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CANCELLING 1-HANDLES AND SOME TOPOLOGICAL IMBEDDINGS

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In this note we use the existence of a certain type of handle decomposition (see corollary) for compact simply connected P. L. 4-manifolds and R. Edwards results on the double suspension conjecture to prove:

THEOREM 2. *Let* $\alpha \in H_2(M; Z)$ *where* M *is a compact simply connected* P. L. 4-*manifold. Then there is a proper topological imbedding* (*possible nonlocally flat*) $\theta: S^2 \times R \to M \times R$ (*mapping ends to ends*) *with* $\theta_*[S^2 \times R] = \bar{\alpha} \in H_2\ (M \times R; Z)$. $\bar{\alpha}$ *is the image of* α *under* $\times R$. *Proper, here, means inverse images of compact sets are compact.*

In [2], we considered the problem of constructing smooth proper imbeddings, θ, and showed that if α is characteristic (dual to $w_2(\tau(M))$, the only obstruction to the existence of θ is an Arf invariant which is equal to the Milnor-Kervaire number $(=(\text{signature}\,(M) - \alpha \cdot \alpha/8)(\text{mod}\,2))$ when M is closed and that if α is ordinary (not dual to $W_2(\tau(M))$ there is no obstruction. This suggests two problems: (1) Can θ always be arranged to be topologically locally flat, and (2) can θ always be arranged to be P. L.?

Here is our "handle cancellation" theorem:

THEOREM 1. *Let* M *be any compact connected* P. L. *manifold of dimension* $= m$ (*assume* M *orientable if* $m = 3$). *Let* N *be a compact connected codimension* 0 *submanifold of* ∂M. *If* $\pi_1(M, N) = 0$, *then there is a codimension* 0 *submanifold*, $\bar{N}$, *of* M *with*: (1) $N \hookrightarrow \bar{N}$, (2) *the inclusion* $N \hookrightarrow \bar{N}$ *is a homotopy equivalence*, (3) $M = \bar{N} \cup$ 2-*handles* $\cup$ 3-*handles* $\cup \cdots \cup$ m-*handles.*

Note. The P. L. category is convenient here since handle decompositions always exists.

Proof. If $n \geqq 5$, the usual arguments for cancelling handles produce the desired $\bar{N} \overset{\text{P. L.}}{=\!=} N \times I$ (see Appendix [3]). We need only consider the cases $m = 3$ or 4.

Let $m = 4$ and let $\mathscr{H}(M, N)$ be a handle decomposition of M relative to N. We may assume $\mathscr{H}(M, N)$ has no zero-handles.

Let $\{h_i^1\} = \{D_i^1 \times D_i^{m-1}\}$ be the 1-handles. Let $\{c_i\}$ be closed curves on L_1, the level after the 1-handles are attached, each consisting of $(D_i^1 \times \text{pt.})$ for some pt. $\in \partial D_i^{m-1}$ and an arc in $N - \{D_i^1 \times \partial D_i^{m-1}\}$. We claim that the latter arcs may be chosen so that each curve, c_i, is null homotopic in $X \overset{\text{def}}{=\!=} \overline{M - (\text{1-handles of } \mathscr{H}(M, N))}$. Since $m = 4$, $\pi_1(X) \to \pi_1(M)$ is an isomorphism. The arcs may be chosen (since $\pi_1(N) \to \pi_1(M)$ is epic) so that each c_i represents $0 \in \pi_1(M)$ and, therefore, $0 \in \pi_1(X)$.

Let $\{\gamma_j\}$ be the disjoint simple closed curves in L_1 along which the 2-handles $\{h_j\}$ are attached. Picking paths to the base pt., $*$, $\{\gamma_i\}$ determines relations $\{r_j\}$ and $\pi_1(X) = \pi_1(L_1)/\langle r_j \rangle$. Choosing a path from c_1 to $*$, we have $[c_i] \in \langle r_j \rangle$. So $[c_i] = \prod_{k=1}^{n} u_k x_k u_k^{-1}$ where $u_k \in \pi_1(L)$ and $x_k \in \{r_j, r_j^{-1}\}$. For each curve c_i, introduce a trivial oriented (2-handle, 3-handle) pair. Let h_i^2 be the new 2-handle. Choose a path from ∂h_i^2 to $*$. Now perform a sequence of n-handle passings. h_i^2 should be passed over the oriented ($+$ or $-$ as $x_k = r_j$ or r_j^{-1}) 2-handles corresponding to $x_1, \cdots, x_n$ along arcs corresponding to the elements $u_1 \cdots u_n$. The framing along each arc is immaterial so long as it restricts at the end points to a framing induced by the orientation of each 2-handle. Let $\{\gamma_i\}$ be the curves along which $\{h_i^2\}$ are attached after the above handle passings. γ_i is homotopic to c_i. By the handle cancellation lemma [3], attaching 2-handles to $\{c_i\}$ would result in a product $N \times I$. Since homotopy type depends only on the homotopy class of attaching maps, $\bar{N} \overset{\text{def}}{=\!=} N \cup \{h_i^1\} \cup \{h_i^2\} \overset{\text{h.e.}}{\simeq} N \times I$. $\bar{N}$ has the desired properties.

Let $m = 3$. If $\pi_1(N) = 0$ then $\pi_1(M) = 0$ and M must be a homotopy (S^3-interior of closed disks). Let $\bar{N} = \overline{M - \{\text{closed disk} \cup \text{thickened arcs to } \partial \text{ components} \not\supset N\}}$, so $M = \bar{N} \cup$ 2-handles $\cup$ 3-handles. We now assume $\pi_1(N) \neq 0$.

If $\pi_1(N) \to \pi_1(M)$ is an isomorphism, every imbedded 2-sphere in M seperates M, one component of the complement being a homotopy B^3 with finitely many punctures. Let $\bar{M} = M \bigcup_{\text{spherical } \partial \text{ components}} (\text{3-cells})$. By the sphere theorem, $\bar{M}$ is a $K(\pi, 1)$ so $(\bar{M}, N)$ is an h-cobordism. But $M \overset{\text{diff}}{=\!=} \bar{M} \cup$ 2-handles, so $\bar{M}$ satisfies the conditions for $\bar{N}$.

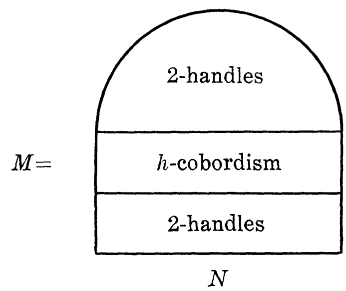
Assume $\pi_1(N) \to \pi_1(M)$ is epi. By Dehn's lemma, if $\pi_1(N) \to \pi_1(M)$ is not injective, there is an essential simple closed curve, $\alpha \subset N$, bounding an imbedded 2-disk $\beta \subset M$. Let (M', N') be the result of ambient surgery (handle subtraction) along β. $\pi_0(N') \to \pi_0(M')$ is an isomorphism. (Proof: $\beta(\alpha)$ disconnects $M(N)$ if and only if there is no curve in $M(N)$ meeting $\beta(\alpha)$ algebraically once. Since $H_1(N) \to H_1(M)$ is epi, there is a dual curve for β if and only if there is a dual curve for α.)

PROPOSITION. *On each component,* $\pi_1(N') \to \pi_1(M')$ *is* epi.

Proof. M is obtained from M' by attaching a 1-handle, and N' is obtained from N by the corresponding 0-surgery. The proposition can be deduced from the following group theoretic fact: Let $\theta: A \to X$, $\phi: B \to Y$ be group homomorphisms. If $\theta * \phi$ is epi, then θ and ϕ are epi.

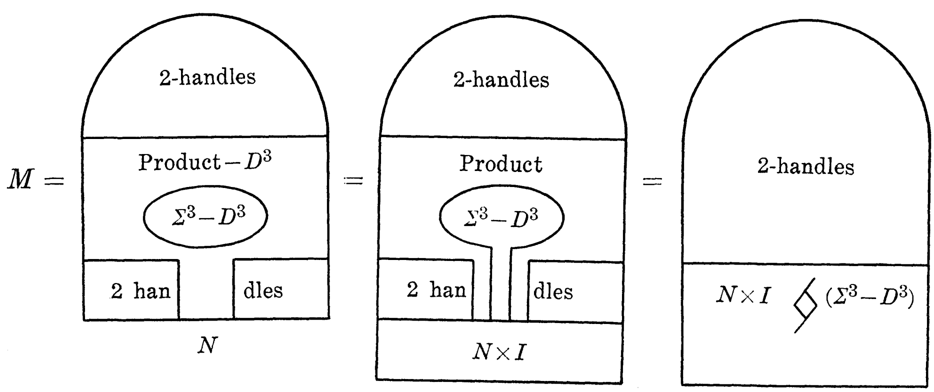
Proceeding inductively (on the genus of the components of N'), we obtain (M'', N'') with $\pi_1(N'') \to \pi_1(M'')$ an isomorphism on each component. This decomposes M as:

Diagram 1:



By a theorem of J. Stallings [5], every h-cobordism between orientable surfaces is the connected sum of a product and a homotopy 3-sphere, Σ^3. So we have:

Diagram 2:



Set $\bar{N} = N \times I \natural (\Sigma^3 - D^3)$.

This completes the proof of Theorem 1.

Note. The orientation restriction in dimension 3 results from ignorance about h-cobordisms on RP^2.

COROLLARY. *Let M be a compact simply connected* P. L. (=*smooth*) *4-manifold.* $M \overset{\text{P. L.}}{=\!=} K \cap$ *2-handles* $\cup$ *3-handles* $\cup$ *4-handles, where K is a compact contractable 4-manifold.*

Proof. Apply Theorem 1 to $(M - D^4, \partial D^4)$. Let $K = \bar{N} \cup D^4$.

REMARK. A. Casson has recently exhibited (unpublished work) a simply connected P. L. 4-manifold with boundary, M, with the property that every handle decomposition of M, $\mathscr{H}(M)$, must contain a 1-handle. This answers negatively a question raised in [4] on the existence of (relative) 2-spines. So the preceding corollary is all one can hope for.

Proof of Theorem 2. Let $M \overset{\text{P. L.}}{=\!=} K \cup$ 2-handles $\cup$ 3-handles $\cup$ 4-handles. Let $\hat{M} = \text{cone}\,(\partial K) \cup$ 2-handles $\cup$ 3-handles $\cup$ 4-handles. $H_2(M; Z) \cong H_2(\hat{M}, \text{cone}\,(\partial K); Z) \cong H_2(\hat{M}; Z)$. Any element of $H_2(\hat{M}, \text{cone}\,(\partial K); Z)$ is represented by a relatively imbedded 2-disk constructed as a linear combination of 2-handles in the above handle decomposition by taking ambient boundary-connected-sums. So every element, α, of $H_2(\hat{M}; Z)$ is represented by a simplicial imbedding, ω, of S^2 in $\hat{M}$. By a theorem of R. Edwards, [1], $(\text{cone}\ \partial K) \times R$ is (topologically) homeomorphic to $K \times R$, $\hat{M} \times R$ is (topologically) homeomorphic to $M \times R$. The composition:

$$S^2 \times R \xrightarrow{\omega \times \text{id}_R} \hat{M} \times R \xrightarrow{\text{top. homeomorphism}} M \times R$$

is the topological imbedding with the claimed properties.

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