DERIVATIONS AND COMMUTATIVITY OF RINGS

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Let R be a ring with center C and Jacobson radical J. Let $\mathscr{I}$ be the additive group of all inner derivations of R and $\mathscr{D}$ be an additive group of derivations of R satisfying:

(i) For any $\partial \in \mathscr{D}$ and $\delta \in \mathscr{I}$, $[\partial, \delta] \in \mathscr{D}$;

(ii) For any $x \in R$, $\partial x = 0$ for all $\partial \in \mathscr{D}$ iff $x \in C$;

(iii) For any prime ideal P in R and any $x \in R$, $\partial x \in P$ for all $\partial \in \mathscr{D}$ iff $\delta x \in P$ for all $\delta \in \mathscr{I}$.

Suppose, for each $x \in R$ and $\partial \in \mathscr{D}$, there is a $p \in R$ which depends upon x and ∂ such that $\partial x = (\partial x)^2 p$. Then the nilpotent elements in R are central and form an ideal N in R, R/N is a subdirect sum of division rings and commutative rings, and R/J is a subdirect sum of division rings. Suppose further that, for each $x \in R$ and $\partial \in \mathscr{D}$, such a p is a polynomial of ∂x with integral coefficients. Then R is necessarily commutative.

1. **Introduction.** A well known theorem of Wedderburn states that all finite division rings are fields. There are various generalizations of this theorem. Jacobson [4] proves that if, for each x in a ring R, there is an integer $n(x) > 1$ such that $x^{n(x)} = x$ then R is commutative. Among others, Herstein [2] weakens Jacobson's condition by assuming $x^{n(x)} = x$ for *only* every commutator x in R. In other directions of research, Putcha, Wilson and Yaqub [8] show that if, for each commutator x in a ring R, there exists an integer $n(x) > 1$ and a central element $z(x) \in R$ such that $x = x^{n(x)}z(x)$ then R/J is a subdirect sum of division rings where J denotes the Jacobson radical of R. Suppose further that such $n(x) = 2$ for each commutator $x \in R$. Then R is commutative. Recently, Ligh [6] proves that if, for each commutator x in a ring R, there exists $p(x) \in R$ such that $x = x^2 p(x)$ then R/N is a subdirect sum of commutative rings and division rings where N denotes the lower nil radical of R. In view of the fact that a commutator is simply in the image of an inner derivation, we introduce in this paper the notion of primary classes of derivations of a ring R. A primary class $\mathscr{D}$ of derivations of a ring R is very much like the group $\mathscr{I}$ of all inner derivations of R but it could be much smaller than $\mathscr{I}$.

In §2, we give some basic properties of primary classes of derivations.

In §3, we consider a ring R having a primary class $\mathscr{D}$ of derivations and satisfying the following conditions:

(A) *For each $x \in R$ and $\partial \in \mathscr{D}$, there is a $p = p(x, \partial) \in R$ such that $\partial x = (\partial x)^2 p$.*

It is shown that the nilpotent elements in R are central and hence form an ideal N in R. Moreover, R/N is a subdirect sum of division rings and commutative rings and R/J is a subdirect sum of division rings. Thus Putcha, Wilson and Yaqub's and Ligh's results follow immediately.

In §4, we study a ring R having a primary class $\mathscr{D}$ of derivations and satisfying the following condition:

(B) *For each $x \in R$ and $\partial \in \mathscr{D}$, there is a polynomial $p(t)$ of t with integral coefficients such that $\partial x = (\partial x)^2 p(\partial x)$.*

We prove that R is necessarily commutative. This generalizes the celebrated result of Herstein cited in the first paragraph.

In §5, some remarks are given and open problems are proposed.

Throughout this paper, R denotes an associative ring, $\mathscr{Z}$ the ring of integers, and $\mathscr{Q}$ the rational field. For $x, y \in R$, $[x, y] = xy - yx$, and δ_x denotes the inner derivation: $r \to [x, r]$ for all $r \in R$. We recall that a mapping $\partial: R \to R$ is called a derivation of R if $\partial(xy) = \partial xy + x\partial y$ and $\partial(x + y) = \partial x + \partial y$ for all $x, y \in R$.

2. **Basic properties.** Let R be a ring, C be its center and $\mathscr{I}$ be the additive group of all inner derivations of R. An additive group $\mathscr{D}$ of derivations of R is said to be a *primary class of derivations of R* if

(i) For any $\partial \in \mathscr{D}$ and $\delta \in \mathscr{I}$, $[\partial, \delta] \in \mathscr{D}$;

(ii) For any $x \in R$, $\partial x = 0$ for all $\partial \in \mathscr{D}$ if and only if $x \in C$;

(iii) For any prime ideal P in R and any $x \in R$, $\partial x \in P$ for all $\partial \in \mathscr{D}$ if and only if $\delta x \in P$ for all $\delta \in \mathscr{I}$.

It is clear that $\mathscr{I}$ itself is a primary class of derivations of R, (iii) $\Rightarrow$ (ii) for any prime ring R, and (i) is equivalent to that, for and $\partial \in \mathscr{D}$ and $x \in R$, $\delta_{\partial x} \in \mathscr{D}$.

It is also easy to see that $\partial x \in N$, the lower nil radical of R, for all $\partial \in \mathscr{D}$ if and only if $\delta x \in N$ for all $\delta \in \mathscr{I}$.

For convenience, we denote by $\mathscr{D}(R)$ the set of all elements ∂x where $\partial \in \mathscr{D}$ and $x \in R$.

EXAMPLE 1. Let $F = GF(2^n)$ where $n \geq 2$ and w be a generator of the multiplicative group of F. Let R be the ring of 2 by 2 matrices over F and $\mathscr{D}$ be the set of inner derivations δ_x where x is of the form $\alpha \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix}$, $\alpha, \beta \in F$. It is easy to see that $[\partial, \partial'] = 0$ for all $\partial, \partial' \in \mathscr{D}$ and $\mathscr{D}$ forms a primary class of deriva-

tions of R which is properly contained in $\mathscr{I}$, the group of all inner derivations of R.

EXAMPLE 2. Let F be a field and $R = F[x, y]$ be the free algebra generated by x and y over F. Let $\mathscr{D}$ be the Lie ideal of $\mathscr{I}$ generated by δ_x and δ_y. Then $\mathscr{D}$ is a primary class of derivations of R. $\mathscr{D} \neq \mathscr{I}$ since $\delta_{x^2}, \delta_{xy}, \delta_{y^2}, \delta_{x^3}, \cdots$, are not in $\mathscr{D}$.

The authors, however, do not know whether a primary class of derivations of a ring R is necessarily contained in the group of all inner derivations of R.

PROPOSITION 2.1. *Let R be a ring and R^* be a prime ring which is a homomorphic image of R under a homomorphism ϕ. Suppose $\mathscr{D}$ is a primary class of derivations of R. For each $\partial \in \mathscr{D}$, define $\partial^*: R^* \to R^*$ by $\partial^*(\phi(x)) = \phi(\partial x)$ for all $x \in R$. Then the set $\mathscr{D}^*$ of all ∂^*, where $\partial \in \mathscr{D}$, forms a primary class of derivations of R^*.*

Proof. For $\partial \in \mathscr{D}$, ∂^* is well defined since $\phi(x) = \phi(y) \Rightarrow x - y \in \operatorname{Ker} \phi \Rightarrow \delta(x - y) \in \operatorname{Ker} \phi$ for all $\delta \in \mathscr{I} \Rightarrow \partial(x - y) \in \operatorname{Ker} \phi$ for all $\partial \in \mathscr{D} \Rightarrow \phi(\partial x) = \phi(\partial y)$. Here we use the fact that $\operatorname{Ker} \phi$ is a prime ideal in R. It is readily to verify that $\mathscr{D}^*$ forms an additive group of derivations of R^* and that $\mathscr{D}^*$ satisfies the condition (i).

We observe that for $\phi(c) \in R^*$, $\phi(c)$ is central in $R^* \Rightarrow \phi(c)\phi(x) = \phi(x)\phi(c)$ for all $x \in R \Rightarrow [c, x] \in \operatorname{Ker} \phi$ for all $x \in R \Rightarrow \partial c \in \operatorname{Ker} \phi$ for all $\partial \in \mathscr{D} \Rightarrow \partial^*(\phi(c)) = 0$ for all $\partial^* \in \mathscr{D}^*$. Thus the condition (ii) holds for $\mathscr{D}^*$.

To see (iii), let P^* be a prime ideal in R^*. Set $P = \{x \in R \mid \phi(x) \in P^*\}$. P is then a prime ideal in R. For $x \in R$, $\partial^*(\phi(x)) \in P^*$ for all $\partial^* \in \mathscr{D}^* \Rightarrow \phi(\partial x) \in P^*$ for all $\partial \in \mathscr{D} \Rightarrow \partial x \in P$ for all $\partial \in \mathscr{D} \Rightarrow \delta x \in P$ for all $\delta \in I \Rightarrow \phi(\delta x) \in P^*$ for all $\delta \in \mathscr{I} \Rightarrow \delta^*(\phi(x)) \in P^*$ for all inner derivations δ^* of R^*. Thus (iii) holds for $\mathscr{D}^*$.

COROLLARY 2.2. *Let R be a ring and P be a prime ideal in R. Suppose $\mathscr{D}$ is a primary class of derivations of R. For each $\partial \in \mathscr{D}$, define $\partial^*: R/P \to R/P$ by $\partial^*(x + P) = \partial x + P$ for all $x \in R$. Then the set $\mathscr{D}^*$ of all ∂^*, where $\partial \in \mathscr{D}$, forms a primary class of derivations of R/P.*

COROLLARY 2.3. *In Corollary 2.2, if P is replaced by the lower nil radical N of R, the statement remains true.*

Proof. Use the fact that N is the intersection of all prime ideals in R.

COROLLARY 2.4. *In Corollary 2.2, if P is replaced by the Jacobson radical J of R the statement remains true.*

Proof. Using the fact that J is the intersection of all prime ideals in R containing J, one can show that for each $\partial \in \mathscr{D}$, ∂^* is well defined and that the conditions (i)–(iii) follow immediately for $\mathscr{D}^*$.

3. **Primary class satisfying (A).** Throughout this sections, we assume that R is a ring having a primary class $\mathscr{D}$ of derivations of R. Furthermore, we assume that $\mathscr{D}$ satisfies the condition (A) stated in the introduction.

Under condition (A), it is clear that zero is the only nilpotent element in $\mathscr{D}(R)$. We shall show first that all nilpotent elements in R are central. We begin with

LEMMA 3.1. *Let $x \in R$ and $\partial \in \mathscr{D}$. If $x^2 = 0$, then $x\partial x + \partial x x = 0$ and $[\partial x, x] = 0$.*

Proof. $\partial x^2 = 0$ implies $x\partial x + \partial x x = 0$, consequently, $[\partial x, x]^2 = (\partial x x - x \partial x)^2 = \partial x x \partial x x - x(\partial x)^2 x + x \partial x x \partial x = 0$. Since $[\partial x, x] \in \mathscr{D}(R)$ by condition (i), in the definition of a primary class of derivations, $[\partial x, x] = 0$.

LEMMA 3.2. *Let $x, z \in R$ and $\partial \in \mathscr{D}$. If $x^2 = 0$ and $[\partial z, x] = 0$, then $[\partial z, xy] = 0$ for all $y \in R$.*

Proof. Since $[\partial z, xyx] = \partial z x y x - x y x \partial z = x[\partial z, y]x \in \mathscr{D}(R)$ and its square is zero, we have $x[\partial z, y]x = 0$. Thus, $[\partial z, xy] = \partial z x y - x y \partial z = x[\partial z, y]$ is a nilpotent element in $\mathscr{D}(R)$ and hence $[\partial z, xy] = 0$.

Let us recall that the *characteristic* of an element $x \in R$, char x, is the order of x in the additive group of R.

LEMMA 3.3. *Let $x \in R$ and $\partial \in \mathscr{D}$. If $x^2 = 0$ and if char $(x\partial x) \neq 2$, then $\partial x = 0$.*

Proof. By Lemma 3.1, $2x\partial x = 0$ and hence $x\partial x = \partial x x = 0$. By expanding $\partial(x\partial x) = 0$, we obtain $(\partial x)^2 + x\partial^2 x = 0$. Pre-multiplying by ∂x yields $(\partial x)^3 = 0$. Thus $\partial x = 0$.

LEMMA 3.4. *Let* $x \in R$ *and* $\partial \in \mathscr{D}$. *If* $x^2 = 0$ *and* $\mathrm{char}\,(x\partial x) = 2$, *then* $2\partial x = 0$ *and* $\partial x \in C$.

Proof. Similar to the proof of Lemma 3.3, we obtain $2\partial x = 0$. Since $[\partial x, x] = 0$, by Lemma 3.2, $x[\partial x, y] = [\partial x, xy] = 0$ for all $y \in R$. It follows that

$$0 = \partial(x[\partial x, y]) = \partial x[\partial x, y] + x[\partial^2 x, y] \,.$$

But $0 = \partial([\partial x, x]) = (\partial x)^2 + x\partial^2 x + \partial^2 xx + (\partial x)^2 = x\partial^2 x + \partial^2 xx = [\partial^2 x, x]$ and hence by Lemma 3.2, $x[\partial^2 x, y] = [\partial^2 x, xy] = 0$ for all $y \in R$. Therefore, $\partial x[\partial x, y] = 0$ and it follows that $[\partial x, y]^2 = \partial xy[\partial x, y] = \partial xy(\partial xy + y\partial x) = (\partial x)^2 y^2 + (\partial x)^2 y^2 = 0$. Thus, $[\partial x, y] = 0$ for all $y \in R$ and $\partial x \in C$.

LEMMA 3.5. *Let* $x, y \in R$ *and* $\partial \in \mathscr{D}$. *If* $x^2 = 0$ *and* $[\partial y, x] = 0$, *then* $\partial_1 x\partial y \in C$ *for all* $\partial_1 \in \mathscr{D}$.

Proof. By Lemma 3.2, $x[\partial y, z] = [\partial y, xz] = 0$ for all $z \in R$ and hence $0 = \partial_1(x[\partial y, z]) = \partial_1 x[\partial y, z] + x[\partial_1\partial y, z]$. Since $\partial_1[\partial y, x] = 0$, $[\partial_1\partial y, x] = 0$ by Lemmas 3.3 and 3.4 and $[\partial_1\partial y, xz] = x[\partial_1\partial y, z] = 0$ for all $z \in R$ by Lemma 3.2. Thus, $0 = \partial_1 x[\partial y, z] = [\partial_1 x\partial y, z]$ for all $z \in R$ and $\partial_1 x\partial y \in C$.

LEMMA 3.6. *Let* $x \in R$. *If* $x^2 = 0$ *then* $\partial_2 x\partial_1 \partial y \in C$ *for all* $\partial, \partial_1, \partial_2 \in \mathscr{D}$ *and* $y \in R$.

Proof. Noting that $\partial x \in C$ for all $\partial \in \mathscr{D}$, we have $[\partial y, x] = \partial_{\partial y}(x) \in C$ which implies $[\partial_1\partial y, x] = \partial_1[\partial y, x] = 0$. Thus, by Lemma 3.5, $\partial_2 x\partial_1 \partial y \in C$.

LEMMA 3.7. *Let* $x \in R$. *If* $x^2 = 0$ *then* $\partial_1 x\partial^2 y = 0$ *for all* $\partial, \partial_1 \in \mathscr{D}$ *and* $y \in R$.

Proof. $\partial_1 xx\partial^2 y = \partial_1 x\partial^2(xy) \in C$ by Lemmas 3.3, 3.4, and 3.6. Hence $0 = \partial_1(\partial_1 xx\partial^2 y) = (\partial_1 x)^2\partial^2 y$. By the condition (A), $\partial_1 x\partial^2 y = 0$.

LEMMA 3.8. *Let* $x, y \in R$ *and* $\partial \in \mathscr{D}$. *If* $\partial x \in C$ *and* $(\partial x)^2 y \in C$ *then* $\partial xy \in C$.

Proof. Suppose $\partial x = (\partial x)^2 p$ where $p \in R$. Then, for any $z \in R$, $0 = p[(\partial x)^2 y, z] = p(\partial x)^2[y, z] = (\partial x)[y, z] = [\partial xy, z]$.

LEMMA 3.9. *Let* $x \in R$. *If* $x^2 = 0$ *then* $\partial_2 x(\partial_1 x\partial y + \partial x\partial_1 y) \in C$ *for all* $\partial, \partial_1, \partial_2 \in \mathscr{D}$ *and* $y \in R$.

Proof. Let $u = \partial_2 x(\partial_1 x\partial y + \partial x\partial_1 y)$. By Lemma 3.6, $\partial_2 x\partial_1\partial(xy) \in C$, i.e., $u + \partial_2 xx\partial_1 y \in C$. Since $\partial_2 x, \partial_2 x\partial_1\partial y \in C$, we have $[\partial_2 xx\partial_1 y, x] = 0$ and hence $[u, x] = 0$. For any $z \in R$, $[u, z] = (\delta_{\partial(\partial_2 x\partial_1 xy)} + \delta_{\partial_1(\partial x\partial xy)})(z) \in \mathscr{D}(R)$ and hence $[u, xzx] = x[u, z]x \in \mathscr{D}(R)$. Since $x[u, z]x$ is nilpotent, $x[u, z]x = 0$ and $x[u, z]$ is nilpotent. But $x[u, z] = [u, xz] \in \mathscr{D}(R)$. So $x[u, z] = 0$. Noting that $\partial_2 u = \partial_2 x(\partial_1 x\partial_2\partial y + \partial x\partial_2\partial_1 y) \in C$ by Lemma 3.6, we obtain $0 = \partial_2(x[u, z]) = \partial_2 x[u, z] = [\partial_2 xu, z]$. So $\partial_2 xu \in C$. By Lemma 3.8, $u \in C$.

LEMMA 3.10.   *Let* $x \in R$. *If* $x^2 = 0$ *then* $\partial_1 x\partial_1\partial y = 0$ *for all* $\partial, \partial_1 \in \mathscr{D}$ *and* $y \in R$.

Proof. Let $\partial \in \mathscr{D}$. Suppose $\partial x = 0$. Then according to Lemma 3.9, $\partial_2 x\partial_1 x\partial y \in C$ for all $\partial_1, \partial_2 \in \mathscr{D}$ and $y \in R$. Particularly, $(\partial_1 x)^2\partial y \in C$ and by Lemma 3.8, $\partial_2 x\partial y \in C$. Thus, $0 = \partial_1(\partial_1 x\partial y) = \partial_1 x\partial_1\partial y$.

Now, suppose $\partial x \neq 0$. Then by Lemma 3.9, $\partial_1(\partial_2 x\partial_1 xy + \partial_2 x\partial y\partial_1 y) = 0$ or $\partial_2 x\partial_1 x\partial_1 y = 0$ for all $\partial_1, \partial_2 \in \mathscr{D}$ and $y \in R$. Particularly, $(\partial_1 x)^2\partial_1\partial y = 0$. Using the condition (A) again $\partial_1 x\partial_1\partial y = 0$.

LEMMA 3.11.   *Let* $x \in R$. *If* $x^2 = 0$ *then* $\partial x\partial y \in C$ *for all* $\partial \in \mathscr{D}$ *and* $y \in R$.

Proof. By Lemma 3.10, $\partial x\partial\partial_1(yx) = 0$ or $0 = \partial x(\partial\partial_1 yx + \partial_1 y\partial x + \partial y\partial_1 x) = \partial x(\partial_1 y\partial x + \partial y\partial_1 x)$. Consequently, $\partial_1(\partial x\partial(yx)) = \partial x(\partial_1\partial yx + \partial_1 y\partial x + \partial y\partial_1 x) = \partial x\partial_1\partial yx \in \mathscr{D}(R)$ while its square is zero by Lemma 3.6. Since zero is the only nilpotent element in $\mathscr{D}(R)$ $\partial x\partial_1\partial yx = 0$, and since $\partial x\partial_1\partial y \in C$ and $\partial x \in C$, $0 = \partial(\partial x\partial_1\partial yx) = (\partial x\partial_1\partial y)\partial x = (\partial x)^2\partial_1\partial y$. Applying the condition (A), we get $\partial x\partial_1\partial y = 0$, i.e., $\partial_1(\partial x\partial y) = 0$. Since ∂_1 is arbitrary in $\mathscr{D}$, $\partial x\partial y \in C$ as we desired.

Now we are in the position to prove the following

PROPOSITION 3.12.   *All nilpotent elements in* R *are central and hence form an ideal in* R.

Proof. Suppose $x \in R$ and $x^n = 0$. We proceed by induction on n. Suppose $n = 2$. By Lemma 3.11, $\partial x\partial(xy) \in C$ for all $\partial \in \mathscr{D}$ and $y \in R$. From $[\partial x\partial(xy), x] = 0$, we arrive $[(\partial x)^2 y + \partial xx\partial y, x] = 0$. This implies that $[(\partial x)^2 y, x] = 0$ by Lemma 3.11, i.e., $[y, (\partial x)^2 x] = 0$ and $(\partial x)^2 x \in C$. Thus, $0 = \partial((\partial x)^2 x) = (\partial x)^3$ and consequently, $\partial x = 0$. This is true for all $\partial \in \mathscr{D}$, so $x \in C$.

Now we assume $x^{n+1} = 0$, where $n > 1$. Since $(x^2)^n = (x^3)^n = 0$, x^2 and x^3 are central by the induction hypothesis. Let $\partial \in \mathscr{D}$.

$0 = \partial x^2 = x\partial x + \partial xx$ and $0 = \partial x^3 = x^2\partial x$ yield $[\partial x, x]^2 = 0$. Since $[\partial x, x] \in \mathscr{D}(R)$, $[\partial x, x] = 0$. Noting that $0 = x(\partial x^2) = x\partial xx$ we obtain $(x\partial x)^2 = x[\partial x, x]\partial x = 0$. Thus, by the induction hypothesis, $x\partial x \in C$ and, consequently, $0 = \partial(x\partial x) = (\partial x)^2 + x\partial^2 x$. Pre-multiplaying by x and $(\partial x)^2$ respectively and by noting that $0 = \partial(x^2\partial x) = \partial x^2\partial x + x^2\partial^2 x = x^2\partial^2 x$, we obtain $x(\partial x)^2 = 0$ and hence $(\partial x)^4 = 0$. Thus $\partial x = 0$ for all $\partial \in \mathscr{D}$ and $x \in C$. This completes the proof.

COROLLARY 3.13. *Let A be a ring. Suppose for each commutator $x \in A$, there exists a $p(x) \in A$ such that $x = x^2 p(x)$. Then all nilpotent elements in A are central and form an ideal in A.*

REMARK. The proof of Proposition 3.12 can be simplified *slightly* by transfinite method. Let ϕ_i be epimorphisms of R onto prime rings R_i with $\bigcap_i \mathrm{Ker}\, \phi_i = N$. First show that each R_i has no nonzero nilpotent elements. Then for each nilpotent element $x \in R$, $\phi_i(x) = 0$ and consequently $\phi_i(\partial x) = 0$ for all i. This means that $\partial x \in N$ and so $\partial x = 0$ for all $\partial \in \mathscr{D}$.

PROPOSITION 3.14. *Suppose R is a prime ring. Then R is either a commutative ring or a division ring.*

Proof. We should note that R has no zero divisors. For $xy = 0$ implies $(yRx)^2 = 0$. Since R has nonzero nilpotent elements according to Proposition 3.12, $yRx = 0$ and whence $x = 0$ or $y = 0$.

Suppose now to the contrary that R is neither a commutative ring nor a division ring. Then $\mathscr{D}(R) \neq (0)$. Let $0 \neq \partial x \in \mathscr{D}(R)$ and $\partial x = (\partial x)^2 p$, where $p \in R$. Then, for any $y \in R$, $\partial x(y - \partial xpy) = 0$ yields $y = \partial xpy$, so ∂xp is a left identity in R. By Proposition 3.12, all nilpotent elements in R are central so all idempotent elements in R are also central and, particularly, $\partial xp \in C$. Therefore, ∂xp is a two-sided identity element in R which will be denoted by 1. This shows that, for $\partial x \in \mathscr{D}(R)$, either $\partial x = 0$ or ∂x is invertible.

Since we also assume that R is not a division ring, there exists $0 \neq a \in R$ which is not invertible. Suppose $a \in C$. Then, for any $\partial \in \mathscr{D}$, $\partial(ax) = \partial ax + a\partial x = a\partial x \neq 0$ for some $x \in R$. Thus $\partial(ax)$ is invertible so is a, a contradiction. Therefore $a \notin C$. Let $0 \neq \partial \in \mathscr{D}$ and let $b \in R$, $\partial b \neq 0$. Observe $\partial(a^2 b) = \partial a^2 b + a^2\partial b$. Suppose ∂a^2 is not invertible. Then $\partial a^2 = 0$ and $\partial(a^2 b) = a^2\partial b \neq 0$. Hence $\partial(a^2 b)$ is invertible, so is a, again contradicting the choice of a. Thus, ∂a^2 must be invertible for each nonzero $\partial \in \mathscr{D}$. Likewise ∂a^3 is invertible for each nonzero $\partial \in \mathscr{D}$. Let u and v be respectively the inverses of ∂a^2 and ∂a^3. Then $\partial(a^2 u - a^3 v) = \partial a^2 u + a^2\partial u - \partial a^3 v - a^3\partial v = a^2(\partial u - a\partial v)$ which is not invertible since a is not. Hence

$\partial(a^2u - a^3v) = 0$.   However, $a^2u - a^3v = a^2(u - av) \neq 0$ since $a \neq 0$ and a is not a unit. It follows that $\partial((a^2u - a^3v)b) = \partial(a^2u - a^3v)b +$ $(a^2u - a^3v)\partial b = (a^2u - a^3v)\partial b \neq 0$.   Consequently, $\partial((a^2u - a^3v)b)$ and hence $a^2u - a^3v$ are invertible. This is clearly impossible since a is not invertible. This completes the proof.

THEOREM 3.15. *Let R be a ring with lower nil radical N. Then R/N is a subdirect sum of division rings and commutative rings.*

Proof. From Corollary 2.3, R/N is a ring having a primary class of derivations $\mathscr{D}^*$ satisfying the condition (A). It is known by Proposition 3.12 that N is the set of all nilpotent elements in R, R/N has no nilpotent elements $\neq 0$, and R/N is a subdirect sum of prime rings R_i. By Proposition 2.1, each R_i possesses a primary class of derivations which satisfies the condition (A). Each R_i is either a division ring or a commutative ring by Proposition 3.14.

COROLLARY 3.16 (Ligh). *Suppose, for each commutator $x \in A$, there exists $p(x) \in A$ such that $x = x^2p$. Then A/N is a subdirect sum of division rings and commutative rings, where N is the lower nil radical of A.*

THEOREM 3.17. *Let R be a ring with Jacobson radical J. Then R/J is a subdirect sum of division rings.*

Proof. R/J is a subdirect sum of primitive rings. Each subdirect summand is a division ring by Proposition 3.14 and the fact that commutative primitive rings are fields.

COROLLARY 3.18 (Putcha, Wilson, Yaqub). *Let A be a ring. Suppose, for each commutator $x \in A$, there are a central element $z \in A$ and an integer $n > 1$ such that $x = x^nz$. Then A/J is a subdirect sum of division rings, where J denotes the Jacobson radical of A.*

In the balance of this section, we shall consider those rings R having a primary class $\mathscr{D}$ of derivations and satisfying a rather stronger condition than the condition (A), namely,

(C) For each $x \in R$ and $\partial \in \mathscr{D}$, there exists an element $z \in C$ such that $\partial x = (\partial x)^2z$.

THEOREM 3.19. *Let R be a ring having a primary class $\mathscr{D}$ of*

derivations and satisfying the condition (C). *Then* R *is commutative.*

Proof. By Theorem 3.15, R/N is a subdirect sum of rings R_i, where each R_i is either a commutative ring or a division ring. Suppose R_i is a division ring. Let $\mathscr{D}_i^*$ be the primary class of derivations of R_i induced from $\mathscr{D}$. For $x \in R_i$ and $\partial^* \in \mathscr{D}_i$, $\partial^* x = (\partial^* x)^2 z$ where z lies in the center of R_i. If $\partial^* x \neq 0$, then $\partial^* x$ has inverse z which is central. Thus $\partial^* x$ is central for all $\partial^* \in \mathscr{D}_i^*$ and $x \in R_i$. If the characteristic of R_i is 2 then, for $\partial^* \in \mathscr{D}_i^*$, $\partial^* (x^2) = 2x\partial^* x = 0$. Thus x^2 is central for all $x \in R_i$. By a well known result of Kaplansky (see [4], p. 219), R_i is commutative. Now suppose the characteristic of R_i is not 2. Then, for $0 \neq x \in R_i$, let y be the inverse of x. We have, for $\partial^* \in \mathscr{D}_i^*$, $0 = \partial^* (xy) = \partial^* x y + x\partial^* y$ and $0 = \partial^{*2}(xy) = 2\partial^* x \partial^* y$ which yield that $0 = \partial^* x(\partial^* xy + x\partial^* y) = (\partial^* x)^2 y$. Thus, $\partial^* x = 0$ for all $\partial^* \in \mathscr{D}_i^*$ and $x \in R_i$. R_i is therefore commutative.

R/N being a subdirect sum of commutative rings is commutative. To see that R is commutative, let $x, y \in R$. Then $xy - yx \in N$ yields $\partial x \in N$ and hence $\partial x = 0$ for all $\partial \in \mathscr{D}$ and $x \in R$. Thus R is commutative.

COROLLARY 3.20 (Putcha, Wilson, Yaqub). *Let* R *be a ring. Suppose, for each commutator* $x \in R$, *there exists a central element* z *which depends upon* x, *such that* $x = x^2 z$. *Then* R *is commutative.*

4. Primary class satisfying (B). Let us first list a useful result which will be used in the sequel of this section.

LEMMA 4.1. *Let* F *be a division ring of characteristic* $p \neq 0$ *and* C *be the center of* F. *Suppose* $a \in F \backslash C$ *and* $a^{p^n} = a$ *for some integer* $n \geq 1$. *Then there exists an* $x \in F$ *such that* $xax^{-1} = a^i \neq a$ *for some integer* i.

Proof. (See [3], p. 70.)

LEMMA 4.2. *Let* R *be a division ring and* $\mathscr{D}$ *be a primary class of derivations of* R. *Suppose, for each* $a \in R$ *and* $\partial \in \mathscr{D}$, *there exists an integer* $n > 1$ *which depends upon* a *and* ∂ *such that* $\partial a = (\partial a)^n$. *Then* R *is a field.*

Proof. Suppose to the contrary that R is not a field. Let $a \in R \backslash C$, where C denotes the center of R. Then there exists $\partial \in \mathscr{D}$

such that $\partial a \neq 0$, and $(\partial a)^m = \partial a$. Let P be the prime field of R. Since $2\partial \in \mathscr{D}$, $2\partial a = (2\partial a)^n = 2^n(\partial a)^n$ for some integer $n > 1$. Let $q = (m-1)(n-1) + 1$. One can see easily that $(2\partial a)^q = 2\partial a$ and $(\partial a)^q = \partial a$. Hence $2^q = 2$ and R is of nonzero characteristic p say.

We shall show now that, for each $0 \neq \partial \in \mathscr{D}$, there is an $a \in R$ such that $\partial a \notin C$. suppose not. Then $[\partial x, y] = 0$ for all $x, y \in R$ and $\partial_1 \partial(x) = 0$ for all $x \in R$, $\partial_1 \in \mathscr{D}$. If $p \neq 2$, we have $0 = \partial(\partial x^2) = \partial(2x\partial x) = 2(\partial x)^2$ and consequently $\partial x = 0$ for all $x \in R$, a contradiction. Thus we may assume now $p = 2$. Since, for each $x \in R$, $\partial(x^2) = 2x\partial x = 0$, we have, for all $x, y \in R$, $0 = [\partial(xx^2), y] = \partial x[x^2, y]$. If $\partial x \neq 0$ then $[x^2, y] = 0$ for all $y \in R$ and $x^2 \in C$. If $\partial x = 0$, then since, for all $y, z \in R$, $0 = [\partial(xz), y] = \partial z[x, y]$, $[x, y] = 0$ for all $y \in R$ and hence $x \in C$. Thus in either case, $x^2 \in C$. Using again the result of Kaplansky cited in the proof of Theorem 3.19, we obtain the commutativity of R, a contradiction. This proves the assertion that, for each $0 \neq \partial \in \mathscr{D}$, an element $a \in R$ exists with $\partial a \notin C$. For this ∂a, $(\partial a)^m = \partial a$ for some integer $m > 1$ and hence ∂a is algebraic over P and hence $(\partial a)^{p^k} = \partial a$ for some integer $k > 0$. By Lemma 4.1, there is an $x \in R$ such that $x\partial a x^{-1} = (\partial a)^i \neq \partial a$, i.e., $x\partial a = (\partial a)^i x \neq \partial a x$. Thus, $[\partial a, x] \neq 0$ and $[\partial a, x]\partial a = \partial a x \partial a - x(\partial a)^2 = (\partial a)^i(\partial a x - x\partial a) = (\partial a)^i[\partial a, x]$. However, by the hypothesis, $[\partial a, x]^h = [\partial a, x]$ for some integer $h > 1$. Thus the ring $S = \{\sum \alpha_{ij}(\partial a)^i[\partial a, x]^j \mid \alpha_{ij} \in P\}$ is a finite division ring. By the Wedderburn theorem for finite division rings, S is a field. Consequently $\partial a[\partial a, x] = [\partial a, x]\partial a$, a contradiction.

In the balance of this section, we assume R is a ring having a primary class $\mathscr{D}$ of derivations. Also we assume $\mathscr{D}$ satisfies the condition (B) stated in the introduction.

Perhaps we should note that the condition (B) implies the condition (A) and hence we can use the results in §3 freely in the sequel of this section.

LEMMA 4.3. *If R is a division ring then R is a field.*

Proof. Suppose the characteristic of R is not zero, and P is the (finite) prime field of R. Since each $\partial a \in \mathscr{D}(R)$ is algebraic over P, there exists an integer $m > 1$ such that $(\partial a)^m = \partial a$. By Lemma 4.2, R is commutative. Suppose the characteristic of R is zero. Suppose to the contrary that R is not commutative and $a \in R$ is not central. Then there is $\partial \in \mathscr{D}$ with $\partial a \neq 0$. For each prime number $p \in \mathscr{X}$, there exists $f_p(t) \in \mathscr{X}[t]$ such that $(p\partial a)^2 f_p(p\partial a) = p\partial a$. Thus $p\partial a f_p(p\partial a) - 1 = 0$. Let $g_p(t) \in \mathscr{X}[t]$ be a polynomial with least degree such that $p\partial a g_p(\partial a) - 1 = 0$. We claim that $ptg_p(t) - 1$ is

irreducible over $\mathscr{X}$ and hence it is also irreducible over $\mathscr{Q}$ by the Gauss lemma. Suppose $ptg_p(t) - 1 = h(t)k(t)$, where $h(t)$, $k(t) \in \mathscr{X}[t]$, $h(\partial a) = 0$ and $k(t) \neq \pm 1$. By comparing the constant terms of $ptg_p(t) - 1$ and $h(t)k(t)$, one can see that $k(t)$ is not a constant and the degree of $k(t)$ is n say. Let α, β, γ be respectively the leading coefficients of $k(t)$, $h(t)$, and $g_p(t)$. Then $\alpha\beta = p\gamma$. Since $h(t)k(t) \equiv -1 \,(\mathrm{mod}\, p)$, $k(t)$ is a constant polynomial modulo p and $p\,|\,\alpha$. Thus $\deg\,(g_p(t) - \alpha p^{-1} t^{n-1} h(t)) < \deg g_p(t)$ and since $p\partial a g_p(\partial a) = 1$ and $h(\partial a) = 0$, $p\partial a (g_p(\partial a) - \alpha p^{-1}(\partial a)^{n-1} h(\partial a)) - 1 = 0$. This contradicts the choice of $g_p(t)$. Therefore, for each prime number p, $ptg_p(t) - 1$ is irreducible over $\mathscr{Q}$ and has ∂a as a root. All these polynomials therefore must be divisible by each other over $\mathscr{Q}$. This clearly is impossible, and the lemma is proved.

From Proposition 3.14 and Lemma 4.3, we immediately have the following

LEMMA 4.4. *If R is a prime ring then R is commutative.*

Now we are in the position to prove our main result of this section.

THEOREM 4.5. *Let R be a ring having a primary class $\mathscr{D}$ of derivations which satisfies the condition (B). Then R is commutative.*

Proof. By Corollary 2.4, $\mathscr{D}$ induces a primary class $\mathscr{D}^*$ of derivations of R/J, where J denotes the Jacobson radical of R. Being semisimple, R/J is a subdirect sum of primitive rings. According to Lemma 4.4, each subdirect summand is commutative. Hence R/J is commutative. Note that J is the intersection of all prime ideals of R which contain J. Since for all $x \in R$ and $\delta \in \mathscr{I}$, the group of all inner derivations of R, $\delta x \in J$, we have $\partial x \in J$ for all $\partial \in \mathscr{D}$. From the condition (B), $\partial x = (\partial x)^2 p(\partial x)$ for some $p(t) \in \mathscr{X}[t]$. Let $e = \partial x p(\partial x)$. Clearly, $e \in J$ and $\partial x = \partial x e$. Let f be the quasi-invere of e. Then $0 = \partial x e(e + f - ef) = \partial x e = \partial x$. Thus, $\partial x = 0$ for all $x \in R$ and $\partial \in \mathscr{D}$ and R is commutative.

The following corollary which generalizes the celebrated result of Herstein cited in the introduction is an immediate consequence of Theorem 4.5.

COROLLARY 4.6. *Let A be a ring. If, for each commutator $x \in A$, there exists a polynomial $p(t) \in \mathscr{X}[t]$ such that $x = x^2 p(x)$, then*

A is commutative.

5. Some remarks and open problems. We have seen that, in many imposed conditions which imply the commutativity or "near" commutativity of rings R, general elements in R may often be replaced by the elements in $\mathscr{D}(R)$, where $\mathscr{D}$ is a primary class of derivations of R. In view of this, we would raise the question as to what extent can we replace elements of R or commutators by elements of $\mathscr{D}(R)$ and still yield the same or similar results? Thus a wide field of questions on commutativity is open. For example, one problem immediately comes to our minds: By the results given by Herstein, Martindale, Utumi and many others ([1], [3], [7], [9], etc.), we would like to propose the following open problems:

Let $\mathscr{D}$ be a primary class of derivations of a ring R and C be the center of R.

(1) Suppose, for each $\partial x \in \mathscr{D}(R)$, there exists a $p \in R$ which depends upon ∂x, such that $\partial x - (\partial x)^2 p \in C$. Do the nilpotent elements in R form an ideal N? If so, is R/N a subdirect sum of commutative rings and division rings?

(2) Suppose, for each $\partial x \in \mathscr{D}(R)$, there exists a polynomial $p(t) \in \mathscr{Z}[t]$ which depends upon ∂x, such that $\partial x - (\partial x)^2 p(\partial x) \in C$. Does $\mathscr{D}(R) \subseteqq C$?

Ligh has recently extended many commutativity theorems for rings including the one by Putcha, Wilson and Yaqub to near rings or d.g. near rings. For this reason we would like to know:

(3) Can the results in the present paper also be extended to d.g. near rings?

Once again, we propose the following:

(4) Does every member in a primary class of derivations have to be an inner derivation?

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