PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

A CHARACTERIZATION OF COVERING DIMENSION BY USE OF $\Delta_k(X)$

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Covering dimension, in the sense of Katětov, of a topological space X is characterized by use of $\Delta_k(X)$ which will be defined in the main discussion in terms of cardinalities of finite open covers of X.

1. Introduction. L. Pontrjagin and L. Schnirelmann [6] characterized dimension of a compact metrizable space X by use of the numbers $N_\rho(\varepsilon, X) = \min\{m \in N \mid$ the metric space (X, ρ) has a cover $\mathscr{U}$ such that $|\mathscr{U}| = m$ and $\operatorname{diam} U \leqq \varepsilon$ for every $U \in \mathscr{U}\}$. Their result is quite interesting in the sense that covering dimension, which is defined in terms of order (a kind of local cardinality) of a cover, is characterized in terms of global cardinality of a cover. J. Bruijning [1] generalized Pontrjagin-Schnirelmanns theorem to separable metric spaces by use of totally bounded metrics and to topological spaces by means of totally bounded pseudometrics.

In the present paper, we shall characterize covering dimension of topological spaces by use of a new function $\Delta_k(X)$, which will be defined later. It seems that $\Delta_k(X)$ can provide us with a neater characterization of dimension, perhaps because it does not involve the metric ρ in its definition while $N_\rho(\varepsilon, X)$ does.

2. Conventions. In the following discussions we frequently consider a finite collection $\mathscr{U} = \{U_1, \cdots, U_i\}$ of subsets of a space X such that $\bigcup\{U_j \mid 1 \leqq j \leqq i\} \supset A$ for a certain subset A of X, and a cover $\mathscr{V} = \{V_1, \cdots, V_i\}$ of A such that $V_j \subset U_j \cap A$ for $1 \leqq j \leqq i$. Then we may say: *$\mathscr{U}$ is shrunk to $\mathscr{V}$ on A.*

If $\mathscr{V}$ consists of open (closed) subsets of A, we *shrink $\mathscr{U}$ to the open (closed) cover $\mathscr{V}$ of A.* If $A = X$, we may drop the words "on A". If $\cap\mathscr{V} = \varnothing$, $\mathscr{V}$ is *vanishing*.

We shall denote by C_m^k the set of all m-element subsets of the set $\{1, 2, \cdots, k\}$ and by $\binom{k}{m}$ its cardinality, i.e., $\binom{k}{m} = k!/m!(k-m)!$.

By the *dimension* of a space X, $\dim X$, we will mean its Katětov dimension, i.e.,

$$\dim X = -1 \qquad \text{iff} \quad X = \varnothing\,;$$
$$\dim X \leqq n (n \geqq 0) \quad \text{iff every finite cover of } X\,,$$

consisting of functionally open sets, has a finite refinement, also consisting of functionally open sets and with order $\leqq n + 1$;

$\dim X = n (n \geqq 0)$ iff $\dim X \leqq n$ but not $\dim X \leqq n-1$;
$\dim X = \infty$ iff not $\dim X \leqq n$, for every n.

We will sometimes use the following, without explicitly mentioning it: for normal spaces, Katětov dimension coincides with ordinary covering dimension [3, p. 268].

For basic concepts in general topology and dimension theory see [3], [4], and [5]. The reader is warned that different authors sometimes mean different numbers by the order of a cover; in our definition, the order is the maximum number of mutually intersecting sets in the cover, but in Engelking [3] the order is defined to be one less. Since we will frequently be referring to [3] the reader should be aware of this.

3. The main theorem: the normal case. Let X be a topological space and k a positive integer. Define

$\Delta_k(X) = \min\{m \in N \mid$ for every functionally open cover $\mathscr{U}$ of X with $|\mathscr{U}| \leqq k$ there exists a functionally open cover $\mathscr{V}$ of X with $|\mathscr{V}| \leqq m$ and $\mathscr{V}^{\Delta} < \mathscr{U}\}$.

Here $\mathscr{V}^{\Delta} = \{\mathrm{St}(x, \mathscr{V}) \mid x \in X\}$ and $<$ means: "refines".

REMARK. If X is a normal space, we may drop the word "functionally" in the above definition either or both times it occurs and still arrive at the same number. This is easily proved using similar techniques as in the proof that Katětov dimension coincides with ordinary covering dimension referred to above. We will use this observation in the sequel without explicitly mentioning it.

PROPOSITION 1. *Let $n \geqq 0$, and X be a topological space with $\dim X \leqq n$. Let $k \in N$. Then*

$$\Delta_k(X) \leqq 2^k - 1 \qquad \text{if} \quad k \leqq n+1$$

$$\Delta_k(X) \leqq \binom{k}{1} + \cdots + \binom{k}{n+1} \qquad \text{if} \quad k \geqq n+1 .$$

Proof. Suppose $\mathscr{U} = \{U_1, \cdots, U_k\}$ is a functionally open cover of X. Since $\dim X \leqq n$, we can shrink $\mathscr{U}$ to a functionally open cover $\mathscr{V} = \{V_1, \cdots, V_k\}$ with $\operatorname{ord} \mathscr{V} \leqq n+1$. Further shrink $\mathscr{V}$ to a functionally closed cover $\mathscr{F} = \{F_1, \cdots, F_k\}$ (as in [3, p. 267]). For every nonempty $A \subset \{1, \cdots, k\}$ we define the following functionally open set:

$$W(A) = [\bigcap\{V_i \mid i \in A\}] \cap [\bigcap\{X \backslash F_i \mid i \notin A\}] .$$

Let $\mathscr{W} = \{W(A) \mid W(A) \neq \varnothing\}$. Since $\operatorname{ord} \mathscr{V} \leqq n+1$, $W(A) = \varnothing$ if $|A| >$

$n+1$. Therefore $|\mathscr{W}| \leqq 2^k - 1$ if $k \leqq n+1$, and $|\mathscr{W}| \leqq \binom{k}{1} + \cdots + \binom{k}{n+1}$ if $k \geqq n+1$. It is also easy to see that $\mathscr{W}^{\Delta} < \mathscr{V} < \mathscr{U}$, because for each $x \in F_i$, $\mathrm{St}(x, \mathscr{W}) \subset V_i$. This proves Proposition 1.

LEMMA 1. *Let X be a normal space and $n \geqq 0$. Then* $\dim X \leqq n$ *iff every open cover $\{W_1, \cdots, W_{n+2}\}$ of X can be shrunk to a vanishing open cover of X.*

Proof. See Engelking [3, p. 282].

LEMMA 2. *Let X be a normal space with* $\dim X \geqq n$, *where either $n \geqq 1$ or $n = 0$ and X infinite.*

Let $k \in N$. Then there exist k disjoint closed subsets of X with dimension $\geqq n$.

Proof. The proof is by techniques similar to those of C. H. Dowker [2] who proved related results.

If $n = 0$ we use the fact that X is infinite to prove our result. If $n > 0$ the result will follow from: there exist two disjoint closed subsets of X with dimension $\geqq n$. Let $\mathscr{U} = \{U_1, \cdots, U_i\}$ be an open cover which has no open shrinking of order $\leqq n$ and has no proper subcover. Since $n > 0$, $i \geqq 2$. Let $\mathscr{F} = \{F_1, \cdots, F_i\}$ be a closed shrinking of $\mathscr{U}$. From [3, p. 276] it follows that some element of $\mathscr{F}$, say F_1, has dimension $\geqq n$. Let V be an open set such that $F_1 \subset V \subset \bar{V} \subset U_1$. We claim: $\dim(X \backslash V) \geqq n$. Indeed, the collection $\{U_1, U_2 \backslash \bar{V}, \cdots, U_i \backslash \bar{V}\}$ is an open cover of $X \backslash V$. If $\dim(X \backslash V) < n$, by standard methods one can prove the existence of a collection $\mathscr{W} = \{W_1, \cdots, W_i\}$ of open subsets of X such that $W_1 \subset U_1$, $W_2 \subset U_2 \backslash \bar{V}, \cdots$, $W_i \subset U_i \backslash \bar{V}$, $\mathrm{ord}\, \mathscr{W} \leqq n$ and $X \backslash V \subset \bigcup \mathscr{W}$. Then define $O_1 = V \cup W_1$, $O_2 = W_2, \cdots, O_i = W_i$ to get an open cover $\{O_1, \cdots, O_i\}$ of X with order $\leqq n$ and shrinking $\mathscr{U}$, contradicting our initial assumption. Thus $\dim F_1 \geqq n$, $\dim X \backslash V \geqq n$, and $F_1 \cap (X \backslash V) = \varnothing$. This proves Lemma 2.

PROPOSITION 2. *Let X be a normal space with* $\dim X \geqq n$ *and let either $n \geqq 1$ or $n = 0$ and X infinite. Let $k \in N$. Then*

$$\Delta_k(X) \geqq 2^k - 1 \qquad \text{if} \quad k \leqq n+1$$

$$\Delta_k(X) \geqq \binom{k}{1} + \cdots + \binom{k}{n+1} \qquad \text{if} \quad k \geqq n+1 .$$

Proof. We will only prove the proposition for the case $k \geqq n+1$, since the case $k \leqq n+1$ then follows by substituting $k-1$ for n. So, let $k \geqq n+1$. Let $\{C(\alpha) \mid \alpha \in C^k_{n+1}\}$ be a collection of disjoint closed subsets of X of dimension $\geqq n$ (Lemma 2). For each $\alpha \in C^k_{n+1}$ we can find, by Lemma 1, an open cover $\mathscr{U}(\alpha) = \{U^\alpha_i \mid i \in \alpha\}$ of $C(\alpha)$ which cannot be shrunk to a vanishing open cover of $C(\alpha)$. Note that $|\mathscr{U}(\alpha)| = n + 1$. Note also that $\mathscr{U}(\alpha)$ cannot be shrunk to a vanishing closed cover of $C(\alpha)$ either, since such a cover could, by using normality of $C(\alpha)$, be expanded to a vanishing open cover of $C(\alpha)$ still refining $\mathscr{U}(\alpha)$. Now, define open subsets U_i $(1 \leqq i \leqq k)$ of X as follows:

$$U_i = [X \setminus \bigcup \{C(\alpha) \mid \alpha \in C^k_{n+1}] \cup [\bigcup \{U^\alpha_i \mid i \in \alpha\}]$$

(note that α, not i, is the free variable in the right hand formula). Then $\mathscr{U} = \{U_i \mid 1 \leqq i \leqq k\}$ is an open cover of X. Suppose $\mathscr{V}^\Delta < \mathscr{U}$ for a finite open cover $\mathscr{V}$. We claim: $|\mathscr{V}| \geqq \binom{k}{1} + \cdots + \binom{k}{n+1}$, which implies $\Delta_k(X) \geqq \binom{k}{1} + \cdots + \binom{k}{n+1}$. We will show this in the following way: let $\beta \subset \{1, \cdots, k\}$ be chosen so that $1 \leqq |\beta| \leqq n + 1$. We will prove the existence of an element $V(\beta) \in \mathscr{V}$ such that

$$\beta = \{j \in \{1, \cdots, k\} \mid V(\beta) \subset U_j\} .$$

In this way we can assign in a one-to-one manner an element $V(\beta) \in \mathscr{V}$ to every β. Since there are $\binom{k}{1} + \cdots + \binom{k}{n+1}$ β's, this gives us $|\mathscr{V}| \geqq \binom{k}{1} + \cdots + \binom{k}{n+1}$. So, let $\beta \subset \{1, \cdots, k\}$ be subject to the condition $1 \leqq |\beta| \leqq n + 1$, and fixed. Let $\gamma \subset \{1, \cdots, k\}$ be so that $\beta \cap \gamma = \emptyset$ and $\beta \cup \gamma \in C^k_{n+1}$. We will write $\alpha = \beta \cup \gamma$. Put $K = C(\alpha) \setminus \bigcup \{U^\alpha_i \mid i \in \gamma\}$. Observe that $\{U^\alpha_j \mid j \in \beta\}$ is a collection of open subsets of $C(\alpha)$ which covers K and cannot be shrunk to vanishing closed cover $\{K_j \mid j \in \beta\}$ on K: namely, suppose it could. For each $j \in \beta$, the set K_j could, by normality of $C(\alpha)$, be expanded to an open set H_j of $C(\alpha)$ such that $K_j \subset H_j \subset U^\alpha_j$ in such a way that $\bigcap \{H_j \mid j \in \beta\} = \emptyset$. Thus $\mathscr{U}(\alpha) = \{U^\alpha_i \mid i \in \alpha\}$ could be shrunk to the vanishing open cover $\{H_j \mid j \in \beta\} \cup \{U^\alpha_i \mid i \in \gamma\}$ of $C(\alpha)$, which is a contradiction. Now define closed sets G_j, $j \in \beta$, as follows.

For any $j \in \beta$, put $G_j = \{x \in K \mid \mathrm{St}\,(x, \mathscr{V}) \subset U_j\}$. Then it is easy to see that $G_j \subset U^\alpha_j$ and that $\bigcup \{G_j \mid j \in \beta\} = K$ (recall that $\mathscr{V}^\Delta < \mathscr{U}$). Hence $\bigcap \{G_j \mid j \in \beta\} \neq \emptyset$. Let $x \in \bigcap \{G_j \mid j \in \beta\}$. Let $V(\beta)$ be an element of $\mathscr{V}$ containing x. Since $x \in G_j$ for $j \in \beta$, $V(\beta) \subset U_j$ for $j \in \beta$. Since $x \notin U_i$ for $i \notin \beta$, $V(\beta) \not\subset U_i$ for $i \notin \beta$. Thus $\beta = \{j \in \{1, \cdots, k\} \mid V(\beta) \subset U_j\}$. As noted above, this suffices to prove the proposition.

Combining Propositions 1 and 2, we have the following

COROLLARY. *Let X be an infinite normal space, with* $\dim X =$

n, $0 \leqq n \leqq \infty$, *and let* k *be a natural number. Then*

$$\Delta_k(X) = 2^k - 1 \qquad \text{if} \quad k \leqq n+1$$

$$\Delta_k(X) = \binom{k}{1} + \cdots + \binom{k}{n+1} \qquad \text{if} \quad k \geqq n+1\,.$$

REMARK. This corollary is nothing but a special case of our main theorem. The first equality holds for finite - as well for infinite dimensional X.

Proof. If X is finite dimensional, this is a combination of Propositions 1 and 2. If $\dim X = \infty$, Proposition 2 gives us $\Delta_k(X) \geqq 2^k - 1$, thus we only have to prove $\Delta_k(X) \leqq 2^k - 1$. To this end, let $\mathscr{O} = \{O_1, \cdots, O_k\}$ be an open cover. Obviously, ord $\mathscr{O} \leqq k$. Now take this $\mathscr{O}$ and substitute it for $\mathscr{V}$ in the proof of Proposition 1. Since in this proof the fact $\dim X \leqq n$ is used only to find this $\mathscr{V}$ with ord $\mathscr{V} \leqq n+1$, everything still works and we find a cover $\mathscr{W}$ with $|\mathscr{W}| \leqq 2^k - 1$ and $\mathscr{W}^\Delta < \mathscr{O}$. This proves our corollary.

4. The main theorem: the completely regular case. In this section we will extend the above result to the class of completely regular spaces. Let, for X in this class, βX be its Čech-Stone compactification.

LEMMA 3. *Let* X *be a Tychonoff space. Then* $\dim X = \dim \beta X$.

Proof. This is well-known and, in fact, may be chosen to be the definition of $\dim X$. See e.g., [3, p. 272].

PROPOSITION 3. *Let* X *be a Tychonoff space, and* $k \geqq 1$. *Then* $\Delta_k(X) = \Delta_k(\beta X)$.

Proof. Proposition 1, applied to X, and Proposition 2, applied to βX, together with Lemma 3 yield $\Delta_k(X) \leqq \Delta_k(\beta X)$. To prove the converse, let $k \in N$ and $\mathscr{U} = \{U_1, \cdots, U_k\}$ be a functionally open cover of βX such that for every (functionally) open cover $\mathscr{V}$ of βX with $\mathscr{V}^\Delta < \mathscr{U}$ the relation $|\mathscr{V}| \geqq \Delta_k(\beta X)$ holds true. Shrink $\mathscr{U}$ to a functionally open cover $\mathscr{U}' = \{U_1', \cdots, U_k'\}$ with $\bar{U}_i' \subset U_i (1 \leqq i \leqq k)$. Define $\mathscr{W} = \{U_1' \cap X, \cdots, U_k' \cap X\}$. Then $\mathscr{W}$ is a functionally open cover of X. Suppose $\mathscr{W}'$ is a functionally open cover of X with $\mathscr{W}'^\Delta < \mathscr{W}$. We will prove: $|\mathscr{W}'| \geqq \Delta_k(\beta X)$.

Let Ex be theop erator which assigns to every open subset O of X the largest open subset, Ex O, of βX with the property that $\text{Ex}\, O \cap X = O$. In [3, p. 269-270] it is proved that $\text{Ex}\,(O_1 \cap O_2) =$

$\operatorname{Ex} O_1 \cap \operatorname{Ex} O_2$ for open sets O_1 and O_2, and that $\operatorname{Ex}(O_1 \cup O_2) = \operatorname{Ex} O_1 \cup \operatorname{Ex} O_2$ whenever O_1 and O_2 are functionally open. Furthermore, it is easily seen that $\operatorname{Ex} O \subset \bar{O}$ (closure taken in βX). Write $\mathscr{W}' = \{W'_1, \cdots, W'_l\}$ and define $\mathscr{V} = \{\operatorname{Ex} W'_1, \cdots, \operatorname{Ex} W'_l\}$. Then $\bigcup \mathscr{V} = \operatorname{Ex} W'_1 \cup \cdots \cup \operatorname{Ex} W'_1 = \operatorname{Ex}(W'_1 \cup \cdots \cup W'_l) = \operatorname{Ex} X = \beta X$, thus $\mathscr{V}$ is an open cover of βX. Let $p \in \beta X$ and consider all elements of $\mathscr{V}$ that contain p.

Let us say that these are $\operatorname{Ex} W'_1, \cdots, \operatorname{Ex} W'_m$ $(m \leqq l)$. Apparently, $\emptyset \neq \operatorname{Ex} W'_1 \cap \cdots \cap \operatorname{Ex} W'_m = \operatorname{Ex}(W'_1 \cap \cdots \cap W'_m)$, which implies $W'_1 \cap \cdots \cap W'_m \neq \emptyset$. Let $q \in W'_1 \cap \cdots \cap W'_m$. Since $\mathscr{W}'^{\Delta} < \mathscr{W}$, $W'_1 \cup \cdots \cup W'_m \subset \operatorname{St}(q, \mathscr{W}') \subset U'_i \cap X$ for some i, $1 \leqq i \leqq k$. Thus $\operatorname{St}(p, \mathscr{V}) = \operatorname{Ex} W'_1 \cup \cdots \cup \operatorname{Ex} W'_m = \operatorname{Ex}(W'_1 \cup \cdots \cup W'_m) \subset \operatorname{Ex}(U'_i \cap X) \subset \overline{U'_i \cap X} \subset \bar{U}'_i \subset U_i \in \mathscr{U}$. Therefore $\mathscr{V}^{\Delta} < \mathscr{U}$, and by the choice of $\mathscr{U}$ and the fact that $|\mathscr{V}| = |\mathscr{W}'| = l$ we conclude $l \geqq \Delta_k(\beta X)$. This proves $\Delta_k(X) \geqq \Delta_k(\beta X)$. Since we already had $\Delta_k(X) \leqq \Delta_k(\beta X)$, the proposition is proved.

COROLLARY. *Let X be an infinite Tychonoff space, and let k be a natural number. Then, if* $\dim X = n$, $0 \leqq n \leqq \infty$,

$$\Delta_k(X) = 2^k - 1 \qquad \text{if} \quad k \leqq n + 1$$

$$\Delta_k(X) = \binom{k}{1} + \cdots + \binom{k}{n+1} \qquad \text{if} \quad k \geqq n + 1 .$$

Proof. This follows immediately from Lemma 3, Proposition 3, and the corollary in the preceding section.

5. The main theorem: the general case. Let X be a topological space. Define a completely regular space $\tilde{X}$ and a continuous mapping $\phi: X \to \tilde{X}$ as follows: if $\mathscr{F} = \{f \mid f: X \to [0, 1],\ f \text{ is continuous}\}$, let $\phi: X \to \prod_{f \in \mathscr{F}} I_f$ be defined by $\phi(x) = (f(x))_{f \in \mathscr{F}}$. (Here $I_f = [0, 1]$ for $f \in \mathscr{F}$.) Define $\tilde{X} = \phi(X)$.

REMARK. The functor which associates $\tilde{X}$ with X was used in dimension theory by K. Morita [7] under the name of Tychnoff functor.

LEMMA 4. (i) *If $U \subset X$ is functionally open, so is $\phi(U) \subset \tilde{X}$;*
(ii) *If $U \subset X$ is functionally open, then $U = \phi^{-1}(\phi(U))$;*
(iii) *If O_1 and $O_2 \subset X$ are functionally open, then $\phi(O_1 \cap O_2) = \phi(O_1) \cap \phi(O_2)$.*

Proof. (i) It suffices to observe, that if $U = f^{-1}((0, 1])$, $\phi(U) = \pi_f^{-1}((0, 1])$, where $\pi_f: X \to I_f$ is projection; (ii) if $U = f^{-1}((0, 1])$, and $x \notin U$, then $f(x) = 0$, thus $x \notin \phi^{-1}(\phi(U))$, (iii) always $\phi(O_1 \cap O_2) \subset \phi(O_1) \cap$

$\phi(O_2)$. Let $p \in \phi(O_1) \cap \phi(O_2)$. Then there are $x \in O_1$, $y \in O_2$ with $\phi(x) = \phi(y) = p$. Thus $y \in \phi^{-1}(\phi(O_1)) = O_1$.

Therefore $p = \phi(y) \in \phi(O_1 \cap O_2)$.

PROPOSITION 4. (i) (K. Morita [7]) $\dim X = \dim \tilde{X}$;
(ii) *$\Delta_k(X) = \Delta_k(\tilde{X})$ for all $k \in N$.*

Proof. (i) Let $\dim X \leqq n$, and let $\mathscr{U} = \{U_1, \cdots, U_m\}$ be a functionally open cover of $\tilde{X}$. Obviously, $\mathscr{U}' = \{\phi^{-1}(U_1), \cdots, \phi^{-1}(U_m)\}$ is a functionally open cover of X. Let $\mathscr{V}' = \{V_1', \cdots, V_l'\}$ be a functionally open refinement of $\mathscr{U}'$ such that $\operatorname{ord} \mathscr{V}' \leqq n+1$ and define $\mathscr{V} = \{\phi(V_1'), \cdots, \phi(V_l')\}$. By (i) of the preceding lemma, $\mathscr{V}$ is a functionally open cover of $\tilde{X}$ and $\mathscr{V} < \mathscr{U}$; from (iii) of the same lemma it follows that $\operatorname{ord} \mathscr{V} = \operatorname{ord} \mathscr{V}' \leqq n+1$. This proves $\dim X \leqq n$. Now let $\dim \tilde{X} \leqq n$, and let $\mathscr{U} = \{U_1, \cdots, U_m\}$ be a functionally open cover of X. Then $\mathscr{U}' = \{\phi(U_1), \cdots, \phi(U_m)\}$ is a functionally open cover of $\tilde{X}$, and, consequently, has a functionally open refinement $\mathscr{V}' = \{V_1', \cdots, V_l'\}$ with $\operatorname{ord} \mathscr{V}' \leqq n+1$. Now $\mathscr{V} = \{\phi^{-1}(V_1'), \cdots, \phi^{-1}(V_l')\}$ is a functionally open cover of X with $\operatorname{ord} \mathscr{V} \leqq n+1$. Take an element of $\mathscr{V}$, e.g., $\phi^{-1}(V_1')$. There is some i, $1 \leqq i \leqq m$, such that $V_1' \subset \phi(U_i)$. But then $\phi^{-1}(V_1') \subset \phi^{-1}(\phi(U_i)) = U_i$, by (ii) of the proceding lemma. Thus $\mathscr{V} < \mathscr{U}$, which completes the proof of the first part of Proposition 4.

(ii) From (i), Proposition 1 and the corollary of §4 it follows that $\Delta_k(X) \leqq \Delta_k(\tilde{X})$. To prove the converse, let $\mathscr{U} = \{U_1, \cdots, U_k\}$ be a functionally open cover of $\tilde{X}$. Define $\mathscr{U}' = \{\phi^{-1}(U_1), \cdots, \phi^{-1}(U_k)\}$. Then there exists a functionally open cover $\mathscr{V}' = \{V_1', \cdots, V_m'\}$ of X with $\mathscr{V}'^{\Delta} < \mathscr{U}'$ and $m = |\mathscr{V}'| \leqq \Delta_k(X)$. Put $\mathscr{V} = \{\phi(V) \mid V \in \mathscr{V}'\}$. Then $\mathscr{V}$ is a functionally open cover of $\tilde{X}$. Let $p \in \tilde{X}$, and consider the elements of $\mathscr{V}$ which contain p. Let us say that these are $\phi(V_1')$, $\cdots$, $\phi(V_l')$. Since $\phi(V_1') \cap \cdots \cap \phi(V_l') \neq \varnothing$, we infer, by (iii) of Lemma 5, $V_1' \cap \cdots \cap V_l' \neq \varnothing$. But $\mathscr{V}'^{\Delta} < \mathscr{U}'$, so there exists $U_i' \in \mathscr{U}_i'$ with $V_1' \cup \cdots \cup V_l' \subset U_i'$. Therefore $\phi(V_1') \cup \cdots \cup \phi(V_l') = \phi(V_1' \cup \cdots \cup V_l') \subset \phi(U_i') = U_i$. Thus $\mathscr{V}^{\Delta} < \mathscr{U}$, and $|\mathscr{V}| \leqq |\mathscr{V}'| = m \leqq \Delta_k(X)$. This proves $\Delta_k(\tilde{X}) \leqq \Delta_k(X)$, and completes the proof of Proposition 4.

Now we will state and prove our

MAIN THEOREM. *Let X be a topological space, such that either $1 \leqq \dim X \leqq \infty$ or $\dim X = 0$ and $\tilde{X}$ is infinite. Let $\dim X = n$, and let k be a natural number. Then*

$$\Delta_k(X) = 2^k - 1 \qquad \text{if} \quad k \leqq n+1$$

$$\Delta_k(X) = \binom{k}{1} + \cdots + \binom{k}{n+1} \qquad \text{if} \quad k \geqq n+1\,.$$

Proof. Note that the conditions of the theorem imply that $\tilde{X}$ is always infinite: for if $\dim X > 0$, so is $\dim \tilde{X}$ by Proposition 4, and a Tychonoff space with positive dimension is always infinite. So we may apply the corollary of §4 to $\tilde{X}$, and this, together with Proposition 4, proves our result.

COROLLARY. *If X satisfies the conditions of the main theorem,*

$$\dim X = \lim_{k\to\infty} \frac{\log \Delta_k(X)}{\log k} - 1 .$$

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Received March 3, 1978.

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