{D} \backslash U$ is compact there exists r, $0 < r < 1$, such that $|\varphi(w)| < r$ for all $w \in \bar{D} \backslash U$. Since $\{\varphi_n\}$ is a bounded sequence and hence a normal family, there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{\varphi_{n_k}\}$ converges uniformly on compact subsets of D. In particular, $\{\varphi_{n_k}\}$ converges uniformly for $|z| \leqq r$. But $\varphi_n(z) \to z_0$ for all $z \in D$. It follows that $\{\varphi_n(z)\}$ converges uniformly to z_0 for $|z| \leqq r$.

Now choose $\delta > 0$ such that $\{s \in \bar{D} \mid |s - z_0| < \delta\} \subset U$. Since $\varphi_n(w) \to z_0$ uniformly for $|w| \leqq r$, there exists a positive integer N such that $|\varphi_n(w) - z_0| < \delta$ if $n \geqq N$ and $|w| \leqq r$. Thus $\varphi_n(\{w \mid |w| \leqq r\}) \subset U$ for $n \geqq N$. Therefore, for each $z \in \bar{D}$ and each positive integer n, at most N elements from $z, \varphi(z), \cdots, \varphi_n(z)$ lie in $\bar{D} \backslash U$. From the definition of U, if $t \in U$, then $|u(t)| < \varepsilon$. Hence for all $z \in \bar{D}$ and $n \geqq N$,

$$|[(uC_\varphi)^n f](z)| = |u(z) \cdots u(\varphi_{n-1}(z))f(\varphi_n(z))| \leqq ||u||^N \varepsilon^{n-N} ||f|| .$$

Therefore $||(uC_\varphi)^n|| \leqq ||u||^N \varepsilon^{n-N}$ and so $||uC_\varphi||_{sp} = \lim_{n\to\infty} ||(uC_\varphi)^n||^{1/n} \leqq \varepsilon$. This holds for all $\varepsilon > 0$; consequently $||uC_\varphi||_{sp} = 0$ as required.

We next show that if uC_φ is a compact operator on A and if φ has a fixed point z_0 in D, then $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n$ is a positive integer$\} \cup \{0, u(z_0)\}$. This will be proved first for $z_0 = 0$ and then, by a standard argument, extended to arbitrary fixed points z_0 in D.

LEMMA 2.3. *Suppose $u \in A$, $\varphi \in A$, $||\varphi|| \leqq 1$ and $\varphi(0) = 0$. Then $u(0) \in \sigma(uC_\varphi)$ and $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$ for every positive integer n.*

Proof. (i) $u(0) \in \sigma(uC_\varphi)$ since no $f \in A$ satisfies $u(0)f(z) - u(z)f(\varphi(z)) = 1$. For, evaluating at $z = 0$ gives $u(0)f(0) - u(0)f(0) = 0 \neq 1$.

(ii) If $\varphi'(0) = 0$, then φ is not a conformal map of D onto D. Therefore if $\varphi'(0) = 0$, the composition operator C_φ is not invertible and so uC_φ is not invertible. Thus if $\varphi'(0) = 0$, then $u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi)$ for every positive integer n.

(iii) If $u(0) = 0$, then again uC_φ is not invertible and therefore if $u(0) = 0$, then $u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi)$ for every positive integer n.

(iv) Finally if $u(0)\varphi'(0) \neq 0$, we will prove that $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$ for every positive integer n by showing that for each such integer n, the function z^n is not in the range of $(u(0)\varphi'(0)^n - uC_\varphi)$.

Suppose the contrary, that for some positive integer n there exists $f \in A$ with $u(0)\varphi'(0)^n f(z) - u(z)f(\varphi(z)) = z^n$. Write $f(z) = z^m f_0(z)$ where $f_0 \in A$ and $f_0(0) \neq 0$. Then $f_0(z) = f_0(0) + \mathcal{O}(|z|)$. Also let $u(z) = u(0) + \mathcal{O}(|z|)$ and $\varphi(z) = \varphi'(0)z + \mathcal{O}(|z|^2)$. Then

$$u(0)\varphi'(0)^n f(z) - u(z)f(\varphi(z)) = z^n$$

is equivalent to

$$\begin{aligned} u(0)\varphi'(0)^n z^m[f_0(0) + \mathcal{O}(|z|)] - (u(0) + \mathcal{O}(|z|))(\varphi'(0)^m z^m + \mathcal{O}(|z|^{m+1})) \\ \times (f_0(0) + \mathcal{O}(|z|)) = z^n \end{aligned}$$

or

$$[u(0)\varphi'(0)^n f_0(0) - u(0)\varphi'(0)^m f_0(0)]z^m + \mathcal{O}(|z|^{m+1}) = z^n \ . \tag{1}$$

If $m \neq n$, then the left side of (1) has order m and the right side has order n, a contradiction. On the other hand, if $m = n$, then the left side of (1) has order at least $n + 1$ since the coefficient of z^n vanishes, while the right side of (1) has order n, which again is a contradiction.

Hence for each positive integer n, $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$.

LEMMA 2.4. *Suppose* $0 \not\equiv u \in A$, $\|\varphi\| \leqq 1$, $\varphi(0) = 0$ *and* φ *is not a constant function. If* λ *is an eigenvalue of* uC_φ, *then* $\lambda \in \{u(0)\varphi'(0)^n | n$ *is a positive integer*$\} \cup \{u(0)\}$.

Proof. Suppose λ is an eigenvalue of uC_φ with f as corresponding eigenvector. Then $\lambda \neq 0$ since φ is not a constant function and the algebra A has no zero divisors. Write $f(z) = az^m + \mathcal{O}(|z|^{m+1})$, $m \geqq 0$, $u(z) = bz^r + \mathcal{O}(|z|^{r+1})$, $r \geqq 0$ and $\varphi(z) = cz^s + \mathcal{O}(|z|^{s+1})$, $s \geqq 1$, where $abc \neq 0$. Then $\lambda f = (uC_\varphi)f$ becomes

$$\lambda[az^m + \mathcal{O}(|z|^{m+1})] = [bz^r + \mathcal{O}(|z|^{r+1})][a(cz^s + \mathcal{O}(|z|^{s+1})^m + \mathcal{O}(|z|^{ms+1}))]$$

or

$$a\lambda z^m + \mathcal{O}(|z|^{m+1}) = abc^m z^{r+ms} + \mathcal{O}(|z|^{r+ms+1}) \ .$$

Equating powers, we get $m = r + ms$ and $a\lambda = abc^m$.

Since r and m are nonnegative integers and s is a positive integer, $m = r + ms$ implies (i) $r = m = 0$ or (ii) $r = 0$ and $s = 1$. In the first case, $b = u(0)$ and so $a\lambda = abc^m$ implies $\lambda = u(0)$, while if $r = 0$ and $s = 1$, then $b = u(0)$, $c = \varphi'(0)$ and $a\lambda = abc^m$ implies $\lambda = u(0)\varphi'(0)^m$ for some positive integer m, concluding the proof.

THEOREM 2.5. *Suppose* $0 \not\equiv u \in A$, $\varphi \in A$, $\|\varphi\| \leqq 1$, $\varphi(0) = 0$, φ *is* *not a constant function and* uC_φ *is a compact operator. Then* $\sigma(uC_\varphi) = \{u(0)\varphi'(0)^n \mid n$ *is a positive integer*$\} \cup \{0, u(0)\}$.

Proof. By the Fredholm alternative for compact operators, every nonzero element in $\sigma(uC_\varphi)$ is an eigenvalue. It follows from Lemma 2.4 that the only possible eigenvalues of uC_φ are $u(0)$ and $u(0)\varphi'(0)^n$ for positive integers n; on the other hand Lemma 2.3 shows that each of these numbers is in $\sigma(uC_\varphi)$. Hence $\sigma(uC_\varphi) = \{u(0)\varphi'(0)^n \mid n$ is a positive integer$\} \cup \{0, u(0)\}$.

I should like to thank the referee for greatly simplifying my original proof of Theorem 2.5.

For arbitrary $z_0 \in \bar{D}$ we have

THEOREM 2.6. *Let* $u \in A$, $\varphi \in A$, $\|\varphi\| \leqq 1$ *and* uC_φ *be a compact operator on* A. *Suppose* z_0 *is the Denjoy-Wolf fixed point of* φ.

(i) *If* φ *is a constant function, then* $\sigma(uC_\varphi) = \{0, u(z_0)\}$.

(ii) *If* φ *is not a constant function and* $|z_0| = 1$, *then* $\sigma(uC_\varphi) = \{0\}$.

(iii) *If* φ *is not a constant function and* $|z_0| < 1$, *then* $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n$ *is a positive integer*$\} \cup \{0, u(z_0)\}$.

Proof. The only statement that has not been proved is (iii). Also if $u \equiv 0$, then certainly $\sigma(uC_\varphi) = \{0\}$.

Thus assume $u \not\equiv 0$, φ is not a constant function and $\varphi(z_0) = z_0 \in D$. Let p be the linear fractional transformation $p(z) = (z_0 - z)/(1 - \bar{z}_0 z)$. Then p maps D onto D and $p \circ p = z$. If we define S by $Sf(z) = f(p(z))$ for $z \in \bar{D}$, then S is an isometry on A and $S = S^{-1}$. Let $\psi = p \circ \varphi \circ p$ and $u^*(z) = u(p(z))$. Then $u^* \in A$ and $S(u^*C_\psi)S^{-1} = uC_\varphi$. Indeed,

$$\begin{aligned}[S(u^*C_\psi)S^{-1}]f &= [S(u^*C_\psi)](f \circ p) = S[u^* \cdot f \circ p \circ \psi] \\ &= (u^* \circ p) \cdot (f \circ p \circ \psi \circ p) = u \cdot (f \circ \varphi) = (uC_\varphi)f\ .\end{aligned}$$

Consequently $\sigma(u^*C_\psi) = \sigma(uC_\varphi)$. Since $\psi(0) = 0$, it follows from Theorem 2.5 that $\sigma(u^*C_\psi) = \{u^*(0)\psi'(0)^n \mid n$ is a positive integer$\} \cup \{0, u^*(0)\}$. But $u^*(0) = u(p(0)) = u(z_0)$ and $\psi'(0) = \varphi'(z_0)$. Thus $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n$ is a positive integer$\} \cup \{0, u(z_0)\}$.

REMARKS. 1. By considering the adjoint $(uC_\varphi)^*$ of uC_φ it can be shown that each nonzero eigenvalue of uC_φ has multiplicity one.

2. Operators of the form uC_φ on A for those φ which are conformal maps of D onto D were considered in [3]. Except for the case where φ has finite orbit, their spectra consist of circles, discs or annuli centered at the origin.

3. Caughran and Schwartz [1], Schwartz [4], and Shapiro and Taylor [5] have considered compact composition operators on H^p. Included in their papers are geometric conditions on φ insuring that C_φ be compact. They also determine $\sigma(C_\varphi)$ when C_φ is compact. It is shown that if C_φ is a compact composition operator, then φ has a fixed point z_0 in D and $\sigma(C_\varphi) = \{\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, 1\}$.

4. The arguments leading to Theorem 2.5 are valid if $u \in H^\infty$ $\varphi \in H^\infty$, $|\varphi(z)| < 1$ for $|z| < 1$ and uC_φ acts on H^p, $1 \leqq p \leqq \infty$. Thus for such u and φ, if $\varphi(z_0) = z_0 \in D$ and uC_φ is a compact operator on H^p, then again $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$.

REFERENCES

1. J. G. Caughran and H. J. Schwartz, *Spectra of compact composition operators*, Proc. Amer. Math. Soc., **51** (1975), 127–130.

2. M. A. Denjoy, *Sur l'itération des fonctions analytiques*, C. R. Acad. Sci. Paris, **182** (1926), 255–257.

3. H. Kamowitz, *The spectra of a class of operators on the disc algebra*, Indiana Univ. Math. J., **27** (1978), 851-610.

4. H. J. Schwartz, *Composition operators on H^p*, Thesis, Univ. of Toledo, 1969.

5. J. H. Shapiro and P. D. Taylor, *Compact, nuclear and Hilbert-Schmidt composition operators on H^2*, Indiana Univ. Math. J., **23** (1973), 471–496.

6. J. Wolf, *Sur l'itération des fonctions*, C. R. Acad. Sci. Paris, **182** (1926), 42–43 and 200–201.

Received December 1, 1977 and in revised May 5, 1978.

UNIVERSITY OF MASSACHUSETTS/BOSTON
DORCHESTER, MA 02125