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WHEN IS A POINT BOREL?

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Let X be a topological space. We investigate the question: When is a point (of X) Borel? In relation to this, we establish the equivalence of (a) Each point (singleton) is Borel, (b) Each point is the intersection of closed set and a G_δ, (c) The derived set of each point is Borel, (d) The derived set of each point is an F_σ, (e) The derived set of each subset is Borel, and (f) The derived set of each subset is an F_σ. Conditions (a), (b), (c), and (d) are also equivalent for a fixed point. As a separation axiom (a) is shown to lie strictly between T_1 and T_0. A number of examples are given and the work of other authors discussed.

0. Introduction.

Consideration of the question posed in the title for a particular case led to the development of the theorem below.

THEOREM 0.1. *The following are equivalent conditions for a topological space X.*

(a) *Each point (singleton) of X is a Borel set.*

(b) *Each point of X is the intersection of a closed set and a G_δ set.*

(c) *The derived set of each point of X is a Borel set.*

(d) *The derived set of each point of X is an F_σ set.*

(e) *The derived set of each subset of X is a Borel set.*

(f) *The derived set of each subset of X is an F_σ set.*

The initial discovery was the implication (a) $\Rightarrow$ (b). Using it, one can show directly[1] that the T_0 separation axiom is satisfied if each point is Borel, with the latter condition certain for T_1 spaces.

In [1], C. E. Aull and W. J. Thron introduce and study a number of separation axioms between T_0 and T_1, each of which is classified by some property of derived sets. In Theorem 3.1 of [1], they prove that $\{p\}'$ is closed (which is taken as a separation axiom, T_D) if and only if there is a closed set F and an open set U such that $\{p\} = F \cap U$, for all $p \in X$. With this as a catalyst, the equivalence of (b) and (d) is established and "each point is Borel" is fit into the classification scheme of Aull and Thron as follows:

[1] We shall do it differently.

a space X is

T_1 if and only if $\{p\}'$ is empty, for each $p \in X$.

T_D if and only if $\{p\}'$ is closed, for each $p \in X$.

Each point of X is Borel if and only if $\{p\}'$ is an F_σ set, for each $p \in X$.

T_0 if and only if $\{p\}'$ is a union of closed sets, for each $p \in X$.[2]

The implication (b) $\Rightarrow$ (f) was suggested to us by the mention in [1] of an observation of C. T. Yang (see [2, p. 56]) to the effect that the derived set of every subset of X is closed if and only if the derived set of every point in X is closed.

The equivalence of (c) and (d) or (e) and (f) might be interpreted as saying that the attempt to classify separation axioms by the Borel complexity of derived sets collapses to just three cases: T_1, T_D, and "each point is Borel".

Section one is devoted to a proof of Theorem 0.1.

In Section two we assemble some other results concerning the property "each point is Borel". First, we show it is necessarily observed in each first countable T_0 space. Next, an example is given of a T_0 space in which no point is Borel[3]. Finally, the property (thus each of (a)-(f)) is shown to be countably (but not generally) productive, hereditary, and not preserved by quotient (even closed) maps. In all of these arguments it is the equivalence (a) $\Leftrightarrow$ (b) that is used.

For purposes of application, we remark that properties (a), (b), (c), and (d) at a fixed point p are equivalent.

1. A proof of Theorem 0.1.

Fix a space X. As is well known the Borel sets in X can be specified by the following recursion.

$\mathscr{B}_0$ is the collection of all sets which are either closed or open.

$\mathscr{B}_{\alpha+1}$ consists of the unions and intersections of countable collections of members of $\mathscr{B}_\alpha$, whenever α is an ordinal with $\alpha < \omega_1$.

$\mathscr{B}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{B}_\alpha$ for each limit ordinal $\lambda \leqq \omega_1$.

The Borel subsets of X are just the members of $\mathscr{B}_{\omega_1}$.

Denote by $\mathscr{C}$ the collection of those subsets of X that can be obtained as the intersection of a closed set and a G_δ set. It is easy to see that $\mathscr{C}$ is closed under countable intersections and that $\mathscr{B}_0 \subset \mathscr{C} \subset \mathscr{B}_1$.

The following is the main lemma.

[2] Aull and Thron state this last equivalence as Theorem 2.3 (e) in [1]. They entertain a number of other separation axioms as well.

[3] In connection with this, we remark that of the fourteen examples in [3] of T_0 spaces that are not T_1, all of them enjoy the above property and twelve are even T_D.

LEMMA 1.0. *For every ordinal $\alpha < \omega_1$ and every $S \in \mathscr{B}_\alpha$ if $p \in S$ then there exists $C \in \mathscr{C}$ such that $p \in C \subset S$.*

Proof. We proceed by transfinite induction on α. When $\alpha = 0$ we can let $C = S$. Our induction hypothesis is

For every ordinal $\beta < \alpha$ and every $T \in \mathscr{B}_\beta$ if $p \in T$ then there exists $C \in \mathscr{C}$ with $p \in C \subset T$.

Now let $S \in \mathscr{B}_\alpha$ with $p \in S$. There are two cases.

Case (i). For each $i \in \omega$, there is an ordinal $\gamma(i) < \alpha$ and sets $B_i \in \mathscr{B}_{\gamma(i)}$ such that $S = \bigcup_{i \in \omega} B_i$. We may assume $p \in B_0$. According to our induction hypothesis there is $C \in \mathscr{C}$ with $p \in C \subset B_0 \subset \bigcup_{i \in \omega} B_i = S$.

Case (ii). For each $i \in \omega$, there is an ordinal $\gamma(i) < \alpha$ and sets $B_i \in \mathscr{B}_{\gamma(i)}$ such that $S = \bigcap_{i \in \omega} B_i$. By our induction hypothesis for each $i \in \omega$ pick $C_i \in \mathscr{C}$ with $p \in C_i \subset B_i$. Put $C = \bigcap_{i \in \omega} C_i$. Then $C \in \mathscr{C}$ and $p \in C \subset \bigcap_{i \in \omega} B_i = S$.

In either case the induction is complete and the lemma is established.

Proof of Theorem 0.1. We shall establish the implications

$$(a) \Longrightarrow (b) \Longrightarrow (d) \Longrightarrow (f) \Longrightarrow (e) \Longrightarrow (c) \Longrightarrow (a) .$$

(a) $\Rightarrow$ (b): If $\{p\}$ is a Borel set, then for some $\alpha < \omega_1$, $\{p\} \in \mathscr{B}_\alpha$. By Lemma 1.0 there is $C \in \mathscr{C}$ such that $p \in C \subset \{p\}$ and so $\{p\} = C \in \mathscr{C}$.

(b) $\Rightarrow$ (d): Choose a closed set F and a countable collection $\{U_i: i \in \omega\}$ of open sets such that $\{p\} = F \cap (\bigcap_{i \in \omega} U_i)$. We may assume that $F = \overline{\{p\}}$. Moreover, we have $\{p\}' = \overline{\{p\}} \sim \{p\}$. Clearly, $\{p\}' = \bigcup_{i \in \omega} [(X \sim U_i) \cap \overline{\{p\}}]$, whereupon $\{p\}'$ is an F_σ set.

(d) $\Rightarrow$ (f): Let A be a nonempty subset of X. We must prove that A' is an F_σ set. For each point $p \in A \sim A'$ we choose an open neighborhood U_p of p satisfying

(i) $U_p \cap A = \{p\}$.

Using (i), it is easy to see that the relation

(ii) $U_p \cap A' \subset \{p\}'$

holds for each $p \in A \sim A'$. Next we assert that

(iii) If $p \neq q$, then $U_p \cap U_q \cap \bar{A}$ is empty, for all $p, q \in A \sim A'$.

To see this, assume $r \in U_p \cap \bar{A}$. If $r = p$, then $r \notin U_q$ by (i). Otherwise $r \in U_p \cap A'$, by (i). Accordingly, $r \in \{p\}'$ by (ii). But U_q is an open set with $p \notin U_q$. Consequently $r \notin U_q$.

Now let $U = \bigcup \{U_p: p \in A \sim A'\}$. For each $p \in A \sim A'$, choose a sequence $\langle F_{p,i}: i \in \omega\rangle$ of closed sets such that $\{p\}' = \bigcup_{i \in \omega} F_{p,i}$. Let $B_i = \cup \{U_p \cap F_{p,i}: p \in A \sim A'\}$. On the basis of the foregoing, it is not difficult to see that both

$$\text{(iv)} \quad A' \cap U = \bigcup_{i \in \omega} B_i$$

and

$$\text{(v)} \quad (A \sim A') \cap \left(\bigcup_{i \in \omega} B_i\right) \text{ is empty.}$$

We now demonstrate the validity of (vi) below.

$$\text{(vi)} \quad B_i = \bar{B}_i \cap U .$$

Since $B_i \subset U$ by definition, it is clear that $B_i \subset \bar{B}_i \cap U$. Suppose that $x \in \bar{B}_i \cap U$. Choose $p \in A \sim A'$ such that $x \in U_p$. If $q \in A \sim A'$ with $q \neq p$, then $U_p \cap U_q \cap F_{q,i} \subset U_p \cap U_q \cap \bar{A}$. But $U_p \cap U_q \cap \bar{A}$ is empty according to (iii). Since $x \in \bar{B}_i$ and U_p is a neighborhood of x, it must be that $x \in \overline{U_p \cap F_{p,i}}$, whereupon $x \in F_{p,i}$. So $x \in U_p \cap F_{p,i} \subset B_i$ and (vi) is established.

Since $A \sim A' \subset U$, in view of (vi) we can extend (v) to

$$\text{(vii)} \quad (A \sim A') \cap \left(\bigcup_{i \in w} \bar{B}_i\right) \text{ is empty.}$$

Since $A' \sim U = \bar{A} \sim U$, we have the equations

$$\begin{aligned} A' &= (A' \cap U) \cup (A' \sim U) \\ &= (A' \cap U) \cup (\bar{A} \sim U) \end{aligned}$$

and then

$$\text{(viii)} \quad A' = \left(\bigcup_{i \in \omega} B_i\right) \cup (\bar{A} \sim U) .$$

But using the relation $\bar{B}_i \subset \bar{A}$, (vi), and (viii) we have

$$\text{(ix)} \quad A' = \left(\bigcup_{i \in w} \bar{B}_i\right) \cup (\bar{A} \sim U)$$

and therefore A' is an F_σ set.
(f) $\Rightarrow$ (e) and (e) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (a): This follows from the equation $\{p\} = \overline{\{p\}} \sim \{p\}'$. This completes the proof of theorem 0.1.

2. Other results.

First we make the observation below.

THEOREM 2.0. *Every point of a first countable T_0 space is Borel.*

Proof. Let X be a first countable T_0 space and $p \in X$ with an (open) neighborhood base $\langle U_i : i \in \omega \rangle$. Put $G = \bigcap_{i \in \omega} U_i$. For each $x \in G \sim \{p\}$ pick an open neighborhood V_x of x with $p \notin V_x$. Let $F = X \sim \bigcup \{V_x : x \in G \sim \{p\}\}$. Notice that $\{p\} = F \cap G$ and accordingly $\{p\}$ is a Borel set.

EXAMPLE 2.1.[4] A T_0 space in which no point is Borel.

We construct a set X and a strict dense linear ordering R on X without endpoints such that

(*) If Y is a countable subset of X and $p \in X$ such that yRp for all $y \in Y$, then there is $q \in X$ with qRp and yRq for $y \in Y$.

Once such a set X and ordering R have been constructed we define a topology by calling $U \subset X$ a basic open set if and only if there is $p \in X$ with $U = \{q : pRq \text{ and } q \in X\}$.

X and R are obtained as direct limits on the basis of the following recursion over the ordinals α.

$X_0 = \{0\}$	R_0 is empty
$X_{\alpha+1} = X_\alpha \times \{-1, 0, 1\}$	$R_{\alpha+1}$ is the lexicographic ordering of X_α induced by R_α and the usual ordering $-1 < 0 < 1$ on $\{-1, 0, 1\}$.

Note. X_α can be identified with $X_\alpha \times \{0\}$, in which case R_α is identified with $R_{\alpha+1}$ restricted to $X_\alpha \times \{0\}$.

If λ is a limit ordinal, then let

$$X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha \qquad R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha .$$

We let $X = X_{\omega_1}$ and $R = R_{\omega_1}$.

Evidently R is a dense linear ordering of X without endpoints. $\langle X, R \rangle$ fulfills (*) since $Y \cup \{p\} \in X_\alpha$ for some $\alpha < \omega_1$ and by taking $q = (p, -1) \in X_{\alpha+1}$ we obtain

$$(y, 0) R_{\alpha+1} q R_{\alpha+1} (p, 0) \quad \text{for all} \quad y \in Y .$$

[4] A simpler example of a T_0 space in which some point is not Borel is $\omega_1 + 1$ endowed with the right ray topology. The space constructed here has cardinality $\aleph_1$. Example 2.3 below also provides a T_0 space in which no point is Borel, but it is of cardinality $2^{\aleph_1}$.

Since $(y, 0)$ is identified with y and $(p, 0)$ is identified with p (in X) (*) is verified.[5]

Now let $p \in X$. In the (right ray) topology described above $\overline{\{p\}} = \{q: qRp \text{ or } q = p\}$. Let $\langle U_i: i \in \omega \rangle$ be any countable system of basic open neighborhoods of p. Accordingly for each $i \in \omega$ pick $r_i \in X$ with $U_i = \{t: t \in X \text{ and } r_i Rt\}$. In particular $r_i Rp$ for all $i \in \omega$. Consequently, by (*) there exists $x \in X$ such that xRp and $r_i Rx$ for all $i \in \omega$. Therefore $x \in \overline{\{p\}} \cap \bigcap_{i \in \omega} U_i$. Since $x \neq p$, we conclude that $\{p\} \neq \overline{\{p\}} \cap \bigcap_{i \in \omega} U_i$ and so that $\{p\}$ is not the intersection of any closed set and any G_δ set. Thus, by Theorem 0.1 ((a) $\Leftrightarrow$ (b)) no point of X is Borel. X is clearly T_0.

THEOREM 2.2. *The property "each point is Borel" is hereditary and countably productive.*

Proof. The first statement is an easy consequence of Theorem 0.1 ((a) $\Leftrightarrow$ (b)) which we leave to the reader.

As for the second, again we use Theorem 0.1 ((a) $\Leftrightarrow$ (b)). Let $\langle X_i: i \in \omega \rangle$ be a sequence of spaces in which each point is Borel and let $p = \langle p_i: i \in \omega \rangle$ be a point in the product space $X = \prod_{i \in \omega} X_i$. For each $i \in \omega$ represent $\{p_i\} = F_i \cap (\bigcap_{j \in \omega} U_{i,j})$ where F_i is closed in X_i and $U_{i,j}$ is open in X_i, for each $i, j \in \omega$. It follows that

$$\{p\} = \bigcap_{i \in \omega} \left[\pi_i^{-1}(F_i) \cap \left(\bigcap_{j \in \omega} \pi_i^{-1}(U_{i,j}) \right) \right],$$

whereupon every point in X is Borel. The case of finite products is similar.

EXAMPLE 2.3. A space X in which each point is Borel and nevertheless every finite indiscrete space is a closed image of X and X^κ has no Borel points provided $\kappa > \aleph_0$.

We take X to be the set of real numbers[6] endowed with the right ray topology. As X is a first countable T_0 space, we know that each point of X is Borel.

Let $Y = \{y_0, y_1, \cdots, y_{n-1}\}$ be a finite indiscrete space. Partition

[5] All of the ambiguities of this construction can be avoided at the expense of involving the relatively complicated construction of the limit of a system of relations directed by a system of embeddings. Heuristically, the argument is: "start with a point and a sharp knife. Chop each point into left, middle, and right pieces. Do this $\aleph_1$ times."

[6] We could use the rationals as easily; even the Sierpinski space $S = \{0, 1\}$, whose open sets are $\{0, 1\}$, $\{1\}$, and the empty set, has the property that $S^{\aleph_1}$ has no Borel points.

the negative integers into n infinite sets $X_0, X_1, \cdots, X_{n-1}$ and define $f: X \to Y$ by

$$f(x) = \begin{cases} y_i & \text{if} \quad x \in X_i \\ y_0 & \text{otherwise} . \end{cases}$$

Then every closed set of X (a closed left ray) maps onto Y. Thus f is closed. Since Y is indiscrete f is continuous.

As for the second assertion about X, let $\kappa > \aleph_0$ and $p \in X^\kappa$. Note that p is a function with domain κ and range included in X. It is easy to see that $\overline{\{p\}} = \{q: q \in X^\kappa \text{ and } q_\alpha \leqq p_\alpha \text{ for all } \alpha \in \kappa\}$. If $\langle U_i: i \in \omega \rangle$ is a countable sequence of basic open sets in X^κ, it is easy to see that $\overline{\{p\}} \cap \bigcap_{i \in \omega} U_i \neq \{p\}$. Thus, any pepresentation of $\{p\}$ as the intersection of a closed set and a G_δ set is impossible. An application of Theorem 0.1 now finishes the argument.

References

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