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# THREE-DIMENSIONAL OPEN BOOKS CONSTRUCTED FROM THE IDENTITY MAP

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**Three-dimensional manifolds are constructed as open books, using the identity diffeomorphism. The open book constructed in this way with (non)orientable page of Euler characteristic $\chi$ is the connected sum of $(1-\chi)$ copies of the (non)orientable $S^2$ bundle over $S^1$**

**Introduction.** We investigate orientable and nonorientable three-dimensional manifolds which are open books according to the following definition of Winkelnkemper [2].

DEFINITION. A manifold of dimension $n$ is said to have an open book description if it can be constructed using a co-dimension 2 submanifold $\partial V$ and a diffeomorphism $h: V \to V$ of an $(n-1)$-dimensional manifold with boundary $\partial V$. $h$ is required to be the identity map in a neighborhood of $\partial V$. The construction is to form the mapping torus $(V \times I)/(v, 0) = (h(v), 1)$ and then to identify $(v, t) = (v, t')$ for all $v$ in $\partial V$ and $t, t'$ in $I$. The image of the copies of $\partial V$ in the resulting manifold is called the binding of the open book and the circle's worth of copies of $V$ are called the pages.

Related results appear in the recent book of Rolfsen [1].

*Statement of results.*

THEOREM 1. *If $V = S_g - n\mathring{B}^2$, the surface of genus $g$ with $n$ disjoint, open discs removed from it, then the open book produced by setting $h$ equal to the identity map is the connected sum of $(2g + (n-1))$ copies of $(S^1 \times S^2)$. (Adopt the convention that zero copies of $(S^1 \times S^2)$ will refer to $S^3$.)*

THEOREM 2. *If $V = P_k - n\mathring{B}^2$, the 2-sphere with $k$ cross-caps attached and $n$ disjoint, open discs removed from it, then the open book produced by setting $h$ equal to the identity map is the connected sum of $(k + (n-1))$ copies of the Klein bottle of dimension three. $(k \geqq 1, n \geqq 1)$*

By the three-dimensional Klein bottle we mean the nonorientable $S^2$ bundle over $S^1$, $(S^2 \times I)/(x, y, z, 0) = (-x, y, z, 1)$.

*Proofs of results.*

LEMMA 1. *Let M be a closed, smooth manifold of dimension* $(n+1)$. *If an unkotted copy of* $(S^1 \times \mathring{B}^n)$ *is removed from a coordinate patch on M and the identification* $(\theta, x) = (\theta', x)$ *is performed for all* $(\theta, x)$ *in* $(S^1 \times S^{n-1})$ *then the resulting manifold is the connected sum* $M \# (S^2 \times S^{n-1})$.

*Proof.* Remove a copy of $\mathring{B}^{n+1}$ which contains the bounding $(S^1 \times S^{n-1})$ and temporarily add a copy of $B^{n+1}$ to it, giving $S^{n+1} - (S^1 \times \mathring{B}^n)$. The identifications glue all the meridian $(n-1)$-spheres to one copy of $S^{n-1}$ on the boundary of the removed torus. On the bounding $(S^1 \times S^{n-1})$ in $S^{n+1} - (S^1 \times \mathring{B}^n) = (B^2 \times S^{n-1})$, the $(n-1)$-spheres are parallels. When these are all identified to one $S^{n-1}$ we obtain $(S^2 \times S^{n-1})$. Now remove the superfluous copy of $\mathring{B}^{n+1}$ and form the connected sum of $M - \mathring{B}^{n+1}$ with $(S^2 \times S^{n-1}) - \mathring{B}^{n+1}$ to finish the proof.

*Proof of Theorem* 1. Consider the polygonal normal form of $S_g$ $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Punch $n$ holes in it and form the Cartesian product with the unit interval.

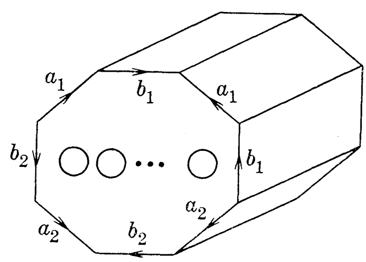


FIGURE 1

We diffeomorph one of the inner cylinders to the outside and form the mapping torus. If we perform the required identifications on the outer copy of $(S^1 \times S^1)$ we obtain $S^3$-$\{n$ solid tori$\}$. The $(n-1)$ copies of $(S^1 \times S^1)$ which do not come from the $a_1 b_1 \cdots a_g^{-1} b g^{-1}$ each contribute a connected sum of $S^3$ with $(S^1 \times S^2)$ when the required identifications are performed. This follows from the absence of linking and Lemma 1.

The remaining $(S^1 \times S^1)$ can be surgered out in a $\mathring{B}^3$ as in Lemma

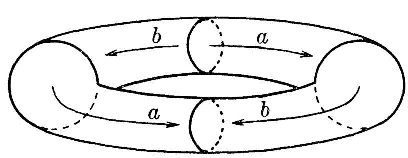


FIGURE 2

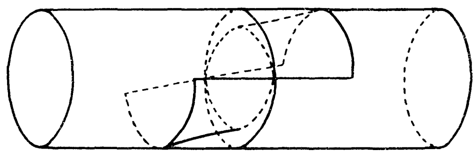

FIGURE 3

1 and an extra $B^3$ added. Since the $a_i$ and $b_i$ were meridians on the removed $(S^1 \times \mathring{B}^2)$ they are parallels on the remaining $(S^1 \times B^2) = S^3 - (S^1 \times \mathring{B}^2)$. An identification such as this, pictured in Figure 2, gives the connected sum of $2g$ copies of $(S^1 \times S^2)$. The four vertical discs give the union of two $S^2$'s joined along a common equator. This configuration is $S^3 - 4\mathring{B}^3$ and we now attach two copies of $S^2 \times I$. A separating $S^2$ between the two handles can be constructed using four of the discs with the flanges shown in Figure 3. One quarter of the $S^2$ consists of the two curved half-flanges, andthe sub-disc in a vertical disc from Figure 2.

We now complete the proof by removing the extra $\mathring{B}^3$ which we added above and forming the required connected sum.

*Proof of Theorem* 2. The proof is analogous. The two extra ingredients are to notice that the connected sum of $(S^1 \times S^n)$ with the $(n + 1)$-dimensional Klein bottle is diffeomorphic to the connected sum of two $(n + 1)$-dimensional Klein bottles and that an identification such as that shown in Figure 4 gives a connected sum of two Klein bottles of dimension 3.

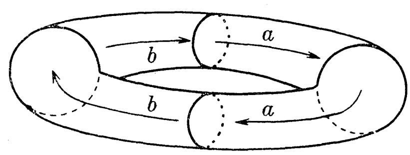


FIGURE 4

## REFERENCES

1. Dale Rolfsen, *Knots and Links*, Publish or Perish, Inc., Berkeley, Calif., 1976.
2. H. E. Winkelnkemper, *Manifolds as open books*, Bull. Amer. Math. Soc., **79** (1973), 45-51.

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