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# HOMOTOPY CONDITIONS WHICH DETECT SIMPLE HOMOTOPY EQUIVALENCES

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**Let $X$, $Y$, and $K$ be compact polyhedra, let $p: Y \times K \to Y$ be the projection map, and let $f: X \to Y \times K$ be a homotopy equivalence which has a homotopy inverse $g: Y \times K \to X$ along with homotopies $fg \simeq \mathrm{id}$, $gf \simeq \mathrm{id}$ such that $p(fg \simeq \mathrm{id})$ and $pf(gf \simeq \mathrm{id})$ are small homotopies. In this paper we prove that if $\pi_1$ of each component of $K$ is free abelian, then $f$ must be a simple homotopy equivalence.**

**1. Introduction.** All spaces in this paper will be locally compact, separable and metric, and a proper map is a map for which preimages of compacta are compact. The following is the main technical definition of this paper. If $\alpha$ is an open cover of $Y$, then a proper map $f: X \to Y$ is said to be an *$\alpha$-equivalence* provided that there is a map $g: Y \to X$, an $\alpha$-homotopy of $f \circ g: Y \to Y$ to the identity, and an $f^{-1}(\alpha)$-homotopy of $g \circ f: X \to X$ to the identity. Here $f^{-1}(\alpha) = \{f^{-1}(U) \mid U \in \alpha\}$, and a *$\beta$-homotopy* is a homotopy for which the *track* of each point lies in some element of $\beta$ (see § 2).

In [14] Ferry used $Q$-manifolds to prove the following result:

*If $Y$ is a polyhedron, then there is an open cover $\alpha$ of $Y$ so that for any polyhedron $X$ and $\alpha$-equivalence $f: X \to Y$, $f$ must be a simple homotopy equivalence.*

(For the definition of a *simple homotopy equivalence* (s.h.e.) for compact polyhedra we refer the reader to [24], and for noncompact polyhedra we refer to [19], where the designation *infinite* s.h.e. is used.) The above result represents the most general homotopy conditions that the author knows of which detect s.h.e.'s. It easily implies half of the Classification Theorem from $Q$-manifold theory [7, p. 88], which gives a homeomorphism condition which detects s.h.e.'s (see Theorem 2 below). On the other hand it follows from [16] that any cell-like map of polyhedra must be an $\alpha$-equivalence, for every $\alpha$. Therefore the above result implies that every cell-like map of polyhedra is a s.h.e., thus recapturing the main result of [5].

The purpose of this paper is to generalize the above result, while at the same time giving a proof which does not rely upon $Q$-manifold theory. In what follows $K$ will be a compact polyhedron for which each Whitehead group $\mathrm{Wh}(K \times T^n)$ vanishes, where $T^n$ is the $n$-torus ($T^0 = \{\text{point}\}$). This includes, for example, all polyhedra $K$ for which $\pi_1$ of each component of $K$ is free abelian or (more generally) poly $Z$ [13]. Here is our main result.

THEOREM 1. *For any polyhedron* $Y$ *with projection map* $p$: $Y \times K \to Y$, *there exists an open cover* $\alpha$ *of* $Y$ *so that if* $X$ *is a polyhedron and* $f$: $X \to Y \times K$ *is a* $p^{-1}(\alpha)$*-equivalence, then* $f$ *is a* s.h.e. *Moreover,* $\alpha$ *depends only on* $Y$.

It is clear that we cannot completely remove the $\pi_1$ restriction on $K$, for if $Y$ is a point we can choose compact polyhedra $X$ and homotopy equivalences $f: X \to K$ which are not s.h.e.'s [11, p. 98]. Note that Theorem 1 implies that any homeomorphism between polyhedra is a s.h.e., thus giving another proof of the topological invariance of simple homotopy type for polyhedra [4].

The proof of Theorem 1 that we give here uses no $Q$-manifold theory. We will work entirely in the PL category of polyhedra, and we rely on torus geometry in the spirit of [21]. The niceness condition on $\pi_1$ of each component of $K$ is used to conclude that some obstructions encountered in certain projective class groups and Whitehead groups vanish. It would be interesting to know if the $\pi_1$ condition on $K$ could be replaced by the assignment of a torsion to $f: X \to Y \times K$ in a nice subgroup of the Whitehead group $\mathrm{Wh}(Y \times K)$.

The author feels that Theorem 1 is not the last word in results of this type. It seems probable that the $p^{-1}(\alpha)$-equivalence condition in Theorem 1 can be replaced by a far more general condition on homotopy equivalences $f: X \to Y$, which would require that there exists a homotopy inverse $g: Y \to X$ of $f$ such that the homotopies $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$ would only "wind around nice elements of $\pi_1$."

As an application of Theorem 1 we give a short proof of the following result, which is half of the Classification Theorem of [7, p. 88]. We use $Q$ to represent the *Hilbert cube*, the countable infinite product of closed intervals. We need nothing at all from $Q$-manifold theory. This is a far cry from the proof of this half of the Classification Theorem given in [7], which uses a lot of $Q$-manifold theory.

THEOREM 2. *If* $X, Y$ *are polyhedra, then a proper map* $f$: $X \to Y$ *is a* s.h.e. *provided that* $f \times \mathrm{id}: X \times Q \to Y \times Q$ *is proper homotopic to a homeomorphism.*

The other half of the Classification Theorem asserts that given any s.h.e. $f: X \to Y$, $f \times \mathrm{id}: X \times Q \to Y \times Q$ is proper homotopic to a homeomorphism. There is a proof of this which uses elementary PL techniques and nothing at all from $Q$-manifold theory [3].

We point out that the splitting theorem of § 7 (Theorem 7.2) is in reality the main result of this paper. Once we have established it, Theorem 1 follows by a more-or-less standard argument. Theorem 7.2 is also the main tool used in [8] to investigate the problem of approximating maps of $Q$-manifolds to $Q$-manifold bundles by homeomorphisms.

Finally the author would like to thank Marshall Cohen for an unusually helpful referee's report. Our goal was to produce a paper that would be readable by expert and nonexpert alike, but without the input of the referee we would have certainly failed in both departments.

Here is a list of the sections to follow:

§ 2. General preliminaries.

§ 3. Preliminaries on equivalences.

§ 4. A finiteness result. Here we show that a certain homotopy domination can be extended to a homotopy equivalence in a well-controlled manner. This result is only needed in § 5.

§ 5. The handle lemma. Here we use torus geometry to establish the main technical result of this paper. The procedure is similar to that of [6], but the absence of cell-like maps makes the constructions much more complicated. The appearance of the factor $K$ appears to be more of a nuisance than a hindrance.

§ 6. The handle theorem. Here the inversion idea of [21] is used to reverse the roles of 0 and $\infty$ in the handle lemma.

§ 7. A splitting theorem. Here the handle theorem is applied to prove a general splitting result. This is the form of the handle theorem that is used in § 8.

§ 8. Proof of Theorem 1.

§ 9. Proof of Theorem 2.

## 2. General preliminaries.

The purpose of this section is to introduce some more notation and to establish some elementary results which will be needed in the sequel.

If $f_t\colon X \to Y$ is a homotopy, $t \in I = [0, 1]$, we use the notation $f_t\colon g \simeq h$ to indicate that $f_0 = g$ and $f_1 = h$. If $\alpha$ is an open cover of $Y$, then $f_t\colon X \to Y$ is an *$\alpha$-homotopy* provided that the *track* of each $x \in X$, $\{f_t(x) \mid 0 \leqq t \leqq 1\}$, lies in some element of $\alpha$. We say that the maps $u, v\colon X \to Y$ are *$\alpha$-close* if each set $\{u(x), v(x)\}$ lies in some element of $\alpha$. We will need the following *estimated* version of the homotopy extension theorem.

PROPOSITION 2.1. *Let $f\colon X \to Y$ be a map, $X_0 \subset X$ be closed, and let $g_t\colon X_0 \to Y$ be an $\alpha$-homotopy such that $g_0 = f \mid X_0$. Assume* *either* (i) $X_0$ *and* $X$ *are* ANRs, *or* (ii) $Y$ *is an* ANR. *Then* $g_t$ *extends to an* $\alpha$*-homotopy* $f_t\colon X \to Y$ *such that* $f_0 = f$.

*Proof.* We proceed in the usual manner.

( i ) Let $r\colon X \times I \to (X \times \{0\}) \cup (X_0 \times I)$ be a retraction obtained as a composition $r = r_2 \circ r_1$ as follows. For $N$ a small neighborhood of $X_0$, $r_1$ is a map of $X \times I$ into $(X \times \{0\}) \cup (N \times I)$ defined by $r_1(x, t) = (x, t\varphi(x))$, where $\varphi\colon X \to I$ is a map which is 0 on $X - N$ and 1 on $X_0$. $r_2$ is a retraction of $(X \times \{0\}) \cup (N \times I)$ onto $(X \times \{0\}) \cup (X_0 \times I)$, which exists because $X$ and $X_0$ are ANRs. For $N$ close to $X_0$, $r_2$ does not move points very far. Define $h\colon (X \times \{0\}) \cup (X_0 \times I) \to Y$ by $h(x, 0) = f(x)$ and $h(x, t) = g_t(x)$, and define $f_t\colon X \to Y$ by $f_t(x) = h \circ r(x, t)$. Note that each track, $\{f_t(x) \mid 0 \leqq t \leqq 1\}$, is a single point for $x \notin N$. For $x \in N$ we may choose $r_2$ and $N$ so that the track $\{f_t(x) \mid 0 \leqq t \leqq 1\}$ is close to some track $\{g_t(x') \mid 0 \leqq t \leqq 1\}$, where $x' \in X_0$. Thus $f_t$ is an $\alpha$-homotopy.

(ii) If $Y$ is an ANR, then there is a small neighborhood $N'$ of $(X \times \{0\}) \cup (X_0 \times I)$ in $X \times I$ and an extension of $h$ to $h'\colon N' \to Y$, where $h = f_0 \cup g$ is as above. If $r_1$ is as above, we may choose $r_1(X \times I) \subset N'$, and $f_t(x) = h' \circ r_1(x, t)$ is therefore our desired $\alpha$-homotopy.

If $\alpha, \beta$ are collections of subsets of a set $Y$ and $A \subset Y$, we define

$$\begin{aligned} &\mathrm{St}\,(A, \beta) = \cup \{A \cup U \mid U \in \beta, A \cap U \neq \emptyset\}\ , \\ &\mathrm{St}^0(\alpha, \beta) = \alpha\ , \\ &\mathrm{St}^{n+1}(\alpha, \beta) = \{\mathrm{St}\,(A, \beta) \mid A \in \mathrm{St}^n(\alpha, \beta)\}\ . \end{aligned}$$

If $\alpha = \beta$, then we simply write $\mathrm{St}^n(\alpha, \beta) = \mathrm{St}^n(\alpha)$.

If $f_t\colon X \to Y$ is a homotopy, $Y$ has a given metric, and $\varepsilon > 0$, then we say that $f_t$ is an $\varepsilon$*-homotopy* provided that the track of each point has diameter $< \varepsilon$. A proper map $f\colon X \to Y$ is said to be an $\varepsilon$*-equivalence* if there is a map $g\colon Y \to X$ such that $f \circ g$ is $\varepsilon$-homotopic to id and $g \circ f$ is $f^{-1}(\varepsilon)$-homotopic to id. This latter statement means that there is a homotopy $\varphi_t\colon g \circ f \simeq \mathrm{id}$ such that $f \circ \varphi_t$ is an $\varepsilon$-homotopy. If $A \subset Y$ is closed, then the proper map $f\colon X \to Y$ is said to be an $\alpha$*-equivalence over* $A$ (or $\varepsilon$*-equivalence over* $A$) if there is a map $g\colon A \to X$ such that $f \circ g$ is $\alpha$-homotopic (or $\varepsilon$-homotopic) to the inclusion $A \hookrightarrow Y$, and $g \circ f \mid f^{-1}(A)$ is $f^{-1}(\alpha)$-homotopic (or $f^{-1}(\varepsilon)$-homotopic) to $f^{-1}(A) \hookrightarrow X$. We call $g$ an $\alpha$*-inverse* of $f$ over $A$.

In general, "id" will be used to represent identity maps and "inc" will be used for inclusion maps. For any $X$ and $A \subset X$, $\mathring{A}$ denotes the (topological) *interior* of $A$ and $\mathrm{Bd}(A)$ denotes the *boundary* of $A$. If $X$ has a specified metric and $x \in X$, then $B_\varepsilon(x)$

is the open $\varepsilon$-ball around $x$. Also, $f|A: A \to Y$ is simply written $f|: A \to Y$.

A proper map $f: X \to Y$ is said to be a *fine equivalence* provided that it is an $\alpha$-equivalence, for all open covers $\alpha$ of $Y$. We say that $f: X \to Y$ is *cell-like* (or CE) if $f$ is surjective and all point-inverses have trivial shape in the sense of Borsuk [2]. We recall the following basic connection between these two notions [16]:

*A proper map $f: X \to Y$ between* ANRs *is a fine equivalence iff it is cell-like.*

A proper map $f: X \to Y$ is said to be *contractible* provided that it is surjective and all point-inverses are contractible (in themselves). Thus the above result implies that any contractible map of ANRs is an $\alpha$-equivalence, for all open covers $\alpha$ of the range. In the following result we collect some basic facts about $\alpha$-equivalences which are easy consequences of the definitions involved.

PROPOSITION 2.2. (i) *If $f: X \to Y$ is an $\alpha$-equivalence and $f$ is $\beta$-homotopic to a proper map $\bar{f}: X \to Y$, then $\bar{f}$ is a* $\mathrm{St}^2(\beta, \alpha)$-*equivalence.*

(ii) *If $\bar{f}: Y \to Z$ is a $\beta$-equivalence and $f: X \to Y$ is an $\bar{f}^{-1}(\alpha)$-equivalence, for any open covers $\alpha, \beta$ of $Z$, then $\bar{f}f: X \to Z$ is a* $\mathrm{St}^2(\beta, \alpha)$-*equivalence.*

*Proof.* (i) If $g: Y \to X$ is an $\alpha$-inverse of $f$, then it is easy to see that $g$ is a $\mathrm{St}^2(\beta, \alpha)$-inverse of $\bar{f}$.

(ii) Let $\alpha$ be any open cover of $Z$ and let $g: Y \to X$ be an $\bar{f}^{-1}(\alpha)$-inverse of $f$. Similarly let $\bar{g}: Z \to Y$ be a $\beta$-inverse of $\bar{f}$. We leave it as an easy exercise for the reader to check that $g\bar{g}$: $Z \to X$ is a $\mathrm{St}^2(\beta, \alpha)$-inverse of $\tilde{f}f: X \to Z$.

REMARKS. There is a version of (ii) above in which $\bar{f}$ is only assumed to be a $\beta$-equivalence over $A \subset Z$. In this case (ii) asserts that if $f: X \to Y$ is an $\bar{f}^{-1}(\alpha)$-equivalence over $A$, then $\bar{f}f: X \to Z$ is a $\mathrm{St}^2(\beta, \alpha)$-equivalence over $A$. Finally we remark that the result from [16] (quoted above), in conjunction with (ii), implies that if $f: X \to Y$ is a cell-like map of ANRs and $\bar{f}: Y \to Z$ is a $\beta$-equivalence, then $\bar{f}f$: $X \to Z$ is also a $\beta$-equivalence.

By a *polyhedron* we will mean a space which admits a PL structure in the sense of [17]. We will use notions from [17] such as subpolyhedron, PL map, PL collapse, etc.

For any map $f: X \to Y$ we let $M(f)$ denote its *mapping cylinder.*

It is the quotient space obtained from the disjoint union, $X \times [0, 1] \amalg Y$, by identifying $(x, 1)$ with $f(x)$. We write $M(f) = X \times [0, 1) \cup Y$ and identify $X$ with its 0-level, $X \times \{0\} \subset M(f)$. By the *rays* of $M(f)$ we mean the intervals $\{x\} \times [0, 1) \cup \{f(x)\} \subset M(f)$. There is a natural *collapse to the base*, $c: M(f) \to Y$, defined by $c | Y = \text{id}$ and $c(x, t) = f(x)$, for all $(x, t) \in X \times [0, 1)$.

We will also need the direct mapping cylinder construction. Let $X$ be a space and $f: X \to X$ a map. The *infinite direct mapping cylinder of f*, denoted $D_f$, is the quotient space obtained from the disjoint union,

$$\cdots \amalg X \times [-1, 0] \amalg X \times [0, 1] \amalg X \times [1, 2] \amalg \cdots ,$$

by identifying $(x, n)$ in $X \times [n - 1, n]$ with $(f(x), n)$ in $X \times [n, n + 1]$. Note that $D_f$ is just a union of countably many copies of $M(f)$. In a natural way $D_f$ may be *set-wise* identified with $X \times R$. We use $D_f[a, b]$ to denote the subset of $D_f$ which corresponds to the subset $X \times [a, b]$ of $X \times R$.

A map $f: X \to Y$ is a *homotopy domination* if there is a map $g: Y \to X$ such that $f \circ g \simeq \text{id}$. Let $(X, X_0)$ be a compact ANR pair, $X_0 \neq \emptyset$, and let $e: X \to X$ be a *homotopy idempotent* rel $X_0$. This means that $e | X_0 = \text{id}$ and there exists a homotopy $e_t: e \simeq e^2$ rel $X_0$. Note that the subset of $D_e$ corresponding to $X_0 \times R$ is actually homeomorphic to $X_0 \times R$. So we identify it with $X_0 \times R$. Define $s: D_e \to X$ by $s(x, t) = e_{t-n}(x)$, for $(x, t) \in D_e[n, n + 1)$. Note that $s$ is continuous. Let $i: X \to D_e$ be the map defined by $i(x) = (x, 0)$. We will need some information concerning this special situation which comes up in § 4. Compare with [9].

PROPOSITION 2.3. *The composition* $i \circ s: D_e \to D_e$ *is a homotopy equivalence. Moreover,* $i$ *is a homotopy domination and we can choose a right inverse of* $i$, $s': D_e \to X$, *and a homotopy* $h_t: i \circ s' \simeq \text{id}$ *such that* $s' | X_0 \times R = \text{proj}: X_0 \times R \to X_0$, $h_t | X_0 \times R$ *is given by* $h_t(x, r) = (x, tr)$, *and* $s' \circ i = e$.

*Proof.* Let $\alpha: D_e \to D_e$ be a map such that $\alpha | A = \text{id}$, where $A = i(X)$, the subset of $D_e$ identified with $X \times \{0\}$. Also let $A_0 = X_0 \times \{0\} \subset A$.

*Assertion.* We can choose a homotopy inverse of $\alpha$, say $\beta: D_e \to D_e$, such that $\beta | A = \text{id}$, $\beta \circ \alpha \simeq \text{id rel } A$, $\alpha \circ \beta \simeq \text{id rel } A$.

*Proof.* It suffices to prove that $\alpha$ induces isomorphisms on all homotopy groups, $\pi_n(D_e)$. If $j$ is the inclusion-induced homomorphism, $\pi_n(A) \xrightarrow{j} \pi_n(D_e)$, then the commutativity of

$$\begin{array}{ccc} \pi_n(D_e) & \xrightarrow{\alpha_*} & \pi_n(D_e) \\ & {}_{j}\nwarrow \quad \nearrow_{j} & \\ & \pi_n(A) & \end{array}$$

implies that all we have to do is prove that $j$ is surjective. To see this choose any element $[\varphi] \in \pi_n(D_e)$. By deforming down the rays of the mapping cylinders in $D_e$, and then using the fact that $e \simeq e^2$, we can easily find an element $[\psi] \in \pi_n(A)$ for which $j([\psi]) = [\varphi]$.

Returning to the proof of Proposition 2.3 consider $s\colon D_e \to X$ and note that $i \circ s|A$ is given by $i \circ s(x, 0) = (e(x), 0)$. We only have a homotopy in $D_e$, $i \circ s|A \simeq \text{inc}$, obtained by deforming down the rays of $D_e[0, 1]$, applying $e^2 \simeq e$, and coming back up the rays of $D_e[0, 1]$. Using Proposition 2.1 we can extend this homotopy $i \circ s|A \simeq \text{id}|A$ to a homotopy $i \circ s \simeq \alpha$, where $\alpha|A = \text{id}$. By the Assertion, $\alpha$ is a homotopy equivalence. Thus $i \circ s$ is a homotopy equivalence as we set out to prove.

Choose $\beta\colon D_e \to D_e$ as in the Assertion above and consider the homotopy

$$h'_t\colon i \circ s \circ \beta \simeq \alpha \circ \beta \simeq \text{id},$$

where the first homotopy comes from $i \circ s \simeq \alpha$, and the second comes from the Assertion. Thus $s'' = s \circ \beta$ is a right inverse of $i$. We note that $s''(x, 0) = e(x)$, for $(x, 0) \in A$, and $h'_t(x, 0) = (x, r)$, for all $x \in X_0$ (i.e., $h'_t$ preserves the $X_0$-coordinate in $A_0$). We will now modify $s''$ and $h'_t$ to get our desired $s'$ and $h_t$.

Since $s''(x, 0) = x$, for all $x \in X_0$, we can find a homotopy of

$$s''|A \cup (X_0 \times R) \text{ to } \text{id}_A \cup \text{proj}|X_0 \times R,$$

where each level of the homotopy agrees with $s''$ on $A$. By Proposition 2.1 we can extend this homotopy to a homotopy $s'' \simeq s'$, where $s'|X_0 \times R = \text{proj}|X_0 \times R$ and $s'(x, 0) = s''(X, 0) = e(x)$, for $(x, 0) \in A$. We then get a homotopy

$$h''_t\colon i \circ s' \simeq i \circ s'' \simeq \text{id},$$

where the first homotopy comes from $s' \simeq s''$, and the second is $h'_t$. Thus $h''_t(x, 0) = (x, r)$, for all $x \in X_0$ and $t \in [0, 1]$. Our final step is to show how $h''_t$ can be modified to obtain our required $h_t$.

Let $H\colon D_e \times I \to D_e$ be defined by $H(z, t) = h''_t(z)$ and let

$$S = (D_e \times \{0, 1\}) \cup (X_0 \times R \times I) \subset D_e \times I.$$

The condition $h_t''(x, 0) = (x, r)$, for $x \in X_0$, permits us to find a homotopy $F_u: H|S \simeq F_1$, where $F_u|D_e \times \{0\} = i \circ s'$, $F_u|D_e \times \{1\} = \mathrm{id}$, and $F_1|X_0 \times R \times I$ is given by $F_1((x, r), t) = (x, tr)$. By Proposition 2.1 we can extend $F_1$ to a map $K: D_e \times I \to D_e$, and $h_t(z) = K(z, t)$ fulfills our requirements.

We will need one more result in § 4. In addition to the above notation let $(Y, Y_0)$ be a compact ANR pair and let $u: (X, X_0) \to (Y, Y_0)$, $v: (Y, Y_0) \to (X, X_0)$ be maps such that $u \circ v | Y_0 = \mathrm{id}$, $v \circ u | X_0 = \mathrm{id}$, $e = v \circ u$, and $u \circ v \simeq \mathrm{id}$ rel $Y_0$.

PROPOSITION 2.4. *The compositions $u \circ s$, $u \circ s'$: $D_e \to X \to Y$ are homotopy equivalences. Moreover $u \circ s'$ has a homotopy inverse $i \circ v$: $Y \to X \to D_e$.*

*Proof.* Here are the homotopies which show that $i \circ v$ is a homotopy inverse of $u \circ s'$. It is equally easy to show that $u \circ s$ is a homotopy equivalence.

(1) $u \circ s' \circ i \circ v = u \circ s \circ i \circ v = u \circ e \circ v = u \circ v \circ u \circ v \simeq \mathrm{id}$, where the homotopy comes from $u \circ v \simeq \mathrm{id}$.

(2) $i \circ v \circ u \circ s' = i \circ e \circ s' \simeq i \circ s' \simeq \mathrm{id}$, where the first homotopy comes from $i \circ e \simeq i$ (by deforming down the rays of $D_e[0, 1]$), and the second is just the homotopy $h_t$ of Proposition 2.3.

REMARK. The statement that $u \circ s: D_e \to Y$ is a homotopy equivalence suffices for the proof of Theorem 4.3. However, in the Addendum to Theorem 4.3 we will need to exercise some more control, and for this we need the explicit construction of $u \circ s'$ in the statement and proof given above.

Finally we introduce one more notational convention which will be commonplace in the sequel. Let $f, g: X \to Y$ be maps and let $A \subset Y$. We say that $f = g$ *over* $A$ if $f^{-1}(A) = g^{-1}(A)$ and $f | f^{-1}(A) = g | f^{-1}(A)$. In general we say that $f$ has property $P$ over $A$ if $f | f^{-1}(A): f^{-1}(A) \to A$ has property $P$.

## 3. Preliminaries on equivalences.

In this section we will establish some general results about $\alpha$-equivalences which will be needed in the sequel.

PROPOSITION 3.1. *Let $(X, Y)$ be a compact* ANR *pair with $i$: $Y \hookrightarrow X$ an $\alpha$-equivalence, for any open cover $\alpha$ of $X$. Then there exists a map $g: X \to Y$ such that $g | Y = \mathrm{id}_Y$ and* $\mathrm{inc} \circ g$ *is* $\mathrm{St}^4(\alpha)$*-homotopic to* $\mathrm{id}_X$ rel $Y$.

*Proof.* If reference to the cover $\alpha$ is omitted, then the result is well-known [22, p. 31]. Let $g_1: X \to Y$ be an $\alpha$-inverse of $i$. This means that we have $\alpha$-homotopies $g_1 \simeq \mathrm{id}_X$ and $g \mid Y \simeq \mathrm{id}_Y$. By Proposition 2.1 there is an $\alpha$-homotopy $g_1 \simeq g$ such that $g \mid Y = \mathrm{id}$. The $\alpha$-homotopies $\mathrm{id} \simeq g_1$ and $g_1 \simeq g$ combine to give us a $\mathrm{St}(\alpha)$-homotopy $\mathrm{id} \simeq g$. Call this $\mathrm{St}(\alpha)$-homotopy $F: X \times [0, 1] \to X$, where $F_0 = \mathrm{id}$ and $F_1 = g$.

Define a homotopy

$$G: [(X \times \{0, 1\}) \cup (Y \times [0, 1])] \times [0, 1] \longrightarrow X$$

by the equations

$$\begin{aligned} G((x, 0), t) &= x, \quad \text{for all } x \in X, \\ G((x, 1), t) &= F(g(x), 1 - t), \quad \text{for all } x \in X, \\ G((x, s), t) &= F(x, (1 - t)s), \quad \text{for all } x \in Y. \end{aligned}$$

Note that $G_0$ extends to $F: X \times [0, 1] \to X$ and $G$ is a $\mathrm{St}(\alpha)$-homotopy. Thus $G_1$ extends to $H: X \times I \to X$ which is a $\mathrm{St}^4(\alpha)$-homotopy of id to $g$ rel $Y$.

PROPOSITION 3.2. *Let $(X, X_0)$, $(Y, Y_0)$ be compact* ANR *pairs and let $f: X \to Y$ be an $\alpha$-equivalence such that $f \mid X_0: X_0 \to Y_0$ is a homeomorphism. Then there exists a map $g: Y \to X$ such that $g \mid Y_0 = f^{-1} \mid Y_0$ and there are homotopies $fg \simeq \mathrm{id}$ rel $Y_0$, $gf \simeq \mathrm{id}$ rel $X_0$, where the former is a $\mathrm{St}^4(\alpha)$-homotopy and the latter is an $f^{-1}(\mathrm{St}^9(\alpha))$-homotopy.*

*Proof.* Form the mapping cylinder $M(f)$ and let $\varphi: X \to [0, 1]$ be a map for which $\varphi^{-1}(1) = X_0$. Define $Z \subset M(f)$ to be the union of the base $Y$ with all $(x, t) \in X \times [0, 1)$ for which $\varphi(x) \leqq t < 1$. Thus

$$Z = Y \cup (\cup\{\{x\} \times [\varphi(x), 1) \mid x \in X - X_0\}) .$$

We have an embedding $f_1: X \to Z$ given by $f_1(x) = f(x)$, for $x \in X_0$, and $f_1(x) = (x, \varphi(x))$, for $x \in X - X_0$. $Z$ is called a *reduced mapping cylinder* with top $f_1(X)$ and base $Y$. There is a natural collapse to the base, $c: Z \to Y$, obtained by restricting the collapse of $M(f)$ to $Z$. $Z$ is an ANR because it is a retract of the ANR $M(f)$.

Since $f: X \to Y$ is an $\alpha$-equivalence, it easily follows that $f_1: X \to Z$ is a $c^{-1}(\alpha)$-equivalence. By Proposition 3.1 there is a map $g_1: Z \to f_1(X)$ such that $g_1 | f_1(X) = \mathrm{id}$ and $g_1 \simeq \mathrm{id}\,\mathrm{rel}\, f_1(X)$ via a $\mathrm{St}^4 c^{-1}(\alpha)$-homotopy. Then the reader can easily check that $g = f_1^{-1} g_1 \mid Y: Y \to X$ is our desired map.

For our next result let $X, Y$ be polyhedra and $f: X \to Y$ a proper map. Let $c: M(f) \to Y$ denote the collapse of the mapping cylinder to its base.

PROPOSITION 3.3. *Let* $\varphi: Y \to [0, 4]$ *be a map and let* $\alpha$ *be an open cover of* $Y$ *such that* $\operatorname{diam} \varphi(U) < 1/2$, *for all* $U \in \alpha$. *If* $f: X \to Y$ *is an* $\alpha$*-equivalence over* $\varphi^{-1}([0, 3])$, *then there is a map* $\tilde{g}: c^{-1}\varphi^{-1}([0, 2]) \to X$ *such that* $\operatorname{inc} \circ \tilde{g}$ *is* $c^{-1}\mathrm{St}^4(\alpha)$*-homotopic to* id rel $f^{-1}\varphi^{-1}([0, 2])$, *with the homotopy taking place in* $M(f)$.

*Proof*. For each $t \in [0, 4]$ let $Y_t = \varphi^{-1}([0, t])$ and choose a map $g: Y_3 \to X$ which is an $\alpha$-inverse of $f$ over $Y_3$. Define a map $g_1: c^{-1}(Y_3) \to X$ by $g_1 = g \circ c | c^{-1}(Y_3)$ and note that $g_1 | f^{-1}(Y_3)$ is $f^{-1}(\alpha)$-homotopic to id. We will show how to perform two modifications of $g_1$ to arrive at our desired $\tilde{g}: c^{-1}(Y_2) \to X$.

Using Proposition 2.1 we see that $g_1$ is $f^{-1}(\alpha)$-homotopic to a map $g_2: c^{-1}(Y_3) \to X$ such that $g_2 | f^{-1}(Y_3) = \mathrm{id}$. We have a $c^{-1}(\alpha)$-homotopy

$$g_1 \simeq f \circ g_1 = f \circ g \circ c | c^{-1}(Y_3) \simeq c | c^{-1}(Y_3) \simeq \mathrm{id} ,$$

where the first homotopy comes from deforming down the rays of $M(f)$, the second comes from $f \circ g \simeq \mathrm{id}$, and the third comes from deforming back up the rays of $M(f)$. Thus we have a $c^{-1}\mathrm{St}(\alpha)$-homotopy $F: c^{-1}(Y_3) \times I \to M(f)$ from id to $g_2$. We define our required $\tilde{g}: c^{-1}(Y_2) \to X$ by $\tilde{g} = g_2 | c^{-1}(Y_2)$. In analogy with [22, p. 31] we now show how to modify $F$ to obtain a $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopy of $\tilde{g}$ to id rel $f^{-1}(Y_2)$.

Define

$$G: [(c^{-1}(Y_2) \times \{0, 1\}) \cup f^{-1}(Y_2 \times I)] \times I \longrightarrow M(f)$$

by the equations

$$\begin{aligned} G((x, 0), t) &= x , \quad \text{for all } x \in c^{-1}(Y_2) , \\ G((x, 1), t) &= F(g_2(x), 1 - t) , \quad \text{for all } x \in c^{-1}(Y_2) , \\ G((x, s), t) &= F(x, (1 - t)s) , \quad \text{for all } x \in f^{-1}(Y_2) . \end{aligned}$$

Observe that in order for the third equation to make sense we must have $g_2 \circ c^{-1}(Y_2) \subset c^{-1}(Y_3)$. This is the reason for choosing $\alpha$ in the prescribed manner.

Note that $G_0$ can be extended to $F | c^{-1}(Y_2) \times I$ and $G$ is a $c^{-1}\mathrm{St}(\alpha)$-homotopy. By Proposition 2.1 we can extend $G_1$ to a map $H: c^{-1}(Y_2) \times I \to M(f)$ which is $c^{-1}\,\mathrm{St}(\alpha)$-homotopic to $F | c^{-1}(Y_2) \times I$, thus implying that $H$ is a $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopy. Then $H$ is our required $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopy of id to $g_2 | c^{-1}(Y_2)$ rel $f^{-1}(Y_2)$.

PROPOSITION 3.4. *There exists a number* $\varepsilon > 0$ *so that if* $f\colon X \to Y$ *is a proper map of polyhedra,* $\varphi\colon Y \to [-4, 4]$ *is a map,* $\alpha$ *is an open cover of* $Y$ *for which* $\operatorname{diam} \varphi(U) < \varepsilon$, *for all* $U \in \alpha$, *and* $f$ *is an* $\alpha$*-equivalence over* $\varphi^{-1}([-3, 1])$ *and* $\varphi^{-1}([-1, 3])$, *then* $f$ *is a* $\mathrm{St}^{19}(\alpha)$*-equivalence over* $\varphi^{-1}([-2, 2])$.

*Proof.* Let $M(f)$ be the mapping cylinder of $f$. By Proposition 3.3 there is a map $g_1\colon c^{-1}\varphi^{-1}([-2.5, .5]) \to X$ for which $\mathrm{inc}\circ g_1$ is $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopic to id rel $f^{-1}\varphi^{-1}([-2.5, .5])$. Similarly there is a map $g_2\colon c^{-1}\varphi^{-1}([-.5, 2.5]) \to X$ for which $\mathrm{inc}\circ g_2$ is $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopic to id rel $f^{-1}\varphi^{-1}([-.5, 2.5])$. The map $g_1$, along with the homotopy $\mathrm{inc}\circ g_1 \simeq \mathrm{id}$, easily give us a map $\tilde{g}_1\colon M(f) \to M(f)$ such that $\tilde{g}_1 = g_1$ on $c^{-1}\varphi^{-1}([-2.4, .4])$, $\tilde{g}_1 = \mathrm{id}$ on $X$, and $\tilde{g}_1 \simeq \mathrm{id}$ rel $X$ via a $\mathrm{St}^4 c^{-1}(\alpha)$-homotopy. Similarly there is a map $\tilde{g}_2\colon M(f) \to M(f)$ such that $\tilde{g}_2 = g_2$ on $c^{-1}\varphi^{-1}([-.4, 2.4])$, $\tilde{g}_2 = \mathrm{id}$ on $X$, and $\tilde{g}_2 \simeq \mathrm{id}$ rel $X$ via a $\mathrm{St}^4 c^{-1}(\alpha)$-homotopy.

Now define $g\colon c^{-1}\varphi^{-1}([-2, 2]) \to X$ by $g = \tilde{g}_2\circ\tilde{g}_1 | c^{-1}\varphi^{-1}([-2, 2])$. (This makes sense if $\varepsilon$ is small enough.) Then $\mathrm{inc}\circ g = \tilde{g}_2\circ\tilde{g}_1| \simeq \tilde{g}_1| \simeq \mathrm{id}$, where these are both $\mathrm{St}^4 c^{-1}(\alpha)$-homotopies. Therefore $\mathrm{inc}\circ g \simeq \mathrm{id}$ rel $f^{-1}\varphi^{-1}([-2, 2])$ via a $\mathrm{St}^9 c^{-1}(\alpha)$-homotopy.

Now define $\bar{g}\colon \varphi^{-1}([-2, 2]) \to X$ by $\bar{g} = g | \varphi^{-1}([-2, 2])$. We leave it for the reader to check that $\bar{g}$ is a $\mathrm{St}^{19}(\alpha)$-inverse of $f$ over $\varphi^{-1}([-2, 2])$. (Compare this with the checking needed in Proposition 3.2.)

**4. A finiteness result.** In this section we prove Theorem 4.3, a result which will only be needed in the next section. Its proof uses some material from Wall's finiteness obstruction theory which we summarize below in Theorems 4.1 and 4.2.

The statements of the following results require the reduced projective class group functor $\tilde{K}_0$. Here is a brief description of just what we will need.

1. For every topological space $X$ there is an abelian group $\tilde{K}_0(X)$. We will not need a definition of $\tilde{K}_0(X)$, but for the interested reader it is the direct sum of all $\tilde{K}_0 Z[\pi_1(C)]$, where $C$ is a path component of $X$ and $Z[\pi_1(C)]$ denotes the integral group ring. (See [23, p. 64] for a definition of $\tilde{K}_0 Z[\pi_1(C)]$.)

2. For each map $f\colon X \to Y$ there is induced a homomorphism $f_*\colon \tilde{K}_0(X) \to \tilde{K}_0(Y)$ so that $\tilde{K}_0$ becomes a covariant functor from the homotopy category of topological spaces (and homotopy classes of maps) to the category of abelian groups (and homomorphisms).

3. It follows from the fundamental theorem of algebraic $K$-theory [1, p. 663] that $\tilde{K}_0(X)$ and $\mathrm{Wh}(X)$ are direct summands of $\mathrm{Wh}(X \times S^1)$, where Wh is the Whitehead group functor [11, p. 39]. Although this fact will not be needed in this section, it will be used in § 5.

Here is the basic geometric problem in which $\tilde{K}_0$ is used. Let $X$ be a compact polyhedron, $Y$ be any ANR, and let $f: X \to Y$ be a homotopy domination (cf. § 2 for a definition). In [23] Wall analyzed the problem of extending $f$ to a homotopy equivalence $\tilde{f}: \tilde{X} \to Y$, where $\tilde{X}$ is a compact polyhedron containing $X$ as a subpolyhedron. Here is the main result from [23] which solves this problem.

THEOREM 4.1. *$f: X \to Y$ can be extended to a homotopy equivalence $\tilde{f}: \tilde{X} \to Y$ (in the above manner) iff an obstruction $\sigma(Y)$ in $\tilde{K}_0(Y)$ vanishes. $\sigma(Y)$ is independent of the choice of $f$ and $X$.*

The main use of this result is the case in which $\pi_1(C)$ is free or free abelian, for each path component $C$ of $Y$; for then $\tilde{K}_0(Y) = 0$ and therefore $f$ extends in the required manner. (See [23, p. 67] for references.)

We now introduce some notation for the next result. Let an ANR $Y$ be written as the union of closed ANRs $Y_1$ and $Y_2$ with $Y_0 = Y_1 \cap Y_2$ also an ANR. Let $j_i$ be the inclusion $Y_i \hookrightarrow Y$, which induces a homomorphism $(j_i)_*: \tilde{K}_0(Y_i) \to \tilde{K}_0(Y)$. The following Sum Theorem computes $\sigma(Y)$ in terms of the $\sigma(Y_i)$ [19, p. 48].

THEOREM 4.2. *If each $Y_i$ is homotopically dominated by a compact polyhedron, then so is $Y$ and*

$$\sigma(Y) = (j_1)_*\sigma(Y_1) + (j_2)_*\sigma(Y_2) - (j_0)_*\sigma(Y_0) .$$

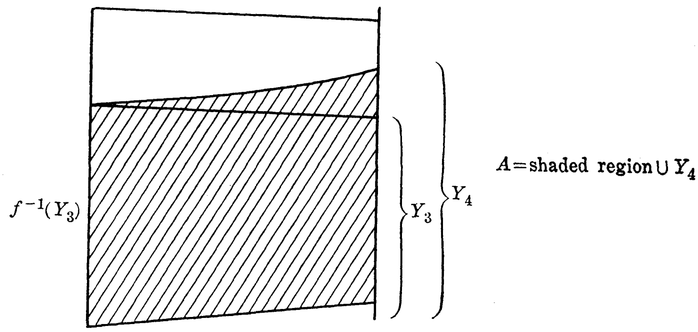
*The main result.* We now introduce some notation for Theorem 4.3, the main result of this section. Consider a compact polyhedral pair $(Y_0, L)$, where $\tilde{K}_0(L) = 0$. Form the polyhedron $Y = Y_0 \cup (L \times [0, 6])$ by sewing $L \times [0, 6]$ to $Y_0$ along $L \equiv L \times \{0\}$. For each $t$ let $Y_t = Y_0 \cup (L \times [0, t])$ and let $\varphi: Y \to [0, 6]$ be the map for which $\varphi^{-1}([0, t]) = Y_t$, for each $t$.

THEOREM 4.3. *There exists an $\varepsilon > 0$ such that if $X$ is any compact polyhedron and $f: X \to Y$ is any $\varphi^{-1}(\varepsilon)$-equivalence over $Y_5$, then we can extend $X$ to a compact polyhedron $\tilde{X}$ and define a map $\tilde{f}: \tilde{X} \to Y$ such that*

(1) *$\tilde{f}$ is a homotopy equivalence,*
(2) *$\tilde{f}(\tilde{X} - X) \subset L \times [1, 6]$,*
(3) *$\tilde{f} = f$ over $Y_1$.*

*Proof.* We will use the direct mapping cylinder construction of § 2 to reduce this problem to one in which Theorems 4.1 and 4.2 are applicable. Adopting the notation of Proposition 3.1 we consider the mapping cylinder $M(f)$, where $X \subset M(f)$ is the top and $Y \subset M(f)$ is the base. If $c: M(f) \to Y$ is the collapse to the base, it follows from Proposition 3.1 that there is a map $g: c^{-1}(Y_4) \to X$ which is $c^{-1}\varphi^{-1}(9\varepsilon)$-homotopic to id rel $f^{-1}(Y_4)$. Consider the following compact subset of $c^{-1}(Y_4)$:

$$A = c^{-1}(Y_3) \cup Y_4 \cup (\cup\{\{x\} \times [\varphi\circ f(x) - 3, 1) \mid x \in f^{-1}(\mathring{Y}_4 - \mathring{Y}_3)\}) .$$




There is a natural retraction $r: M(f) \to A$ obtained by first retracting $M(f)$ to $A \cup Y$ by retracting down the rays of $M(f)$, and then retracting $Y$ to $Y_4$ by retracting down the rays of $L \times [0, 6]$.

The map $X \hookrightarrow M(f) \xrightarrow{r} A$ is a homotopy domination with right inverse $A \xrightarrow{g} X$. Thus we get a homotopy idempotent $e: X \to X$ rel $f^{-1}(Y_3)$,

$$e: X \hookrightarrow M(f) \xrightarrow{r} A \xrightarrow{g} X .$$

That is, $e \simeq e^2$ rel $f^{-1}(Y_3)$. Note that for a sufficiently small choice of $\varepsilon$, the homotopy $e \simeq e^2$ takes $f^{-1}(L \times [3, 6])$ into $f^{-1}(L \times [2, 6])$ at each level. Thus the restriction $e \mid f^{-1}(L \times [t, 6])$ is a homotopy idempotent of $f^{-1}(L \times [t, 6])$, for $0 \leqq t \leqq 2$. In what follows we will need the maps $D_e \underset{i}{\overset{s}{\rightleftarrows}} X$ which were described in § 2 preceding

Proposition 2.3.

Note that the composition $fs: D_e \to Y$ is homotopic to the composition

$$D_e \xrightarrow{s} X \hookrightarrow M(f) \xrightarrow{r} A \xrightarrow{c|} Y .$$

By Proposition 2.4, $D_e \xrightarrow{s} X \hookrightarrow M(f) \xrightarrow{r} A$ is a homotopy equivalence. Since $c|: A \to Y$ is clearly a homotopy equivalence we conclude that $fs: D_e \to Y$ is a homotopy equivalence. Our strategy is to define $\tilde{f} = (fs)\tilde{i}$, where $\tilde{i}: \tilde{X} \to D_e$ is an extension of $i: X \to D_e$ to a homotopy equivalence. In order to extend $i$ to such a homotopy equivalence we will have to invoke the condition $\tilde{K}_0(L) = 0$.

Now choose compact subpolyhedra $K_1$ and $K_2$ of $X$ so that

$$f^{-1}(L \times [2, 6]) \subset K_1 \subset f^{-1}(L \times [5/3, 6]) \subset f^{-1}(L \times [4/3, 6]) \subset K_2 \subset f^{-1}(L \times [1, 6]) .$$

Let $e_j$ denote the restriction of $e$ to $K_j$, $j = 1, 2$. By restriction the homotopy domination $i: X \to D_e$ gives us homotopy dominations $i_j: K_j \to D_{e_j}$.

*Assertion.* $i_2: K_2 \to D_{e_2}$ can be extended to a homotopy equivalence (in the manner of Theorem 4.1).

*Proof.* Using Theorem 4.1 it suffices to prove that the obstruction $\sigma(D_{e_2})$ is zero. To do this first decompose $D_{e_2}$ as $D_{e_2} = D_{e_1} \cup (D_{e_2} - \mathring{D}_{e_1})$, and note that $D_{e_1}$, $D_{e_2} - \mathring{D}_{e_1}$, and $D_{e_1} \cap (D_{e_2} - \mathring{D}_{e_1})$ are all homotopically dominated by compact polyhedra. In fact, $D_{e_1}$ is dominated by $K_1$, $D_{e_2} - \mathring{D}_{e_1} = (K_2 - \mathring{K}_1) \times R$, and $D_{e_1} \cap (D_{e_2} - \mathring{D}_{e_1}) = \mathrm{Bd}(K_1) \times R$. Using Theorem 4.2 we have

$$\sigma(D_{e_2}) = \sigma(D_{e_1}) + \sigma(D_{e_2} - \mathring{D}_{e_1}) - \sigma(D_{e_1} \cap (D_{e_2} - \mathring{D}_{e_1})) ,$$

where we have omitted writing down the obvious inclusion-induced homomorphisms on the right hand side. Since $D_{e_2} - \mathring{D}_{e_1} = (K_2 - \mathring{K}_1) \times R$ and $D_{e_1} \cap (D_{e_2} - \mathring{D}_{e_1}) = \mathrm{Bd}(K_1) \times R$ we observe that the latter two terms on the right hand side vanish. Thus $\sigma(D_{e_2})$ is an element of the inclusion-induced image of $\tilde{K}_0(D_{e_1})$ in $\tilde{K}_0(D_{e_2})$.

To finish the proof of our Assertion it suffices to prove that the inclusion-induced image of $\tilde{K}_0(D_{e_1})$ in $\tilde{K}_0(D_{e_2})$ is zero. Let $s_1: D_{e_1} \to K_1$ be the restriction of $s: D_e \to X$ to $D_{e_1}$, and note that Proposition 2.2 implies that the composition

$$\theta: D_{e_1} \xrightarrow{s_1} K_1 \xrightarrow{e_1} K_1 \xrightarrow{i_1} D_{e_1}$$

is a homotopy equivalence (because $i_1 \circ e_1$ is homotopic to $i_1$). Therefore $\theta$ induces an isomorphism $\theta_*: \tilde{K}_0(D_{e_1}) \to \tilde{K}_0(D_{e_1})$. So it suffices to prove that the composition

$$\theta': D_{e_1} \xrightarrow{\theta} D_{e_1} \hookrightarrow D_{e_2}$$

induces the 0-map from $\tilde{K}_0(D_{e_1})$ to $\tilde{K}_0(D_{e_2})$. Clearly $\theta'$ is equivalent to the composition

$$\begin{aligned} D_{e_1} \xrightarrow{s_1} K_1 \hookrightarrow f^{-1}(L \times [5/3, 6]) \xrightarrow{r} A \cap c^{-1}(L \times [5/3, 6]) \\ \xrightarrow{g} f^{-1}(L \times [4/3, 6]) \hookrightarrow K_2 \xrightarrow{i_2} D_{e_2} . \end{aligned}$$

Applying the functor $\tilde{K}_0$ we conclude from this that $(\theta')_*: \tilde{K}_0(D_{e_1}) \to \tilde{K}_0(D_{e_2})$ factors through $\tilde{K}_0(A \cap c^{-1}(L \times [5/3, 6]))$. But $A \cap c^{-1}(L \times [5/3, 6])$ is homotopy equivalent to $L$. Thus $\tilde{K}_0(A \cap c^{-1}(L \times [5/3, 6])) = 0$, which implies that $(\theta')_*(\tilde{K}_0(D_{e_1})) = 0$.

Using the above Assertion we can extend $i_2: K_2 \to D_{e_2}$ to a homotopy equivalence $\tilde{i}_2: \tilde{K}_2 \to D_{e_2}$, where $\tilde{K}_2$ is a compact polyhedron containing $K_2$ as a subpolyhedron. This implies that $i: X \to D_e$ extends to $\tilde{i}: \tilde{X} \to D_e$ by defining $\tilde{X} = X \cup \tilde{K}_2$ (sewn together along $K_2$) and setting $\tilde{i} = \tilde{i}_2$ on $\tilde{K}_2$. It is easy to prove that $\tilde{i}$ is itself a homotopy equivalence because $D_e - \mathring{D}_{e_2} = (X - \mathring{K}_2) \times R$. Define $\tilde{f}$: $\tilde{X} \to Y$ to be the following composition:

$$\tilde{f}: \tilde{X} \xrightarrow{i} D_e \xrightarrow{s} X \xrightarrow{f} Y .$$

We know from Proposition 2.4 that $D_e \xrightarrow{s} X \hookrightarrow M(f) \xrightarrow{r} A$ is a homotopy equivalence, and this easily implies that $D_e \xrightarrow{s} X \xrightarrow{f} Y$ is a homotopy equivalence. Thus $\tilde{f}: \tilde{X} \to Y$ is a homotopy equivalence and it is clear that $\tilde{f} = f$ over $Y_1$.

Finally, in the following Addendum we improve the above result so that a certain homotopy inverse of $\tilde{f}$ is constructed subject to restrictions. For additional notation let $\alpha$ be an open cover of $Y$ and assume that $f: X \to Y$ is also an $\alpha$-equivalence over $Y_5$.

*Addendum to Theorem* 4.3. *We can choose the homotopy equivalence* $\tilde{f}: \tilde{X} \to Y$ *so that in addition to satisfying* (1)-(3) *of the statement of Theorem* 4.3, *it has a homotopy inverse*, $\tilde{g}: Y \to \tilde{X}$, *and homotopies* $\theta_t: \tilde{f} \circ \tilde{g} \simeq \mathrm{id}$, $\varphi_t: \tilde{g} \circ \tilde{f} \simeq \mathrm{id}$ *which satisfy the following properties*:

(1) $\theta_t$ *is a* $\mathrm{St}^9(\alpha)$-*homotopy on* $Y_1$, *and on* $Y - \mathring{Y}_1$ *it takes place in* $Y - \mathring{Y}_0$.

(2) $\varphi_t$ *is a* $f^{-1}\,\mathrm{St}^4(\alpha)$-*homotopy on* $f^{-1}(Y_1)$, *and on* $\tilde{f}^{-1}(Y - \mathring{Y}_1)$ *it takes place in* $\tilde{f}^{-1}(Y - \mathring{Y}_0)$.

*Proof*. We will redefine $\tilde{f}$ slightly so that we can write down a homotopy inverse $\tilde{g}$ in terms of the control given in Proposition 2.4. Using the notation set up in the proof of Theorem 4.3 we know that $e: X \to X$ is a homotopy idempotent rel $(X - \mathring{K}_1)$. If $s': D_e \to X$ is defined as in Proposition 2.3, then $s' | (X - \mathring{K}_1) \times R = \mathrm{proj}$: $(X - \mathring{K}_1) \times R \to X - \mathring{K}_1$. By Proposition 2.4 we know that $D_e \xrightarrow{s'} X \hookrightarrow M(f) \xrightarrow{r} A$ is a homotopy equivalence with homotopy inverse $A \xrightarrow{g} X \xrightarrow{i} D_e$, where $g: c^{-1}(Y_4) \to X$ is chosen so that it is $c^{-1}\,\mathrm{St}^4(\alpha)$-homotopic to id rel $f^{-1}(Y_4)$. Moreover, by Proposition 2.4 we can choose homotopies $r \circ s' \circ i \circ g \simeq \mathrm{id}$ and $i \circ g \circ r \circ s' \simeq \mathrm{id}$ subject to the following restrictions:

(1) $r \circ s' \circ i \circ g \simeq \mathrm{id}$ via a homotopy (in $A$) which is a $c^{-1}\,\mathrm{St}^9(\alpha)$-homotopy on $c^{-1}(Y_1)$, and on $A - c^{-1}(\mathring{Y}_1)$ it takes place in $A - c^{-1}(\mathring{Y}_0)$.

(2) $i \circ g \circ r \circ s' \simeq \mathrm{id}$ via a homotopy (in $D_e$) which takes $D_{e_1}$ into $D_{e_1}$ and on $D_e - \mathring{D}_{e_1} = (X - \mathring{K}_1) \times R$ it preserves the $(X - \mathring{K}_1)$-coordinate.

Now define $\tilde{f}: \tilde{X} \to Y$ to be the following composition:

$$\tilde{f}: \tilde{X} \xrightarrow{\tilde{i}} D_e \xrightarrow{s'} X \xrightarrow{f} Y .$$

It is clear that $\tilde{f}$ satisfies properties (1)-(3) of the statement of Theorem 4.3.

Using the fact that $D_e - \mathring{D}_{e_2} = (X - \mathring{K}_2) \times R$ it is easy to construct a homotopy inverse of $\tilde{i}$, $j: D_e \to \tilde{X}$, so that $j: | (X - \mathring{K}_2) \times R = \text{proj}: (X - \mathring{K}_2) \times R \to X - \mathring{K}_2$ and so that we have homotopies $\tilde{i} \circ j \simeq \text{id}$, $j \circ \tilde{i} \simeq \text{id}$ subject to the following restrictions:

(1) $\tilde{i} \circ j \simeq \text{id}$ via a homotopy which takes $D_{e_2}$ into $D_{e_2}$, and on $(X - \mathring{K}_2) \times R$ it preserves the $(X - \mathring{K}_2)$-coordinate.

(2) $j \circ \tilde{i} \simeq \text{id}$ via a homotopy which is the identity on $X - \mathring{K}_2$, and on $K_2$ it takes place in $K_2$. (Indeed, $j$ can be taken to be $s'$: $D_e \to X \hookrightarrow \tilde{X}$, for $s'$ is a right inverse of $i$ and $\tilde{i}$ extends $i$ to a homotopy equivalence.)

Then $\tilde{f}: \tilde{X} \to Y$ has a homotopy inverse,

$$\tilde{g}: Y \hookrightarrow M(f) \xrightarrow{r} A \xrightarrow{g} X \xrightarrow{i} D_e \xrightarrow{j} \tilde{X} .$$

We leave it to the reader to check that $\tilde{g}$ fulfills our requirements.

## 5. The handle lemma.

In this section we use Theorem 4.3 to prove the handle lemma, which is the main technical step of this paper. It is essentially an "extension theorem" for $\varepsilon$-equivalences. The proof uses torus geometry in the customary manner (cf. [6] and [21]). For notation, let $B_r^n$ denote the standard $n$-ball in Euclidean $n$-space $R^n$. Throughout this section $K$ will denote a compact polyhedron such that $\text{Wh}(K \times T^n) = 0$, for all $n \geqq 0$. Also $p: Z \times K \to Z$ will always denote projection to $Z$, for any space $Z$.

HANDLE LEMMA. *For each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $X$ is a polyhedron and $f: X \to R^n \times K$ is a proper map which is a $p^{-1}(\delta)$-equivalence over $B_3^n \times K$, then there exists a polyhedron $\tilde{X}$, a proper map $\tilde{f}: \tilde{X} \to R^n \times K$, and a PL homeomorphism $\varphi: \tilde{f}^{-1}(\mathring{B}_1^n \times K) \to f^{-1}(\mathring{B}_1^n \times K)$ such that*

*(1) $\tilde{f}$ is a $p^{-1}(\varepsilon)$-equivalence,*

*(2) $\tilde{f}$ is a PL homeomorphism over $(R^n - B_5^n) \times K$,*

*(3) $f\varphi = \tilde{f}$ over $\mathring{B}_1^n \times K$.*

REMARK. $\delta$ is independent of $K$.

*Proof.* For convenience let $B_r^n = [-r, r]^n \subset R^n$ and omit the subscript when $r = 1$. We will use the metric on $R^n$ defined by

$$d((x_i), (y_i)) = \max \{|x_i - y_i|\}_{i=1}^{n} .$$

Let $e: R \to S^1$ be defined by $e(x) = \exp(\pi i x/4)$, where $S^1$ is the set of complex numbers of absolute value 1. Note that $e$ is a covering map, and if $T^n = S^1 \times \cdots \times S^1$ is the *n-torus*, then $e^n = e \times \cdots \times e: R^n \to T^n$ is also a covering map. Represent the *punctured torus* by $T_0^n = T^n - \{x_0\}$, where $x_0 \notin e^n(B_2^n)$.

There are three large steps in the proof.

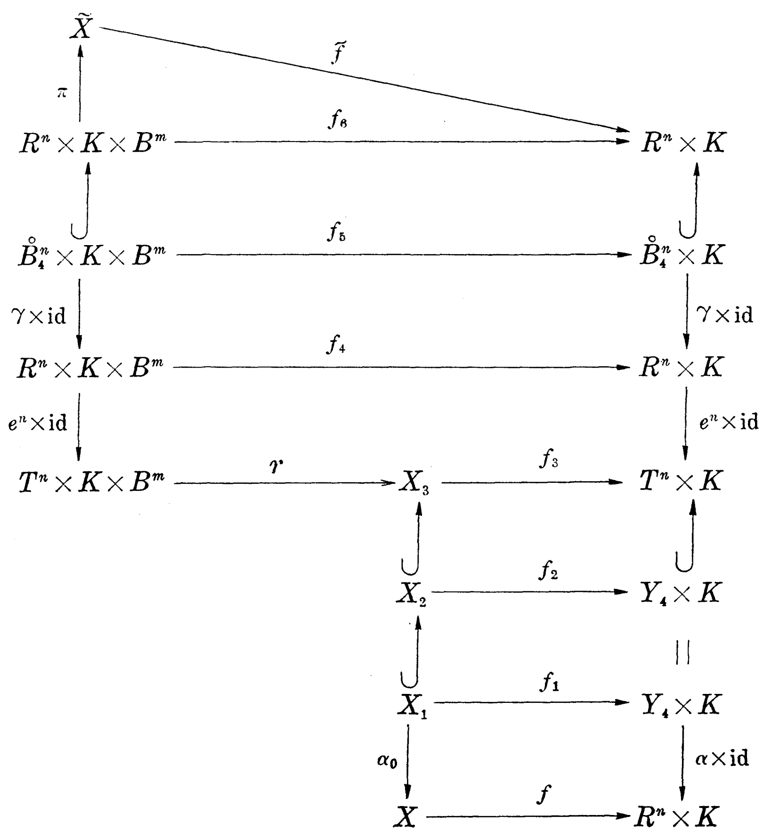
A. Pulling back to $T_0^n \times K$

B. Capping off to get $T^n \times K$

C. Lifting to $R^n \times K$.

There are also intermediate steps which are displayed in the diagram below. We remark here that steps A and C are easy in comparison with B. B requires Theorem 4.3 along with some results from simple homotopy theory. We assume that the reader is familiar with some of the standard results from simple homotopy theory such as those found in [11]. We will not require any infinite simple homotopy theory in this section.

Here is our "main diagram."



The goal of step A is the construction of the map $f_1$, while the goal of $B$ is the construction of the maps $r$ and $f_3$. Finally, in step C we produce our desired "extension" $\tilde{f}$ at the top of the diagram.

A. Pulling back to $T_0^n \times K$. We will first need an immersion $\alpha: T_0^n \to R^n$. (An *immersion* is a local open embedding.) For the construction of $\alpha$ we refer the reader to [14] for an elementary proof. We may clearly assume that $\alpha(T_0^n) \subset B_3^n$, and by using the Schoenflies theorem we can adjust $\alpha$ to obtain the additional restriction, $\alpha \circ e^n | B_2^n = \mathrm{id}$ (see [18, p. 48]). In what follows we assume that $n \geqq 2$. The case $n = 1$ is much simpler (it does not require torus geometry).

Form the pull-back diagram,

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & T_0^n \times K \\ {\scriptstyle\alpha_0}\downarrow & & \downarrow{\scriptstyle\alpha\times\mathrm{id}=\bar{\alpha}} \\ X & \xrightarrow{f} & R^n \times K \end{array}$$

where $X_0 = \{(x, y) | f(x) = \bar{\alpha}(y)\} \subset X \times T_0^n \times K$ and $\alpha_0, f_0$ are projection maps. We leave it as an exercise for the reader to prove that $\alpha_0$ is also an immersion and that $f_0$ is a proper map. Then it is easy to define a PL structure on $X_0$ making $X_0$ into a polyhedron. (See [17, pp. 76-77] for the definition of a PL structure.) Write $T_0^n = Y_0 \cup (S^{n-1} \times [0, \infty))$, where $S^{n-1} \times [0, \infty)$ is attached to $Y_0$ along $S^{n-1} \equiv S^{n-1} \times \{0\}$ and $e^n(B_2^n) \subset Y_0$. Let $Y_t = Y_0 \cup (S^{n-1} \times [0, t])$. With this notation, $T^n - \mathring{Y}_t$ is an $n$-ball.

*Assertion.* For any $\delta_1 > 0$ we can choose $\delta$ small enough so that $f_0$ is a $p^{-1}(\delta_1)$-equivalence over $Y_3 \times K$.

*Proof.* Let $g: B_3^n \times K \to X$ be a map which is a $p^{-1}(\delta)$-inverse of $f$ over $B_3^n \times K$. We want to define a map $g_0: Y_3 \times K \to X_0$ which is a $p^{-1}(\delta_1)$-inverse of $f_0$ over $Y_3 \times K$. Choose any $z \in Y_3 \times K$ and consider $g \circ \bar{\alpha}(z) \in X$. Note that $f \circ g \circ \bar{\alpha}(z)$ is $p^{-1}(\delta)$-close to $\bar{\alpha}(z)$. For any fixed metric on $T_0^n$ choose $\mu < \delta_1/2$ small enough so that $\alpha | B_\mu(y)$ is an open embedding, for each $y \in Y_3$. Then $\delta$ is chosen small enough so that $B_\delta(\alpha(y)) \subset \alpha B_\mu(y)$, for each $y \in Y_3$. We therefore define $g_0$ by

$$g_0(y, k) = (g \circ \bar{\alpha}(y, k), (\bar{\alpha} | B_\mu(y) \times K)^{-1}(f \circ g \circ \bar{\alpha}(y, k))) .$$

$g_0$ is well-defined because $\mathrm{im}\,(\bar{\alpha}) \subset B_3^n \times K = \mathrm{domain}\ (g)$.

To see that $f_0 \circ g_0$ is $p^{-1}(\delta_1)$-homotopic to id we first choose a $p^{-1}(\delta)$-homotopy $\theta_t: f \circ g \simeq \mathrm{id}$. Define a homotopy $\bar{\theta}_t: Y_3 \times K \to T_0^n \times$

$K$ by

$$\bar{\theta}_t(y, k) = (\bar{\alpha} \mid B_\mu(y) \times K)^{-1}(\theta_t \circ \bar{\alpha}(y, k)) .$$

It is clear that $\bar{\theta}_t$ is a $p^{-1}(\delta_1)$-homotopy of $f_0 \circ g_0$ to id.

To see that $g_0 \circ f_0 \mid f_0^{-1}(Y_3 \times K)$ is $(p \circ f_0)^{-1}(\delta_1)$-homotopic to id let $\varphi_t$: $f^{-1}(B_3^n \times K) \to X$ be a $(p \circ f)^{-1}(\delta)$-homotopy of $g \circ f \mid f^{-1}(B_3^n \times K)$ to id. Define a homotopy $\bar{\varphi}_t$: $f_0^{-1}(Y_3 \times K) \to X_0$ by

$$\bar{\varphi}_t(x, (y, k)) = (\varphi_t(x), (\bar{\alpha} \mid B_\mu(y) \times K)^{-1}(f \circ \varphi_1(x))) .$$

This is a $(p \circ f_0)^{-1}(\delta_1)$-homotopy of $g_0 \circ f_0 \mid f_0^{-1}(Y_3 \times K)$ to id.

Now, since $f_0$ is proper we may choose a compact subpolyhedron $X_1$ of $X_0$ so that

$$f_0^{-1}(Y_{3.5} \times K) \subset X_1 \subset f_0^{-1}(Y_4 \times K) .$$

Note that if $\delta_1$ is small enough, then $f_1 = f_0 \mid X_1$: $X_1 \to Y_4 \times K$ is a $p^{-1}(\delta_1)$-equivalence over $Y_3 \times K$.

B. Capping off to get $T^n \times K$. We first construct $f_3$: $X_3 \to T^n \times K$, a $p^{-1}(\delta_3)$-equivalence which agrees with $f_1$ over $Y_0 \times K$. This is done in two steps.

I. Construction of $X_2$. By Theorem 4.3 we can choose $\delta_1$ small enough so that we can add a compact polyhedron to $f_1^{-1}(S^{n-1} \times [2, 4] \times K)$ and thereby replace $f_1$ by a homotopy equivalence $f_2$: $X_2 \to Y_4 \times K$ which agrees with $f_1$ over $Y_2 \times K$. Moreover we have $f_2(X_2 - \mathring{X}_1) \subset (Y_4 - \mathring{Y}_2) \times K$, and $f_2$ has a homotopy inverse $g_2$: $Y_4 \times K \to X_2$ which behaves in the following well-controlled manner (for $\delta_2$ small and dependent on $\delta_1$):

(1) $f_2 \circ g_2 \simeq$ id via a homotopy which is a $p^{-1}(\delta_2)$-homotopy on $Y_2 \times K$, and on $(Y_4 - \mathring{Y}_2) \times K$ it takes place in $(Y_4 - Y_1) \times K$.

(2) $g_2 \circ f_2 \simeq$ id via a homotopy which is a $(p \circ f_2)^{-1}(\delta_2)$-homotopy on $f_2^{-1}(Y_2 \times K)$, and on $f_2^{-1}((Y_4 - \mathring{Y}_2) \times K)$ it takes place in $f_2^{-1}((Y_4 - Y_1) \times K)$.

II. Construction of $X_3$. Choose a PL map $\beta$: $S^{n-1} \times K \to X_2$ such that $\beta(x, k)$ is close to $g_2(x, 4, k)$, for all $(x, k) \in S^{n-1} \times K$. Form the mapping cylinder $M(\beta)$, which is a polyhedron containing $S^{n-1} \times K$ and $X_2$ as subpolyhedra ([10, p. 224]). If $c$: $M(\beta) \to X_2$ is the collapse to the base, then $f_2 \circ c$: $M(\beta) \to Y_4 \times K$ is a homotopy equivalence. The restriction $f_2 \circ c \mid S^{n-1} \times K$ is just $f_2 \circ \beta$: $S^{n-1} \times K \to Y_4 \times K$. Since $\beta(x, k)$ is close to $g_2(x, 4, k)$ we have $f_2 \circ \beta$ close to $f_2 \circ g_2 \mid S^{n-1} \times \{4\} \times K$, which is homotopic to id with the homotopy taking place in $(Y_4 - Y_1) \times K$. By the homotopy extension theorem we have $f_2 \circ c \simeq \tilde{f}_2$, where $\tilde{f}_2 = f_2$ on $X_2$ (the base of $M(\beta)$) and $\tilde{f}_2(x, k) = (x, 4, k)$, for all $(x, k) \in S^{n-1} \times K$. Define $X_3 = M(\beta) \cup (T^n - \mathring{Y}_4) \times K$, where $(T^n - \mathring{Y}_4) \times K$ is sewn to $M(\beta)$ by identifying $(x, k)$ in $S^{n-1} \times K \subset M(\beta)$ with $(x, 4, k)$ in $(T^n - \mathring{Y}_4) \times K$. Then $\tilde{f}_2: M(\beta) \to Y_4 \times K$ extends to $f_3: X_3 \to T^n \times K$ by defining $f_3 | X_3 - M(\beta) = \text{id}$.

*Assertion.* For every $\delta_3' > 0$ we can choose $\delta_2$ small enough so that $f_3: X_3 \to T^n \times K$ has a homotopy inverse $g_3: T^n \times K \to X_3$ which behaves in the following well-controlled manner:

(1) $f_3 \circ g_3 \simeq \text{id}$ via a homotopy which is a $p^{-1}(\delta_3')$-homotopy on $Y_1 \times K$, and on $(T^n - \mathring{Y}_1) \times K$ it takes place in $(T^n - \mathring{Y}_0) \times K$.

(2) $g_3 \circ f_3 \simeq \text{id}$ via a homotopy which is a $(p \circ f_3)^{-1}(\delta_3')$-homotopy on $f_3^{-1}(Y_1 \times K)$, and on $f_3^{-1}((T^n - \mathring{Y}_1) \times K)$ it takes place in $f_3^{-1}((T^n - \mathring{Y}_0) \times K)$.

*Proof.* Consider the homotopy equivalence $\tilde{f}_2: M(\beta) \to Y_4 \times K$ with inverse $g_2: Y_4 \times K \to X_2 \hookrightarrow M(\beta)$. Since $\tilde{f}_2 | S^{n-1} \times K: S^{n-1} \times K \to S^{n-1} \times \{4\} \times K$ is a homeomorphism we can produce a new inverse, $\tilde{g}_2: Y_4 \times K \to M(\beta)$, subject to the following restrictions (for $\delta_3'$ small):

(1) $\tilde{g}_2(x, 4, k) = (x, k)$, for every $(x, 4, k) \in S^{n-1} \times \{4\} \times K$,

(2) $\tilde{f}_2 \circ \tilde{g}_2 \simeq \text{id}$ rel $S^{n-1} \times \{4\} \times K$ via a homotopy which is a $p^{-1}(\delta_3')$-homotopy on $Y_1 \times K$, and on $(Y_4 - \mathring{Y}_1) \times K$ it takes place in $(Y_4 - Y_0) \times K$.

(3) $\tilde{g}_2 \circ \tilde{f}_2 \simeq \text{id}$ rel $S^{n-1} \times K$ via a homotopy which is a $(p \circ \tilde{f}_2)^{-1}(\delta_3')$-homotopy on $\tilde{f}_2^{-1}(Y_1 \times K)$, and on $\tilde{f}_2^{-1}((Y_4 - \mathring{Y}_1) \times K)$ it takes place in $\tilde{f}_2^{-1}((Y_4 - Y_0) \times K)$.

All of this is a consequence of Proposition 3.2. Then our desired $g_3: T^n \times K \to X_3$ extends $\tilde{g}_2$ by defining $g_3 = \text{id}$ on $(T^n - Y_4) \times K$.

Using the above Assertion we conclude that if $T^n - Y_0$ has a small diameter, then $f_3$ is a $p^{-1}(\delta_3)$-equivalence. Moreover $\delta_3$ can be chosen small corresponding to a small choice of $\delta_1$. This completes the construction of $f_3$.

To finish step B we must construct $r: T^n \times K \times B^m \to X_3$. Since $\text{Wh}(K \times T^n) = 0$, it follows that the homotopy equivalence $f_3$ is a s.h.e. Thus the homotopy inverse $g_3: T^n \times K \to X_3$ is also a simple homotopy equivalence. It follows from [12] that there is a PL homeomorphism $h$ of $X_3$ to a subpolyhedron $X_3'$ of $T^n \times K \times B^m$, for some $m \geqq 0$, and a PL collapse $T^n \times K \times B^m \to X_3'$. (See [17, p. 42] for the definition of a PL collapse.) Moreover if $c: T^n \times K \times B^m \to X_3'$ is the contractible PL retraction arising from the collapse, then $f_3 \circ h^{-1} \circ c: T^n \times K \times B^m \to T^n \times K$ is homotopic to the projection map. Then let $r = h^{-1}\circ c\colon T^n \times K \times B^m \to X_3$. It is a contractible PL map for which $f_3\circ r$ is homotopic to the projection. By Proposition 2.2 (ii) of § 2, $f_3\circ r\colon T^n \times K \times B^m \to T^n \times K$ is a $p^{-1}(\delta_3)$-equivalence. This completes step B.

C. Lifting to $R^n \times K$. Since $f_3\circ r$ is homotopic to the projection, it follows from elementary covering space theory that $f_3\circ r$ can be covered by a proper map $f_4'\colon R^n \times K \times B^m \to R^n \times K$ for which there is a *bounded* homotopy to the projection map. This means that there is a homotopy of $f_4'$ to proj: $R^n \times K \times B^m \to R^n \times K$, and $p$ composed with this homotopy yields a bounded homotopy into $R^n$. (Recall our convention regarding the map $p$.) By using an argument similar to that of the Assertion in step A we conclude that $f_4'$ is a $p^{-1}(\varepsilon_1)$-equivalence (where $\varepsilon_1$ is small corresponding to a small choice of $\delta_3$). It is easy to check that $(e^n \times \mathrm{id})|\colon (f_4')^{-1}(U \times K) \to (f_3\circ r)^{-1}(e^n(U) \times K)$ is $1-1$ and onto, for any open set $U \subset R^n$ for which $e^n | U\colon U \to e^n(U)$ is $1-1$.

Now choose a large $d$ (to be specified later on) and use the bounded homotopy of $f_4'$ to the projection map to construct a bounded homotopy of $f_4'$ to $f_4\colon R^n \times K \times B^m \to R^n \times K$ for which

(1) $f_4 = f_4'$ on $B_d^n \times K \times B^m$,

(2) $q\circ f_4 = q$ on $(R^n - B_{d+1}^n) \times K \times B^m$, where $q$ is the projection map to $K$,

(3) $p\circ f_4 = p\circ f_4'$.

(The homotopy $f_4' \simeq f_4$ is easily constructed by applying the homotopy $f_4' \simeq$ proj only in the $K$-coordinate.) If we choose $(f_4')^{-1}(B_2^n \times K) \subset B_d^n \times K \times B^m$, then we see that $f_4 = f_4'$ over $B_2^n \times K$.

Let $\gamma\colon \mathring{B}_4^n \to R^n$ be a radially-defined homeomorphism which is the identity on $B_2^n$. Then $f_5$ is defined to make the appropriate rectangle commute. The map $f_6$ extends $f_5$ by defining $f_6$ to be the projection map of $(R^n - \mathring{B}_4^n) \times K \times B^m$ to $(R^n - \mathring{B}_4^n) \times K$. This is continuous because $p\circ f_4$ is a bounded distance from $p$ and also because $q\circ f_4 = q$ near $\infty$. We note that $f_6 = f\circ\alpha_0\circ r\circ(e^n \times \mathrm{id})\circ(\gamma\times\mathrm{id})$ over $B_2^n \times K$, because $\alpha\circ e^n | B_2^n = \mathrm{id}$.

*Assertion.* For every $\varepsilon > 0$ we can choose $\varepsilon_1$ small enough so that $f_6$ is a $p^{-1}(\varepsilon)$-equivalence.

*Proof.* We will use Proposition 3.4 by showing that for some $u$ and $v$, $0 < u < v < 4$, $f_6$ is a "small" equivalence over $B_v^n \times K$ and over $(R^n - \mathring{B}_u^n) \times K$, where the "smallness" is measured in the $R^n$-coordinate. Choose $v'$ so that $(f_4')^{-1}(B_{v'}^n \times K) \subset B_d^n \times K \times B^m$. Then $f_4 = f_4'$ over $B_{v'}^n \times K$ and therefore $f_4$ is a $p^{-1}(\varepsilon_1)$-equivalence over $B_{v'}^n \times K$. If we let $\gamma^{-1}(B_{v'}^n) = B_{v}^n$ then $f_6$ is a $p^{-1}(\varepsilon)$-equivalence over $B^n_v \times K$. We can make $v$ close to 4 by choosing $d$ large and we can make $\varepsilon$ small by choosing $\varepsilon_1$ small.

*Now for the other half we must prove that* $f_6$ is a "small" equivalence over $(R^n - \mathring{B}^n_u) \times K$, where $u < v < 4$. We will construct a proper map $g_4: R^n \times K \to R^n \times K \times \{0\} \subset R^n \times K \times B^m$ so that

(1) $g_4$ is a bounded distance from $p$,

(2) $q \circ g_4 = q$ on the complement of a large compactum,

(3) there are bounded homotopies $\theta_t: f_4 \circ g_4 \simeq \mathrm{id}$ and $\varphi_t: g_4 \circ f_4 \simeq P$ so that $q \circ \theta_t = q$ and $q \circ \varphi_t = q$ on the complements of large compacta, where $P(x, k, y) = (x, k, 0)$ and the image of $\varphi_t$ lies in $R^n \times K \times \{0\}$.

Then define $g_5 = (\gamma \times \mathrm{id})^{-1} \circ g_4 \circ (\gamma \times \mathrm{id})$ along with homotopies $\theta'_t = (\gamma \times \mathrm{id})^{-1} \circ \theta_t \circ (\gamma \times \mathrm{id})$ and $\varphi'_t = (\gamma \times \mathrm{id})^{-1} \circ \varphi_t \circ (\gamma \times \mathrm{id})$, which are homotopies of $f_5 \circ g_5$ to id and $g_5 \circ f_5$ to $P$, respectively. By conditions (1) and (2) we see that $g_5$ extends via the identity to a map $g_6: R^n \times K \to R^n \times K \times \{0\}$. Similarly, $\theta'_t$ extends via the identity to a homotopy $\tilde{\theta}_t: f_6 \circ g_6 \simeq \mathrm{id}$. The restriction of $\tilde{\theta}_t$ to $(R^n - \mathring{B}^n_u) \times K$ is "small" for $u$ close to 4. Also $\varphi'_t$ extends via $P$ to a homotopy of $g_6 \circ f_6$ to $P$, and combining this with a homotopy of $P$ to id we obtain a homotopy $\tilde{\varphi}_t: g_6 \circ f_6 \simeq \mathrm{id}$. We have $p \circ \varphi_t = p$ on $(R^n - \mathring{B}^n_4) \times K \times B^m$, so $f_6 \circ \varphi_t$ is a "small" homotopy on $(R^n - \mathring{B}^n_u) \times K \times B^m$, for $u$ close to 4. This suffices to prove that $f_6$ is a "small" equivalence over $(R^n - \mathring{B}^n_u) \times K$. We now give the details for the construction of $g_4$ and the homotopies $\theta_t, \varphi_t$.

First choose a $p^{-1}(\varepsilon_1)$-inverse $g'_4: R^n \times K \to R^n \times K \times B^m$ of $f'_4$ and collapse out its $B^m$-component so that $g'_4(R^n \times K) \subset R^n \times K \times \{0\}$. Let $i_0: R^n \times K \to R^n \times K \times \{0\}$ be defined by $i_0(x, k) = (x, k, 0)$. Then we have bounded homotopies

$$i_0 \simeq i_0 f'_4 g'_4 \simeq i_0 \pi g'_4 = g'_4 \quad (\pi = \mathrm{proj}: R^n \times K \times B^m \longrightarrow R^n \times K) .$$

Using this we can construct a bounded homotopy of $g'_4$ to a map $g_4: R^n \times K \to R^n \times K \times B^m$ for which $q \circ g_4 = q$ and $p \circ g'_4 = p \circ g_4$. (Recall that $p$ means projection to everything except $K$, and in this case it is to $R^n \times B^m$.) Moreover, this homotopy affects only the component of $g'_4$ in the $K$-coordinate.

Since $f_4 \simeq f'_4$ and $g_4 \simeq g'_4$, we have obvious bounded homotopies $\theta_t: f_4 \circ g_4 \simeq \mathrm{id}$ and $\varphi_t: g_4 \circ f_4 \simeq P$. However, we want to construct $\theta_t$ and $\varphi_t$ so that condition (3) above is fulfilled. For this we must do a little more work. Because of the similarity of the cases, we will only give the details for $\theta_t$. Let $f''_4 = (p \circ f'_4, q): R^n \times K \times B^m \to R^n \times K$ and let $h_t: q \simeq q \circ g'_4$ be a homotopy. Define a homotopy $\alpha_t$ of $R^n \times K$ to $R^n \times K$ by

$$\alpha_t = (p \circ f'_4 \circ (p \circ g'_4, h_t), q) .$$

Note that $\alpha_0 = f_4''\circ g_4$ and $\alpha_1 = (p\circ f_4'\circ g_4', q)$. If $\theta_t'\colon f_4'\circ g_4' \simeq \mathrm{id}$ is a bounded homotopy, then $\beta_t = (p\circ\theta_t', q)$ gives us a bounded homotopy from $\beta_0 = \alpha_1$ to $\beta_1 = \mathrm{id}$. Thus

$$(*)\qquad f_4''\circ g_4 = \alpha_0 \simeq \alpha_1 = \beta_0 \simeq \beta_1 = \mathrm{id}$$

is a bounded homotopy of $f_4''\circ g_4$ to id. Finally we only need to notice that there is a bounded homotopy of $f_4$ to $f_4''$ so that $q$ of the homotopy is constantly $q$ on the complement of $B_{d+1}^n \times K \times B^m$. So this gives a bounded homotopy $f_4\circ g_4 \simeq f_4''\circ g_4$, and using (*) above we get our desired homotopy $\theta_t\colon f_4\circ g_4 \simeq \mathrm{id}$.

We now enter into the final phase of the proof of the Handle lemma. Our first task is to construct $\tilde{X}$. Let

$$\varphi_1 = (e^n\times \mathrm{id})(\gamma\times \mathrm{id})\,|\,f_6^{-1}(\mathring{B}_2^n\times K)\colon f_6^{-1}(\mathring{B}_2^n\times K) \longrightarrow r^{-1}f_1^{-1}(e^n(\mathring{B}_2^n)\times K)\ ,$$

$$\varphi_2 = \alpha_0\,|\,f_1^{-1}(e^n(\mathring{B}_2^n)\times K)\colon f_1^{-1}(e^n(\mathring{B}_2^n)\times K) \longrightarrow f^{-1}(\mathring{B}_2^n\times K)\ ,$$

which are easily seen to be PL homeomorphisms. Choose a compact subpolyhedron $C$ of $f_1^{-1}(e^n(\mathring{B}_2^n)\times K)$ which contains $f_1^{-1}(e^n(B_1^n)\times K)$. Then $\tilde{X}$ is defined to be the decomposition space $\tilde{X} = R^n \times K \times B^m/\mathscr{D}$, where the nondegenerate elements of $\mathscr{D}$ are

$$\{\{x\}\times\{k\}\times B^m \,|\, x\in R^n - \mathring{B}_5^n, k\in K\} \cup \{\varphi_1^{-1}\circ r^{-1}(x)\,|\,x\in C\}\ .$$

Let $\pi\colon R^n\times K\times B^m \to \tilde{X}$ be the natural quotient map, which is clearly a contractible map as defined in § 2. Now define $\tilde{f}\colon \tilde{X}\to R^n\times K$ by $\tilde{f} = f_6\circ\pi^{-1}$, which is well-defined. By [10, p. 241], $\tilde{X}$ supports a PL structure for which $\tilde{f}\ |\colon \pi((R^n - B_5^n)\times K\times B^m) \to (R^n - B_5^n)\times K$ is a PL homeomorphism. Also if $\varphi\colon \tilde{f}^{-1}(\mathring{B}_1^n\times K)\to f^{-1}(\mathring{B}_1^n\times K)$ is defined by $\varphi = \varphi_2\circ r\circ\varphi_1$, then $\varphi$ is a PL homeomorphism and $f\circ\varphi = \tilde{f}$ over $\mathring{B}_1^n\times K$. Finally we leave it as an exercise for the reader to prove that $\tilde{f}$ is a $p^{-1}(\varepsilon)$-equivalence. (If $g_6\colon R^n\times K\to R^n\times K\times B^m$ is an $\varepsilon$-inverse for $f_6$, then $\pi\circ g_6\colon R^n\times K\to\tilde{X}$ is an $\varepsilon$-inverse for $\tilde{f}$.)

REMARK. In the sequel it will be convenient to use $\tilde{f}$ to identify $\tilde{f}^{-1}((R^n - \mathring{B}_5^n)\times K)$ with $(R^n - \mathring{B}_5^n)\times K$ and $\varphi$ to identify $\tilde{f}^{-1}(\mathring{B}_1^n\times K)$ with $f^{-1}(\mathring{B}_1^n\times K)$. In this case, conditions (2) and (3) are replaced by

(2)′ $\tilde{f} = \mathrm{id}$ over $(R^n - B_5^n)\times K$,

(3)′ $\tilde{f} = f$ over $\mathring{B}_1^n\times K$.

**6. The handle theorem.** In this section we use the handle lemma to establish the handle theorem. In Proposition 6.1 we first establish a weak version of the handle theorem. It is here that we use the inversion idea of [21]. As in § 5, $K$ will denote a compact polyhedron such that $\mathrm{Wh}(K \times T^n) = 0$, all $n \geqq 0$. Also $p: Z \times K \to Z$ will always denote projection to $Z$.

PROPOSITION 6.1. *For each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $X$ is a polyhedron and $f: X \to R^n \times K$ is a proper map which is a $p^{-1}(\delta)$-equivalence over $B_3^n \times K$, then there is a polyhedron $X'$ and a proper map $f': X' \to R^n \times K$ such that*

(1) *$f'$ is a $p^{-1}(\varepsilon)$-equivalence over $B_{2.5}^n \times K$,*

(2) *$f' = f$ over $(R^n - B_2^n) \times K$,*

(3) *$f' = \mathrm{id}$ over $\mathring{B}_1^n \times K$.*

(We use the conventions of the Remark following the proof of the handle lemma.)

*Proof*. For a given $\delta_1 > 0$ we can use the handle lemma to find a $p^{-1}(\delta_1)$-equivalence $f_1: X_1 \to R^n \times K$ such that $f_1 = \mathrm{id}$ over a neighborhood of $\infty$ and $f_1 = f$ over $\mathring{B}_{2.5}^n \times K$. We can extend $f_1$ to a map $\tilde{f}_1: \tilde{X}_1 \to S^n \times K$ so that $\tilde{f}_1 | \tilde{X}_1 - X_1$ is a PL homeomorphism of $\tilde{X}_1 - X_1$ onto $(S^n - R^n) \times K$. (We regard $R^n$ as $S^n - \{\text{point}\}$.) By Proposition 3.4 we conclude that $\tilde{f}_1$ is a $p^{-1}(\delta_1')$-equivalence, for some $\delta_1'$ which is small corresponding to a small choice of $\delta_1$. By restriction we get

$$\tilde{f}_1 | \tilde{X}_1 - f^{-1}(\{0\} \times K): \tilde{X}_1 - f^{-1}(\{0\} \times K) \longrightarrow (S^n - \{0\}) \times K ,$$

which is a $p^{-1}(\delta_1')$-equivalence over any chosen compactum in $(S^n - \{0\}) \times K$ by choosing $\delta_1'$ correspondingly small. Moreover, $\tilde{f}_1 | \tilde{X}_1 - f^{-1}(\{0\} \times K)$ equals $f$ over $(\mathring{B}_{2.5}^n - \{0\}) \times K$.

By the handle lemma, for any $\delta_2 > 0$ we can choose $\delta_1'$ small enough so that there is a $p^{-1}(\delta_2)$-equivalence $f_2: X_2 \to (S^n - \{0\}) \times K$ satisfying $f_2 = \mathrm{id}$ over $(B^n - \{0\}) \times K$ and $f_2 = \tilde{f}_1$ over $(S^n - B_2^n) \times K$. Consider the restriction

$$f_2 | : f_2^{-1}((\mathring{B}_{2.5}^n - \{0\}) \times K) \longrightarrow (\mathring{B}_{2.5}^n - \{0\}) \times K ,$$

which is the identity over $(B_1^n - \{0\}) \times K$ and which agrees with $f$ over $(\mathring{B}_{2.5}^n - B_2^n) \times K$. The polyhedra $f_2^{-1}((\mathring{B}_{2.5}^n - \{0\}) \times K)$, $f^{-1}((R^n - B_2^n) \times K)$ and $\mathring{B}_1^n \times K$ can therefore be added together to form a polyhedron $X'$. In a natural manner we can define $f': X' \to R^n \times K$ which agrees with $f$ over $(R^n - B_2^n) \times K$, agrees with $f_2$ over $(\mathring{B}_{2.5}^n - \{0\}) \times K$, and which is the identity over $\mathring{B}_1^n \times K$. By Proposition

3.4 we conclude that $f'$ is a $p^{-1}(\varepsilon)$-equivalence over $B^n_{2.5} \times K$.

HANDLE THEOREM. *For each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $f: X \to R^n \times K$ is a proper map which is a $p^{-1}(\delta)$-equivalence over $B^n_3 \times K$, then there exists a proper map $f_1: X \times B^m \to R^n \times K$, for some $m \geqq 0$, such that*

(1) *$f_1$ is a $p^{-1}(\varepsilon)$-equivalence over $B^n_{2,5} \times K$,*

(2) *$f_1 = f \circ \text{proj}$ over $(R^n - B^n_2) \times K$,*

(3) *$f_1$ is a contractible PL map over $\mathring{B}^n_1 \times K$.*

*Proof.* Let $f': X' \to R^n \times K$ be the map of Proposition 6.1. Choose a compact subpolyhedron $L$ of $X$ so that

$$f^{-1}(B^n_{2.3} \times K) \subset L \subset f^{-1}(\mathring{B}^n_{2.4} \times K) .$$

Also $\text{Bd}(L)$ bounds a compact subpolyhedron $L'$ of $X'$ so that

$$(f')^{-1}(B^n_{2.3} \times K) \subset L' \subset (f')^{-1}(\mathring{B}^n_{2.4} \times K) .$$

*Assertion* 1. There is a homotopy equivalence $\alpha: L \to L'$ such that $\alpha | \text{Bd}(L) = \text{id}$.

*Proof.* Let $g: B^n_3 \times K \to X$ be a $p^{-1}(\delta)$-inverse of $f$ over $B^n_3 \times K$ and let $g': B^n_{2.5} \times K \to X'$ be a $p^{-1}(\varepsilon)$-inverse of $f'$ over $B^n_{2.5} \times K$. Choose a $(p \circ f')^{-1}(\varepsilon)$-homotopy $\theta'_t$ of $g' \circ f' | (f')^{-1}(B^n_{2.5} \times K)$ to id and define $\alpha: L \to L'$ as follows:

$$\alpha = \begin{cases} \text{id}, & \text{on } L - f^{-1}(\mathring{B}^n_{2.2} \times K) . \\ \theta'_{10t-21}, & \text{on } f^{-1}(\partial B^n_t \times K) = (f')^{-1}(\partial B^n_t \times K), \quad 2.1 \leqq t \leqq 2.2. \\ g' \circ f, & \text{on } f^{-1}(B^n_{2.1} \times K). \end{cases}$$

This makes sense provided that $\delta$ and $\varepsilon$ are small. To show that $\alpha$ is a homotopy equivalence we invoke Proposition 3.4. Specifically we show that $\alpha$ is a "small" equivalence over $L' - (f')^{-1}(\mathring{B}^n_{2.1} \times K)$ and over $(f')^{-1}(B^n_{2.2} \times K)$, where the "small" measurement is made in $R^n$ upon application of $p \circ f'$.

To see that $\alpha$ is an equivalence over $L' - (f')^{-1}(\mathring{B}^n_{2.1} \times K)$ we define $g_1: L' - (f')^{-1}(B^n_{2.1} \times K) \to L$ by $g_1 = \text{id}$. Using the homotopy $\theta'_t$ we easily see that $g_1$ is an inverse of $\alpha$ over $L' - (f')^{-1}(\mathring{B}^n_{2.1} \times K)$. To see that $\alpha$ is an equivalence over $(f')^{-1}(B^n_{2.2} \times K)$ we define $g_2$: $(f')^{-1}(B^n_{2.2} \times K) \to L$ by $g_2 = g \circ f'$. Then

$$\alpha \circ g_2 = \alpha \circ g \circ f' \simeq g' \circ f \circ g \circ f' \simeq g' \circ f' \simeq \text{id} ,$$

where the first homotopy arises from $\theta'_t$, the second from $f \circ g \simeq \text{id}$, and the third from $g' \circ f' \simeq \mathrm{id}$. Similarly,

$$g_2 \circ \alpha \mid \alpha^{-1}(f')^{-1}(B^n_{2.2} \times K) \simeq \mathrm{id} .$$

It is easy to check the "smallness" condition provided that $\varepsilon$ and $\delta$ are chosen small.

How choose a compact subpolyhedron $L_1$ of $L$ so that

$$f^{-1}(B^n_{2.1} \times K) \subset L_1 \subset f^{-1}(B^n_{2.2} \times K)$$

and let $L_1'$ be the corresponding subpolyhedron of $L'$ bounded by $\mathrm{Bd}(L_1)$. It is clear from the proof of Assertion 1 that the homotopy equivalence $\alpha: L \to L'$ may be constructed so that $\alpha \mid L - \mathring{L}_1 = \mathrm{id}$ and $\alpha \mid L_1: L_1 \to L_1'$ is also a homotopy equivalence.

*Assertion* 2. $\alpha$ is a s.h.e.

*Proof*. The proof is similar to Theorem 4.3. To show that $\alpha$ is a s.h.e. we need to show that its Whitehead torsion $\tau(\alpha)$, which lies in the Whitehead group $\mathrm{Wh}(L')$, vanishes [11, p. 72]. Using the Sum Theorem for Whitehead torsion [11, p. 76] we have

$$\tau(\alpha) = \tau(\alpha \mid L_1) + \tau(\alpha \mid L - \mathring{L}_1) + \tau(\alpha \mid \mathrm{Bd}(L_1)) ,$$

where we have omitted writing down inclusion-induced maps. Now $\alpha \mid L - \mathring{L}_1$ and $\alpha \mid \mathrm{Bd}(L_1)$ are identity maps, so their torsion vanishes. Thus we have $\tau(\alpha) = i_* \tau(\alpha \mid L_1)$, where $i$ is the inclusion $L_1' \hookrightarrow L'$ and $i_*$ is the induced map on Whitehead groups, $i_*: \mathrm{Wh}(L_1') \to \mathrm{Wh}(L')$. But $i$ is homotopic to the composition

$$L_1' \xrightarrow{f'} B^n_{2.2} \times K \xrightarrow{g'} (f')^{-1}(B^n_{2.3} \times K) \hookrightarrow L' ,$$

and since $\mathrm{Wh}(K) = 0$ we have $\mathrm{Wh}(B^n_{2.2} \times K) = 0$. Thus $i_*$ factors through 0, implying that $i_*$ is the 0-map. This gives $\tau(\alpha) = i_* \tau(\alpha \mid L_1) = 0$.

Using Assertion 2 we can find a compact polyhedron $J$ which collapses to $L$ and a contractible PL map $u: J \to L'$ such that $u \mid \mathrm{Bd}(L) = \mathrm{id}$. This follows easily from the fact that $\alpha$ is a s.h.e. (See [11, p. 16] for the CW case.) Then $u$ extends to $\tilde{u}: X \cup J \to X'$ by defining $\tilde{u} = \mathrm{id}$ on $X - \mathring{L}$, where $X \cup J$ is the polyhedron formed by sewing $X$ to $J$ along $L$. It follows from [12] that any collapse $A \searrow B$ can be reversed to obtain a collapse $B \times B^m \searrow A'$, for some $m \geqq 0$ and some PL copy $A'$ of $A$. Applying this to the collapse $X \cup J \searrow X$ we obtain a contractible PL map $v: X \times B^m \to X \cup J$. It follows directly from the proof in [12] that $v$ can be constructed so that it is the projection map from $f^{-1}((R^n - \mathring{B}^n_{2.4}) \times B^m$ to $f^{-1}((R_n - \mathring{B}^n_{2.4}) \times K)$. Then

$$f_1 \colon X \times B^m \xrightarrow{v} X \cup J \xrightarrow{\tilde{u}} X' \xrightarrow{f'} R^n \times K$$

fulfills our requirements (except that $f_1 = f \circ \mathrm{proj}$ over $(R^n - B^n_{2.4}) \times K$ rather than over $(R^n - B^n_2) \times K$).

7. **A splitting theorem.** We will use the Handle theorem to establish Theorem 7.2, a result which will be needed in § 8. In Lemma 7.1 we start with a very special case. For notation let $Y$ be a polyhedron with a fixed triangulation and let $\Delta \subset Y$ be a simplex which is not the face of any other simplex. It will be convenient to identify the combinatorial interior of $\Delta$ with $R^n$, and we will use $\partial\Delta$ for its combinatorial boundary. Also $K$ and $p \colon Z \times K \to Z$ will be as in § 6. Choose an open cover $\gamma$ which contains $R^n$ as one of its elements.

LEMMA 7.1. *For every open cover $\alpha$ of $Y$ there is an open cover $\beta$ of $Y$ so that if $X$ is a polyhedron and $f \colon X \to Y \times K$ is a $p^{-1}(\beta)$-equivalence, then there is an $m \geqq 0$, a closed subpolyhedron $X_1$ of $X \times B^m$, and a proper map $f_1 \colon X_1 \to (Y - R^n) \times K$ such that*

(1) *$f_1$ is a $p^{-1}(\alpha')$-equivalence, where $\alpha'$ is the restriction of $\alpha$ to $Y - R^n$,*

(2) *$f_1$ is $p^{-1}(\alpha)$-homotopic to $f \circ \mathrm{proj} | X_1$ (with the homotopy taking place in $Y \times K$).*

REMARKS. There is also a generalization of this result when $\Delta$ is replaced by a finite union of $n$-simplexes in the given triangulation of $Y$, each of which is not the face of any other simplex. Let $\{\Delta_i\}_{i=1}^k$ be this collection of $n$-simplexes, where $\Delta_i$ has combinatorial interior $R^n_i$. Also $\gamma$ is chosen to be any open cover which contains each $R^n_i$ as one of its elements. The generalization goes as follows: *For each $\alpha$ there is a $\beta$ so that each $p^{-1}(\beta)$-equivalence $f \colon X \to Y \times K$ yields a proper map $f_1 \colon X_1 \to (Y - \bigcup_{i=1}^k R^n_i) \times K$ (for $X_1 \subset X \times B^m$) such that*

(1) *$f_1$ is a $p^{-1}(\alpha')$-equivalence,*

(2) *$f_1$ is $p^{-1}(\alpha)$-homotopic to $f \circ \mathrm{proj} | X_1$.*

There are almost no changes in the proof to obtain this generalization. We have treated this special case here only for simplicity of notation.

*Proof.* By restriction we get a proper map

$$f|f^{-1}(R^n \times K): f^{-1}(R^n \times K) \longrightarrow R^n \times K .$$

Note that $f|f^{-1}(R^n \times K)$ is a $p^{-1}(\varepsilon)$-equivalence over any $B_r^n \times K$ we choose, for a sufficiently fine choice of $\beta$. Therefore, if $\varepsilon > 0$ is given, then the Handle theorem implies that $\beta$ can be chosen fine enough so that there is proper map $f': f^{-1}(R^n \times K) \times B^m \to R^n \times K$ which agrees with $f\circ\text{proj}$ over $(R^n - \mathring{B}_{r-1}^n) \times K$, is a $p^{-1}(\varepsilon)$-equivalence over $B_r^n \times K$, and which is a contractible PL map over $B_{r-2}^n \times K$. Then $f'$ naturally extends to $\tilde{f}: X \times B^m \to Y \times K$ by defining $\tilde{f} = f\circ\text{proj}$ over $(Y - R^n) \times K$. Our desired $X_1$ is defined to be

$$X_1 = (X \times B^m) - \tilde{f}^{-1}(\mathring{B}_{r-2}^n \times K) .$$

Let $s: Y - \{0\} \to Y - R^n$ be a radially-defined retraction and define $f_1: X_1 \to (Y - R^n) \times K$ to be

$$f_1 = (s \times \text{id})\circ\tilde{f}|X_1: X_1 \xrightarrow{\tilde{f}} (Y - \{0\}) \times K \xrightarrow{s\times\text{id}} (Y - R^n) \times K .$$

We must show that $X_1$ and $f_1$ meet our requirements (1) and (2). We examine them one-by-one.

(1) It is a nontrivial matter to show that $f_1$ is a $p^{-1}(\alpha)$-equivalence. (For simplicity, $\alpha'$ now becomes $\alpha$.) Let $u: (B_{r-2}^n - \{0\}) \times K \to \partial B_{r-2}^n \times K$ be the radially-defined retraction and let $u_t: u \simeq \text{id}$ be the radially-defined homotopy of $u$ to id. Since $\tilde{f}|\tilde{f}^{-1}(B_{r-2}^n \times K)$ is a contractible map we can "lift" $u$ to a retraction $\tilde{u}: \tilde{f}^{-1}((B_{r-2}^n - \{0\}) \times K) \to \tilde{f}^{-1}(\partial B_{r-2}^n \times K)$ such that $\tilde{f}\circ\tilde{u}$ is as close to $u\circ\tilde{f}$ as we please. Also $u_t$ "lifts" to a homotopy $\tilde{u}_t: \tilde{u} \simeq \text{id}$ such that $\tilde{f}\circ\tilde{u}_t$ is close to $u_t\circ\tilde{f}$ and such that $\tilde{u}_t|\tilde{f}^{-1}(\partial B_{r-2}^n \times K) = \text{id}$, for each $t$ (proof same as Proposition 3.1). Then $\tilde{u}_t$ extends trivially to a homotopy $v_t: \tilde{f}^{-1}((Y - \{0\}) \times K) \to X \times B^m$ such that $v_0$ is a retraction of $\tilde{f}^{-1}((Y - \{0\}) \times K)$ onto $X_1$. We are now ready to construct a $p^{-1}(\alpha)$-inverse of $f_1$. It follows from Proposition 3.4 that $\tilde{f}$ is a $p^{-1}(\alpha_1)$-equivalence, where $\alpha_1$ is fine corresponding to a fine choice of $\beta$. Let $\tilde{g}: Y \times K \to X \times B^m$ be a $p^{-1}(\alpha_1)$-inverse of $\tilde{f}$ and define $g_1: (Y - R^n) \times K \to X_1$ by $g_1 = v_0\circ\tilde{g}$. It is now easy to show that $g_1$ is a $p^{-1}(\alpha)$-inverse of $f_1$. We have

$$f_1\circ g_1 = (s \times \text{id})\circ\tilde{f}\circ v_0\circ\tilde{g} \simeq (s \times \text{id})\circ\tilde{f}\circ\tilde{g} \simeq s \times \text{id} = \text{id} ,$$

where the maps are all restricted to $(Y - R^n) \times K$. The first homotopy comes from $v_0 \simeq \text{id}$, the second from $\tilde{f}\circ\tilde{g} \simeq \text{id}$. We also have

$$g_1\circ f_1 = v_0\circ\tilde{g}\circ(s \times \text{id})\circ\tilde{f} \simeq v_0\circ\tilde{g}\circ\tilde{f} \simeq v_0 = \text{id} ,$$

where the maps are all restricted to $X_1$. The first homotopy comes from the natural radial homotopy $s \simeq \text{id}$, the second from $\tilde{g}\circ\tilde{f} \simeq \text{id}$.

If $r$ is large, then we conclude that $f_1 \circ g_1 \simeq \mathrm{id}$ is a $p^{-1}(\alpha)$-homotopy and $g_1 \circ f_1 \simeq \mathrm{id}$ is a $(p \circ f_1)^{-1}(\alpha)$-homotopy. Thus $f_1$ is a $p^{-1}(\alpha)$-equivalence. (2) It is clear that

$$\tilde{f}|\colon \tilde{f}^{-1}((\Delta - \mathring{B}^n_{r-2}) \times K) \to \Delta \times K$$

is homotopic to $f \circ \mathrm{proj}$ rel $\tilde{f}^{-1}(\partial \Delta \times K)$. This deformation can be constructed by letting $R\colon (B^n_r - \{0\}) \times K \to \partial B^n_r \times K$ be defined via the radial retraction, and using Proposition 2.1 to homotope the identity on $\tilde{f}^{-1}((Y - \mathring{B}^n_{r-2}) \times K)$ to a map which equals $\tilde{g}R\tilde{f}$ on $\tilde{f}^{-1}((B^n_r - \mathring{B}^n_{r-2}) \times K)$ and is the identity on $\tilde{f}^{-1}((Y - R^n) \times K)$. Since $\tilde{f} = f \circ \mathrm{proj}$ over $(R^n - \mathring{B}^n_{r-1}) \times K$, we clearly get our desired deformation of $\tilde{f}|$ to $f \circ \mathrm{proj}$ rel $\tilde{f}^{-1}(\partial \Delta \times K)$. Since $f_1 = (s \times \mathrm{id}) \circ \tilde{f} | X_1$ and $s \times \mathrm{id} \simeq \mathrm{id}$, we conclude that $f_1 \simeq f \circ \mathrm{proj}$ in $Y \times K$ as desired.

We are now ready for our main result. For notation let $Y$ be a polyhedron which is written as the union of closed subpolyhedra $Y_1$ and $Y_2$, where $Y_1 \cap Y_2$ is compact. We also assume $Y_1 - Y_2 \neq \varnothing$, $Y_2 - Y_1 \neq \varnothing$.

THEOREM 7.2. *For each open cover $\alpha$ of $Y$ there exists an open cover $\beta$ of $Y$ so that if $X$ is a polyhedron and $f\colon X \to Y \times K$ is a $p^{-1}(\beta)$-equivalence, then there is an $m \geqq 0$, a subdivision of $X \times B^m$ into closed subpolyhedra, $X \times B^m = X_1 \cup X_2$, and a proper map $f'\colon X \times B^m \to Y \times K$ such that*

(1) *$f' | X_1\colon X_1 \to Y_1 \times K$ is a $p^{-1}(\alpha)$-equivalence,*
(2) *$f' | X_2\colon X_2 \to Y_2 \times K$ is a $p^{-1}(\alpha)$-equivalence,*
(3) *$f' | X_1 \cap X_2\colon X_1 \cap X_2 \to (Y_1 \cap Y_2) \times K$ is a $p^{-1}(\alpha)$-equevalence,*
(4) *$f'$ is $p^{-1}(\alpha)$-homotopic to $f \circ \mathrm{proj}\colon X \times B^m \to Y \times K$.*

*Proof.* Let $N \subset Y$ be a compact subpolyhedron containing $Y_1 \cap Y_2$ in its interior. Consider the open set $\mathring{N} - Y_1$. If we inductively remove the interiors of simplexes in $\mathring{N} - Y_1$ in order of decreasing dimension, by repeatedly applying Lemma 7.1, we produce a closed subpolyhedron $P_1$ of $X \times B^m$ and a proper map $f_1'\colon P_1 \to (Y - (\mathring{N} - Y_1)) \times K$ such that

(1) $f_1'$ is a $p^{-1}(\alpha_1)$-equivalence (where $\alpha_1$ is fine corresponding to $\beta$ fine),

(2) $f_1'$ is $p^{-1}(\alpha_1)$-homotopic to $f \circ \mathrm{proj} | P_1$.

Let $X_1$ be the subpolyhedron $(f_1')^{-1}(Y_1 \times K)$ and note that $f_1'$ restricts to give a $p^{-1}(\alpha_1)$-equivalence $f_1\colon X_1 \to Y_1 \times K$. Using the same trick on a neighborhood of $(Y_1 \cap N) \times K$ in $Y_1 \times K$ we can produce a compact subpolyhedron $X_0$ of $X_1$ and a map $f_0\colon X_0 \to (Y_1 \cap N) \times K$ which is a $p^{-1}(\alpha_1)$-equivalence and which is $p^{-1}(\alpha_1)$-homotopic to $f\circ\text{proj}|X_0$. (For the sake of simplicity we ignore the stabilization of $X_1$ by multiplication with some $B^{m_1}$, and we assume the homotopies are all controlled by the same cover $\alpha_1$ as above.) Then we define $X_2$ to be

$$X_2 = X_0 \cup (X \times B^m - X_1)\,.$$

Since $f_0$ is $p^{-1}(\alpha_1)$-homotopic to $f\circ\text{proj}|X_0$ we can use Proposition 2.1 to find a map $\bar{f}\colon X \times B^m \to Y \times K$ such that $\bar{f}|X_0 = f_0$ and $\bar{f}$ is $p^{-1}(\alpha_1)$-homotopic to $f\circ\text{proj}$. By Proposition 2.2, $\bar{f}$ is a $p^{-1}(\alpha_1')$-equivalence, where $\alpha_1'$ is fine if $\alpha_1$ is fine.

If $\alpha_1$ is sufficiently fine, then $\bar{f}(X_1) \subset Y_1 \times K$ and $\bar{f}(X_2) \subset (Y_2 \cup N) \times K$. Moreover, by Proposition 3.4 we conclude that $\bar{f}|X_1\colon X_1 \to Y_1 \times K$ and $\bar{f}|X_2\colon X_2 \to (Y_2 \cup N) \times K$ are $p^{-1}(\alpha_1)$-equivalences (where again we use the same cover $\alpha_1$ for simplicity). If $N$ is chosen nicely, then $Y_1 \cap N$ collapses to $Y_1 \cap Y_2$. Let $c\colon Y_1 \cap N \to Y_1 \cap Y_2$ be a contractible retraction arising from this collapse. This can be set up so that $c$ extends to a contractible map of $Y_1$ to $Y_1$, and $c$ automatically extends to a contractible map of $Y_2 \cup N$ to $Y_2$. Piecing together these extensions we get an extension of $c$ to a contractible map $\tilde{c}\colon Y \to Y$, and $f' = (\tilde{c} \times \text{id})\circ\bar{f}$ is our desired map by use of Proposition 2.2 (ii). Clearly it is proper if $\alpha$ is sufficiently fine.

## 8. Proof of Theorem 1.

We now use Theorem 7.2 to establish Theorem 1. We first treat the compact case.

THEOREM 8.1. *For every compact polyhedron $Y$ there is an $\varepsilon > 0$ so that for any compact polyhedron $X$ and $p^{-1}(\varepsilon)$-equivalence $f\colon X \to Y \times K$, $f$ must be a* s.h.e.

*Proof.* We induct on the simplexes in a triangulation of $Y$ as follows. If $Y = \{\text{point}\}$, then $f$ is essentially a homotopy equivalence from $X$ to $K$ and it must therefore be a s.h.e. by the niceness condition on $\pi_1$ of each component of $K$. Passing to the inductive step write $Y = Y_1 \cup \Delta$, where $\Delta$ is an $n$-simplex which is not the face of any other simplex in $Y$ and $Y_1$ is the subpolyhedron of $Y$ which meets $\Delta$ in $\partial\Delta$. Assuming the result to be true for $Y_1$, we will prove that it is also true for $Y$. This will suffice to prove our result.

By the induction hypothesis, there exists a $\delta > 0$ such that any $p^{-1}(\delta)$-equivalence $Z \to Y_1 \times K$ is a s.h.e. By Theorem 7.2 we can choose $\varepsilon > 0$ so that if $f\colon X \to Y \times K$ is a $p^{-1}(\delta)$-equivalence then we can subdivide, $X \times B^m = X_1 \cup X_2$, and find a map $f'\colon X \times B^m \to Y \times K$ so that

(1) $f'|X_1: X_1 \to Y_1 \times K$ is a $p^{-1}(\delta)$-equivalence (where $\delta = \delta(\varepsilon)$ is small),

(2) $f'|X_2: X_2 \to \Delta \times K$ is a $p^{-1}(\delta)$-equivalence,

(3) $f'|X_1 \cap X_2: X_1 \cap X_2 \to \partial\Delta \times K$ is a $p^{-1}(\delta)$-equivalence,

(4) $f'$ is $p^{-1}(\delta)$-homotopic to $f \circ \text{proj}: X \times B^m \to Y \times K$.

By (4) all we have to do is to prove that $f'$ is a s.h.e. (recall that proj: $X \times B^m \to X$ is always s.h.e.). Using the Sum Theorem it suffices to prove that the restrictions in (1), (2), and (3) are all s.h.e.'s, The second and third are s.h.e.'s because of the niceness condition on $\pi_1$ of each component of $K$, and because $\Delta$, $\partial\Delta$ also have nice $\pi_1$'s. The first is a s.h.e. because of our inductive assumption. Thus $f'$ is a s.h.e.

*Proof of Theorem* 1. Given a polyhedron $Y$ we want to prove that there is an open cover $\alpha$ of $Y$ so that any $\alpha$-equivalence $f: X \to Y \times K$ is a s.h.e. For the first step write $Y = Y_1 \cup Y_2 \cup \cdots$, where the $Y_i$ are compact subpolyhedra such that $Y_i \cap Y_j = \emptyset$, for $|i - j| \geqq 2$. We are going to apply Theorem 7.2 an infinite number of times. Because of this, a fixed $m$ will not suffice for our subdivisions $X \times B^m = X_1 \cup X_2$. So we introduce the following notation. Let $R^\infty$ denote the direct limit $\varinjlim \{R^n\}$, where the bonding maps are the injections $R^n \overset{\times 0}{\to} R^{n+1}$. Identify $R^n$ with the subset $R^n \times \{0\}$ of $R^\infty$. Then each $X \times B^m$ becomes a subpolyhedron of $X \times R^\infty$.

Applying Theorem 7.2 there is a subdivision, $X \times B^{m_{2i}} = X_{2i} \cup X'_{2i}$, and a proper map $f_{2i}: X \times B^{m_{2i}} \to Y \times K$ so that

(1) $f_{2i}|X_{2i}: X_{2i} \to Y_{2i} \times K$ is a $p^{-1}(\alpha_{2i})$-equivalence,

(2) $f_{2i}|X'_{2i}: X'_{2i} \to (Y_1 \cup Y_2 \cup \cdots \cup Y_{2i-1} \cup Y_{2i+1} \cup \cdots) \times K$ is a $p^{-1}(\alpha_{2i})$-equivalence,

(3) $f_{2i}|X_{2i} \cap X'_{2i}: X_{2i} \cap X'_{2i} \to [(Y_{2i-1} \cup Y_{2i+1}) \cap Y_{2i}] \times K$ is a $p^{-1}(\alpha_{2i})$-equivalence,

(4) $f_{2i}$ is $p^{-1}(\alpha_{2i})$-homotopic to $f \circ \text{proj}: X \times B^{m_{2i}} \to Y \times K$.

Here $\alpha_{2i}$ is an open cover of $Y$ which can be chosen fine corresponding to a fine choice of $\alpha$.

Now choose compact subpolyhedra $L_{2i}$ of $X$ so that

(1) $X_{2i} \subset L_{2i} \times B^{m_{2i}}$,

(2) $L_{2i} \subset f^{-1}(Y_{2i-1} \cup Y_{2i} \cup Y_{2i+1})$,

(3) the $L_{2i}$ are pairwise-disjoint subpolyhedra of $X$.

Form a subpolyhedron $Z$ of $X \times R^\infty$ as follows:

$$Z = X \cup (L_2 \times B^{m_2}) \cup (L_4 \times B^{m_4}) \cup \cdots .$$

We have a natural decomposition $Z = Z_2 \cup Z_4 \cup \cdots$, where $Z_{2i} \cap$

$Z_{2i+2} = X_{2i}$ and $Z_{2i} \subset f^{-1}(Y_{2i-3} \cup Y_{2i-2} \cup \cdots \cup Y_{2i+1}) \times R^\infty$. Using Propositions 2.1 and 3.4 we can easily construct a proper map $F: Z \to Y \times K$ so that

(1) $F | X_{2i} = f_{2i}: X_{2i} \to Y_{2i} \times K$,

(2) $F | Z_{2i}: Z_{2i} \to (Y_{2i-2} \cup Y_{2i-1} \cup Y_{2i}) \times K$ is a $p^{-1}(\varepsilon_{2i})$-equivalence, where $\varepsilon_{2i}$ is small,

(3) $F$ is proper homotopic to $f \circ r$, where $r: Z \to X$ is the contractible PL retraction defined by $r(x, t) = x$.

Now $f$ factors as follows:

$$f: X \xhookrightarrow{i} Z \xrightarrow{r} X \xrightarrow{f} Y \times K .$$

Since $i$ and $r$ are s.h.e.'s (an easy consequence of the definition), it suffices to prove that $f \circ r: Z \to Y \times K$ is a s.h.e. By (3) above we only need to prove that $F$ is a s.h.e.

To see that $F$ is a s.h.e. we decompose $Z$ and $Y \times K$ as follows:

$$Z = Z_2 \cup Z_4 \cup Z_6 \cup \cdots ,$$

$$Y \times K = [(Y_1 \cup Y_2) \cup (Y_2 \cup Y_3 \cup Y_4) \cup (Y_4 \cup Y_5 \cup Y_6) \cup \cdots] \times K .$$

Then $F | Z_{2i}: Z_{2i} \to (Y_{2i-2} \cup Y_{2i-1} \cup Y_{2i}) \times K$ is a s.h.e. by Theorem 8.1 and $F | Z_{2i} \cap Z_{2i+2}: Z_{2i} \cap Z_{2i+2} \to Y_{2i} \times K$ is also a s.h.e. by Theorem 8.1. By the Sum Theorem [20, p. 482] we conclude that $F$ is a s.h.e.

9. **Proof of Theorem 2.** We are given a proper map $f: X \to Y$ such that $f \times \mathrm{id}: X \times Q \to Y \times Q$ is proper homotopic to a homeomorphism $h: X \times Q \to Y \times Q$. We want to prove that $f$ is a s.h.e. We first treat the compact case.

Represent $Q$ by the product $\prod_{i=1}^\infty I_i$, where $I_i = [0, 1]$, and identify $I^n = I_1 \times \cdots \times I_n$ with $I^n \times \{(0, 0, \cdots)\}$ in $Q$. Consider the map $f': X \to Y$ defined by

$$f': X \hookrightarrow X \times Q \xrightarrow{h} Y \times Q \xrightarrow{\text{proj}} Y ,$$

which is certainly homotopic to $f$. So it suffices to prove that $f'$ is a s.h.e.

Let $i$ be the inclusion $X \hookrightarrow X \times I^n$ and define $u: X \times I^n \to Y$ by

$$u: X \times I^n \hookrightarrow X \times Q \xrightarrow{h} Y \times Q \xrightarrow{p} Y ,$$

where $p = \text{proj}$. Then $f' = u \circ i$, and since $i$ is clearly a s.h.e. all we have to do is prove that $u$ is a s.h.e. For this we use Theorem 1 by showing if $\alpha$ is the open cover of $Y$ which comes from Theorem 1, then $n$ can be chosen large enough so that $u$ is an $\alpha$-equivalence.

Define $v: Y \to X \times I^n$ by

$$v: Y \hookrightarrow Y \times Q \xrightarrow{h^{-1}} X \times Q \xrightarrow{q} X \times I^n ,$$

where $q = \text{proj}$. To see that $u \circ v$ is $\alpha$-homotopic to id we first note that $u \circ v = p \circ h \circ q \circ h^{-1} | Y$. Using a homotopy $q \simeq \text{id}$ which affects only the $Q$-factor we get a homotopy $u \circ v \simeq p \circ h \circ h^{-1} | Y = \text{id}$, which must be an $\alpha$-homotopy for $n$ large. To see that $v \circ u$ is $u^{-1}(\alpha)$-homotopic to id we have $v \circ u = q \circ h^{-1} \circ p \circ h | X \times I^n$. Using a homotopy of $p$ to id which affects only the $Q$-factor we have $v \circ u \simeq q \circ h^{-1} \circ h | X \times I^n = \text{id}$. This is certainly a $u^{-1}(\alpha)$-homotopy for $n$ large. Thus $u$ is an $\alpha$-equivalence and this completes the proof of the compact case.

For the noncompact case let $w_i: X \to [0, 1]$, $i \geqq 1$, be a sequence of PL maps such that for each $x$, $w_i(x) = 0$ for $i$ sufficiently large. Define

$$\tilde{X} = \cup \{\{x\} \times \prod_{i=1}^{\infty} [0, w_i(x)] \mid x \in X\} \subset X \times Q .$$

If this is done properly, then $\tilde{X}$ is a polyhedron which contains $X$ as a subpolyhedron. Moreover, $\tilde{X}$ collapses to $X$, thus $X \hookrightarrow \tilde{X}$ is a s.h.e. For each $t \in I$ let $r_t: I \to [0, t]$ be the retraction which sends $[t, 1]$ to $\{t\}$. Define $q: X \times Q \to \tilde{X}$ by

$$q(x, (t_i)) = (x, r_{w_i(x)}(t_i)) .$$

Then $q$ is a retraction which is proper homotopic to id, with a homotopy which affects only the $Q$-factor. This implies that we may repeat the proof of the compact case above by replacing $X \times I^n$ with $\tilde{X}$. If the $w_i$ are chosen properly, then $u$ is still an $\alpha$-equivalence.

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