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EXAMPLES OF SOLVABLE AND NON-SOLVABLE CONVOLUTION EQUATIONS IN $\mathscr{K}'_p$, $p \geqq 1$

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For $p \in [1, +\infty)$ let $\mathscr{K}'_p$ be the space of distributions on R^n not growing faster than some power of $\exp(|\cdot|^p)$, and let $\mathscr{K}'_\infty$ be the space of distributions on R^n of finite order. For every $p \in (1, +\infty]$ the existence of convolutors f is proved such that $f * \mathscr{K}'_p = \mathscr{K}'_p$ but $f * \mathscr{K}'_s \neq \mathscr{K}'_s$ for every $s < p$. The main step in the proof is a construction of slowly decreasing entire functions which satisfy suitable estimates of Paley-Wiener type and which have countably many zeros of orders as high as possible.

1. **Introduction.** Let $\mathscr{D}'_F$ (resp. $\mathscr{E}'$) be the space of Schwartz distributions on R^n of finite order (resp. of compact support). $\mathscr{E}'$ is the space of convolution operators on $\mathscr{D}'_F$. Recall that a function $F: C^n \to C$ is said to be *very slowly decreasing* if it satisfies an inequality of the form

$$(1) \qquad \sup\{|F(x+w)|;\, w \in C^n,\, |w| \leqq r(x)\} \geqq \text{const} \exp(-N\omega(x)), \quad x \in R^n,$$

where $r: R^n \to R_+$ is a function satisfying

$$(2) \qquad r(x) = o(\omega(x)) \quad \text{as} \quad x \longrightarrow \infty$$

and where ω equals $\log(1 + |\cdot|)$ and N is a constant. The following theorem is well-known.

THEOREM 1 (*Ehrenpreis* [3], *Hörmander* [6]). *For every $f \in \mathscr{E}'$ the following conditions are equivalent.*

(i) $f * \mathscr{D}'_F = \mathscr{D}'_F$.

(ii) *$f *$ has a fundamental solution in $\mathscr{D}'_F$.*

(iii) *The Fourier transform $\hat{f}$ of f is very slowly decreasing.*

Let $\mathscr{K}'_p$, $p \geqq 1$, be the space of distributions on R^n which do not grow faster than some power of $\exp(|\cdot|^p)$, and let $\mathscr{O}'_C(\mathscr{K}'_p)$ be the space of distributions which decrease faster than any negative power of $\exp(|\cdot|^p)$ (for the precise definitions see [8, 9]). $\mathscr{O}'_C(\mathscr{K}'_p)$ is the space of convolution operators on $\mathscr{K}'_p$. A function $F: C^n \to C$ is said to be *q-slowly decreasing*, $q \in (1, +\infty]$, if it satisfies (1) with

$$(3) \qquad r = A\omega^{1/q} + B$$

where A, B, N are constants and ω is equal to $\log(1 + |\cdot|)$; a ∞-

slowly decreasing function is also called *extremely slowly decreasing*. Recently Sznajder and Zielezny proved an analogue of Theorem 1 for the spaces $\mathscr{K}'_p$; for the case $p = 1$ see also Grudzinski [5]:

THEOREM 2 (*Sznajder and Zielezny* [8, 9]). *For arbitrary* $p \in [1, +\infty)$ *and* $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ *the following conditions are equivalent.*

(i) $f * \mathscr{K}'_p = \mathscr{K}'_p$.

(ii) $f *$ *has a fundamental solution in* $\mathscr{K}'_p$.

(iii) $\hat{f}$ *is q-slowly decreasing where* $1/p + 1/q = 1$.

In the present note we are concerned with the relations between the various types of slowly decreasing functions defined above. Observe that we have the following inclusions:

$$\mathscr{K}'_p \subset \mathscr{K}'_s \subset \mathscr{D}'_F \quad \text{for} \quad 1 \leqq p \leqq s < +\infty$$

and

$$\mathscr{O}'_C(\mathscr{K}'_p) \supset \mathscr{O}'_C(\mathscr{K}'_s) \supset \mathscr{E}' \quad \text{for} \quad 1 \leqq p \leqq s < +\infty \; .$$

For a few moments let us denote $\mathscr{D}'_F$ and $\mathscr{E}'$ by $\mathscr{K}'_\infty$ and $\mathscr{O}'_C(\mathscr{K}'_\infty)$. Let $p, s \in [1, +\infty]$ and $f \in \mathscr{O}'_C(\mathscr{K}'_r)$ where $r := \max\{p, s\}$. Since trivially a q-slowly decreasing function is q_1-slowly decreasing for every $q_1 < q$ and also very slowly decreasing, Theorems 1 and 2 show: *if* $s > p$ *then* $f * \mathscr{K}'_p = \mathscr{K}'_p$ *implies* $f * \mathscr{K}'_s = \mathscr{K}'_s$. Now, the main question we are dealing with in this note is whether or not this assertion remains valid if $s < p$. Under additional assumptions on f the answer is positive as is shown by the following result to be proved below.

THEOREM 3. *Let* $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ *such that* $\hat{f}$ *is q-slowly decreasing where* $1/p + 1/q = 1$. *If* f *belongs to* $\mathscr{O}'_C(\mathscr{K}'_t)$ *for some* $t > p$ *then* $\hat{f}$ *is extremely slowly decreasing.*

Combined with Theorem 2 this leads to

COROLLARY 1. *Let* $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ *such that* $f * \mathscr{K}'_p = \mathscr{K}'_p$. *If* f *belongs to* $\mathscr{O}'_C(\mathscr{K}'_t)$ *for some* $t > p$ *then* $f * \mathscr{K}'_s = \mathscr{K}'_s$ *for every* $s \in [1, p)$.

In general, however, the answer to our question is negative. For the space $\mathscr{D}'_F$ this was established by Malliavin who —according to Hörmander [6, Remark on p. 168]—proved the following unpublished result.

THEOREM 4. *There exist distributions $f \in \mathscr{E}'$ such that $\hat{f}$ is very but not extremely slowly decreasing.*

Combining this with Theorems 2 and 3 we obtain

COROLLARY 2. *There exist distributions $f \in \mathscr{E}'$ such that $f * \mathscr{D}'_F = \mathscr{D}'_F$ but $f * \mathscr{K}'_p \neq \mathscr{K}'_p$ for every $p \in [1, +\infty)$.*

According to a personal communication by Prof. Ehrenpreis a proof of Theorem 4 can be obtained by modifying the construction in [4, §2]. Below we give a proof which is based on the slightly simpler method of [4, §4]. It is the same method which we use to establish the negative answer to our question for the spaces $\mathscr{K}'_p$, $p < +\infty$, namely we shall prove

THEOREM 5. *For every $p \in (1, +\infty)$ there exist distributions $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ such that $\hat{f}$ is q-slowly decreasing but not q_1-slowly decreasing for any $q_1 > q$. Here $1/p + 1/q = 1$.*

COROLLARY 3. *For every $p \in (1, +\infty)$ there exist distributions $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ such that $f * \mathscr{K}'_p = \mathscr{K}'_p$ but $f * \mathscr{K}'_s \neq \mathscr{K}'_s$ for every $s \in [1, p)$. In view of Theorem 3 these distributions do not belong to $\bigcup_{t>p} \mathscr{O}'_C(\mathscr{K}'_t)$.*

The proofs of Theorems 4 and 5 follow the pattern of Ehrenpreis and Malliavin [4]: For the different kinds of slowly decreaing Fourier transforms of convolutors we derive estimates of the orders of their zeros (see §4); then we construct examples showing that these estimates are sharp (see §6).

Since analogues of Theorems 1 and 2 hold for Beurling distributions as well (see for example Grudzinski [5] for the case $p = 1$) we replace the function $\log(1 + |\cdot|)$ by an arbitrary Beurling weight function ω throughout the rest of the paper. This requires hardly any additional effort in the proofs.

2. Notation. We stick to the customary notation of Schwartz' distribution theory. If $F: \boldsymbol{C}^n \to \boldsymbol{C}$ is an entire function we write

$$M_F(z, r) := \sup \{|F(z + w)|; w \in \boldsymbol{C}^n, |w| \leqq r\}, \quad z \in \boldsymbol{C}^n, \ r > 0,$$

and denote by $\operatorname{ord}(z, F)$ the order of z as a zero of F (which is by definition equal to zero if $F(z) \neq 0$). By $\mathfrak{M}_n$ we denote the set of continuous functions $\omega: \boldsymbol{R}^n \to [0, \infty)$ such that

$$(\alpha) \qquad 0 = \omega(0) \leqq \omega(x + y) \leqq \omega(x) + \omega(y), \quad x, y \in \boldsymbol{R}^n,$$

$$(\beta) \qquad (1+|\cdot|)^{-n-1}\omega \in L_1(\boldsymbol{R}^n)\ ,$$

$$(\gamma) \qquad \omega \geqq a + b\log(1+|\cdot|)\ ,$$

where $a \in \boldsymbol{R}$ and $b > 0$ are constants (see Björck [1, Definition 1.3.22]). It follows from (β) (see [1, Corollary 1.2.8]) that

$$(4) \qquad \omega(x) = o(|x|/\log|x|) \quad \text{as} \quad x \longrightarrow \infty\ .$$

Note that $\log(1+|\cdot|)$ belongs to $\mathfrak{M}_n$. $\mathscr{E}'_\omega$ is the space of Beurling distributions with compact support, $\omega \in \mathfrak{M}_n$ (see [1]).

We call a function $F: \boldsymbol{C}^n \to \boldsymbol{C}$ *q-slowly decreasing* (*resp. very slowly decreasing*) *with respect to* $\omega \in \mathfrak{M}_n$ if (1) holds with r being of the form (3) (resp. (2)). If F is ∞-slowly decreasing with respect to ω it is also called *extremely slowly decreasing with respect to* ω.

3. Proof of Theorem 3. We prove the following version of Theorem 3.

THEOREM 3′. *Let* $\omega \in \mathfrak{M}_n$ *and* $q \in [1, +\infty)$, *and let* $F: \boldsymbol{C}^n \to \boldsymbol{C}$ *be an entire function satisfying an inequality of the form*

$$(5) \qquad |F(x+iy)| \leqq C\exp(\tilde{N}\omega(x) + S|y|^q)\ , \quad x, y \in \boldsymbol{R}^n\ ,$$

where C, $\tilde{N}$, S *are constants. If there exists a number* $Q \in (q, +\infty)$ *such that* F *is* Q*-slowly decreasing with respect to* ω *then* F *is extremely slowly decreasing with respect to* ω.

COROLLARY 4. *Let* $f \in \mathscr{E}'_\omega$. *If* $\hat{f}$ *is* q*-slowly decreasing with respect to* $\omega \in \mathfrak{M}_n$ *for some* $q > 1$ *then* $\hat{f}$ *is extremely slowly decreasing with respect to* ω.

This contradicts an assertion at the end of [2].

Proof of Theorem 3′. We follow the proof of [5, Satz 11] which is identical with Corollary 4. Let us fix $r > 1$ and $\eta > 0$, and set $R := r^{1+1/\eta}$. For $\gamma := \log(R/r)/\log R$ we have $1/\gamma = 1+\eta$ and $(1-\gamma)/\gamma = \eta$. Choose $x, w \in \boldsymbol{C}^n$ such that $|w| = 1$ and $|F(x+rw)| = M_F(x, r)$. By applying Hadamard's Three-Circles-Theorem to the function $\boldsymbol{C} \ni \lambda \mapsto F(x+\lambda w)$ we obtain

$$(6) \qquad M_F(x, 1) \geqq M_F(x, r^{1+1/\eta})^{-\eta} M_F(x, r)^{1+\eta}\ , \quad x \in \boldsymbol{C}^n\ .$$

From (5) and (α) we conclude

$$(7) \qquad M_F(x, R) \leqq C\exp(\tilde{N}\omega(x) + 2SR^q + d)\ , \quad x \in \boldsymbol{R}^n\ ,$$

where d is the constant defined by $d := \sup\{\tilde{N}\omega(x) - S|x|; x \in \boldsymbol{R}^n\}$ which is finite by (4). Setting $\eta := q/(Q-q)$ we see that $R^q = r^Q$; hence for $r = A\omega(x)^{1/Q}$ we have $R^q = A'\omega(x)$ where A' is another constant. Inserting this into (7) and combining the resulting inequality with (6) and with (1) where r is of the form (3) with q replaced by Q we arrive at the desired conclusion.

Note that $F = \hat{f}$ satisfies (5) with $\omega = \log(1 + |\cdot|)$ if $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ and $1/p + 1/q = 1$. So Theorem 3 is a special case of Theorem 3′.

4. Estimates for the orders of the zeros of slowly decreasing entire functions. In this section we derive the estimates for the orders of the zeros of slowly decreasing entire functions which we need for the proof of Theorems 4 and 5. For simplicity we consider the real zeros only.

THEOREM 6. *Let $\omega \in \mathfrak{M}_n$ and $f \in \mathscr{E}'_\omega$. If $\hat{f}$ is very slowly decreasing with respect to ω then*

$$\limsup_{x \to \infty, x \in \boldsymbol{R}^n} \operatorname{ord}(x, \hat{f})/\omega(x) = 0 .$$

THEOREM 7. *Let $\omega \in \mathfrak{M}_n$ and $q \in (1, +\infty]$, and let $F: \boldsymbol{C}^n \to \boldsymbol{C}$ be an entire function. If $q < +\infty$ we suppose that F satisfies an inequality of the form* (5); *if $q = +\infty$ we suppose that F satisfies*

$$(8) \qquad |F(x+iy)| \leqq C_S \exp(\tilde{N}_s\omega(x)) , \quad x, y \in \boldsymbol{R}^n, \ |y| \leqq S, \ S > 0 ,$$

where C_S and $\tilde{N}_S$ are constants depending on S.

(i) *If F is q-slowly decreasing with respect to ω then*

$$\limsup_{x \to \infty, x \in \boldsymbol{R}^n} \operatorname{ord}(x, F)/\omega(x) < +\infty .$$

This limit is equal to zero if in addition the following condition holds:

(9) *F satisfies* (5) (*resp.* (8)) *for every $S > 0$ with a constant $\tilde{N} = \tilde{N}_S$ being independent from S.*

(ii) *If there is $Q > q$ such that F is Q-slowly decreasing with respect to ω then*

$$\limsup_{x \to \infty, x \in \boldsymbol{R}^n} \operatorname{ord}(x, F) \log \omega(x)/\omega(x) < +\infty .$$

Note that if $f \in \mathscr{E}'_\omega$ and if $\hat{f}$ is extremely slowly decreasing with respect to ω then assertion (ii) of Theorem 7 applies to $F = \hat{f}$. This special case of Theorem 7 as well as Theorem 6 are contained in [5, Satz 13]. The proof of Theorem 7 is similar and is based on the following lemma.

LEMMA 1. *Let F be an entire function, and let $\psi\colon \boldsymbol{R}^n \to [0, +\infty)$ and $\phi\colon [0, +\infty) \to [0, +\infty]$ be weight functions such that*

$$|F(x + iy)| \leqq \exp(\psi(x) + \phi(|y|)) , \quad x, y \in \boldsymbol{R}^n . \tag{10}$$

Suppose that ψ satisfies (α) and that ϕ is increasing. Then for arbitrary $\theta, r > 0$ and $x \in \boldsymbol{R}^n$ the following estimate holds:

$$\operatorname{ord}(x, F) \leqq \theta(\psi(x) - \log M_F(x, r) + \tilde{\psi}(r_\theta) + \phi(r_\theta))$$

where $\tilde{\psi}(t)\colon = \max\{\psi(y); y \in \boldsymbol{R}^n, |y| \leqq t\}$ *and* $r_\theta\colon = r\exp(1/\theta)$.

Proof. Choose $w \in \boldsymbol{C}^n$ such that $|w| = r$ and $|F(x + w)| = M_F(x, r)$, and define entire functions $G, g\colon \boldsymbol{C} \to \boldsymbol{C}$ by $G(\lambda)\colon = F(x + \lambda w)$ and $g(\lambda)\colon = \lambda^{-\operatorname{ord}(0,G)}G(\lambda)$. The maximum principle yields $R^{-\operatorname{ord}(0,G)}M_G(0, R) = M_g(0, R) \geqq M_g(0, 1) = M_G(0, 1)$, $R > 1$. Since $\operatorname{ord}(x, F) \leqq \operatorname{ord}(0, G)$, $M_F(x, rR) \geqq M_G(0, R)$ and $M_G(0, 1) \geqq M_F(x, r)$ we obtain by taking logarithms and setting $R\colon = \exp(1/\theta)$

$$\operatorname{ord}(x, F) \leqq \theta(\log M_F(x, rR) - \log M_F(x, r)) .$$

An application of (10) and (α) leads to desired inequality.

Proof of Theorem 7, (i). If $q < +\infty$ fix $\theta > 0$ and set $\psi = \tilde{N}\omega$, $\phi = S|\cdot|^q$ and $r = A(\omega(x) + 1)^{1/q}$. Then $\phi(r_\theta) = (A_\theta)^q S(\omega(x) + 1)$. Note that by (4) there is a constant $d(\theta)$ such that $\tilde{\psi}(r_\theta) \leqq \omega(x) + d(\theta)$. Hence applying Lemma 1 and using (1) we see that the lim sup in (i) is not greater than $\theta(\tilde{N} + N + 1 + S(A_\theta)^q)$. If (5) is valid for every $S > 0$ with a constant $\tilde{N}$ being independent from S we choose S to be $(A_\theta)^{-q}$. Since θ can be taken arbitrarily small the lim sup in (i) is equal to zero. If $q = +\infty$ fix θ and set

$$\phi(y)\colon = \begin{cases} \log C_{r\theta} & \text{for } |y| \leqq r_\theta \\ +\infty & \text{otherwise} \end{cases} ,$$

and argue similarly as in the case $q < +\infty$.

Proof of Theorem 7, (ii). Here $q < +\infty$. Apply Lemma 1 to $\psi = \tilde{N}\omega$, $\phi = S|\cdot|^q$, $r = A(\omega(x) + 1)^{1/Q}$ and $\theta = t/\log\omega(x)$ where $t\colon = q/(1 - q/Q)$. Note that $e^{q/\theta}r^q \leqq A^q(\omega(x) + 1)$.

5. The main lemma for the construction of slowly decreasing functions with high order zeros. The lemma of this section is essentially due to Ehrenpreis and Malliavin [4, §4].

Let $(t_k)_{k\in N}$ be a sequence of real numbers, and let (m_k) and (l_k) be sequences of positive numbers. Our main assumption for the construction of slowly decreasing entire functions postulates the existence of a sequence of numbers $\nu_k \geqq 1$ with

$$\nu := \sum_{k=1}^{\infty} m_k \exp\left(-\frac{\pi}{2}\nu_k\right) < +\infty \tag{11}$$

such that the points t_k lie so far apart from each other that

(12) the intervals $J_k := [t_k - \tau_k, t_k + \tau_k]$ (where $\tau_k := \nu_k l_k$) are pairwise disjoint having distance greater than 1.

Moreover, we assume that the sequence (l_k) converges to $+\infty$ so that we can define a continuous function $h: \boldsymbol{R} \to [0, +\infty)$ by

$$h(y) := \frac{1}{4}\sum_{k\in N} \max\{|y| - l_k, 0\}\frac{m_k}{l_k}, \quad y \in \boldsymbol{R}. \tag{13}$$

LEMMA 2. *Under the preceding hypotheses there exists an entire function* $F: \boldsymbol{C} \to \boldsymbol{C}$ *having the following properties*:

$$F\cdot(1 + |\cdot|)^{-4}\exp(-h(\operatorname{Im}(\cdot)) - 2\pi|\operatorname{Im}(\cdot)|) \in L_2(\boldsymbol{R}^2), \tag{14}$$

$$\operatorname{ord}(t_k, F) \geqq \frac{m_k}{2\pi} - 1, \quad k \in \boldsymbol{N}, \tag{15}$$

$$\sup\{|F(x + u)|; u \in \boldsymbol{R}, |u| \leqq 1\} \geqq \begin{cases} 1 & \text{if} \quad x \in \boldsymbol{R}\setminus\bigcup_{k=1}^{\infty} J_k \\ e^{-m_k} & \text{if} \quad x \in J_k\setminus I_k \end{cases}, \tag{16}$$

where

$$I_k := \left[t_k - \frac{2}{\pi}l_k, t_k + \frac{2}{\pi}l_k\right].$$

Proof. The idea of the proof is as follows: First a suitable subharmonic function is constructed which reflects the desired properties of F (this step is essentially [4, Lemma 4]); then F is found by means of the theory of solution of the Cauchy-Riemann equations as developed in Hörmander [7]. The second step is suggested by a result of Bombieri's (see for example Hörmander [7, Theorem 4.4.4]).

Step 1. Define $g: \boldsymbol{C} \to [-\infty, 0]$ by

$$g(z) := \begin{cases} 0 & \text{if} \quad |\operatorname{Im} z| \geqq 1 \\ \dfrac{1}{2\pi}\log\left|\dfrac{e^{\pi z/2} - 1}{e^{\pi z/2} + 1}\right| & \text{if} \quad |\operatorname{Im} z| \leqq 1 \end{cases}.$$

As is easily calculated, g has the following properties:

(17) $\qquad g - (1/2\pi)\log|\cdot|$ is harmonic in the strip $\{z \in \boldsymbol{C};\ |\mathrm{Im}\, z| < 1\}$

(note that g restricted to the strip is Green's function for the strip),

$$g(x+iy) \geqq -\frac{2}{\pi}\exp\left(-\frac{\pi}{2}|x|\right), \quad x, y \in \boldsymbol{R},\ |x| \geqq \frac{2}{\pi}, \tag{18}$$

and

$$\Delta g = \delta - \psi \otimes (\delta_{+1} + \delta_{-1}) \tag{19}$$

where $\delta \in \mathscr{E}'(\boldsymbol{R}^2)$ and $\delta_{+1}, \delta_{-1} \in \mathscr{E}'(\boldsymbol{R}^1)$ are the Dirac-distributions at $0 \in \boldsymbol{R}^2$ and $+1, -1 \in \boldsymbol{R}$ respectively and where $\psi(x) := (4\cosh(\pi x/2))^{-1}$, $x \in \boldsymbol{R}$. We set

$$v(z) := \sum_{k \in N} m_k g\left(\frac{z - t_k}{l_k}\right), \quad z \in \boldsymbol{C}. \tag{20}$$

If $\mathrm{Re}\, z \notin J_k$ then $|\mathrm{Re}\,(z - t_k)/l_k| \geqq \nu_k$. So by (12), (18) and (11) we obtain

$$v(z) \geqq -\nu + \begin{cases} 0 & \text{if } \ \mathrm{Re}\, z \notin \bigcup_{k \in N} J_k \\ m_k g\left(\dfrac{z - t_k}{l_k}\right) & \text{if } \ \mathrm{Re}\, z \in J_k \end{cases}. \tag{21}$$

Hence $v: \boldsymbol{C} \to [-\infty, 0]$ is a well-defined upper semicontinuous function. Since the series (20) converges in the space $L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$, differentiation (in the distribution sense) and summation commute, and using (19) we obtain

$$\Delta v = \sum_{k \in N} m_k \delta_{t_k} - \sum_{k \in N} \frac{m_k}{l_k} \psi_k \otimes (\delta_{+l_k} + \delta_{-l_k}), \tag{22}$$

where $\delta_{t_k} \in \mathscr{E}'(\boldsymbol{R}^2)$ and $\delta_{+l_k}, \delta_{-l_k} \in \mathscr{E}'(\boldsymbol{R}^1)$ are Dirac-distributions and

$$\psi_k(x) := \left(4\cosh\left(\frac{\pi(x - t_k)}{2l_k}\right)\right)^{-1}, \quad x \in \boldsymbol{R}.$$

Since $\psi_k \leqq 1/4$ and since

$$\frac{d^2 h}{dy^2} = \frac{1}{4}\sum_{k \in N} \frac{m_k}{l_k}(\delta_{+l_k} + \delta_{-l_k})$$

it follows from (22) that the function $w: \boldsymbol{C} \to [-\infty, +\infty)$ defined by $w(z) := v(z) + h(\mathrm{Im}\, z)$, $z \in \boldsymbol{C}$, is subharmonic.

Step 2. For $x \in \boldsymbol{R}$ and $r > 0$ we denote by $S(x, r)$ the open square $\{t + iy;\ |t - x| < r,\ |y| < r\}$. Let us choose a test function

$\chi \in C_0^\infty(\boldsymbol{R}^2)$ such that

$$(23) \qquad \text{(a)} \quad \operatorname{supp} \chi \subset S\left(0, \frac{1}{4}\right), \quad \text{(b)} \quad \chi|_{S(0,1/8)} \equiv 1, \quad \text{(c)} \quad \chi \geqq 0 .$$

By (x_n) we denote the sequence of numbers $x \in (1/2)\boldsymbol{Z}$ such that the distance of x to every I_k is greater than 1/4. Define $G: \boldsymbol{C} \to [0, +\infty)$ by

$$(24) \qquad G: = \begin{cases} 0 & \text{on} \quad \boldsymbol{R}^2 \setminus \bigcup_{n=1}^{\infty} S\left(x_n, \frac{1}{4}\right) \\ e^{-m_k} \tau_{x_n} \chi & \text{on} \quad S\left(x_n, \frac{1}{4}\right) \quad \text{if} \quad S\left(x_n, \frac{1}{4}\right) \cap J_k \neq \emptyset \\ \tau_{x_n} \chi & \text{on} \quad S\left(x_n, \frac{1}{4}\right) \quad \text{if} \quad S\left(x_n, \frac{1}{4}\right) \cap \bigcup_{k \in N} J_k = \emptyset . \end{cases}$$

Note that G is well defined by (23.a) and (12). Since G is constant on every square $S(x, 1/8)$, $x \in (1/2)\boldsymbol{Z}$, $\partial G/\partial \bar{z}$ is identically zero there, and we define by

$$(25) \qquad H(z): = \begin{cases} \dfrac{\partial G}{\partial \bar{z}}(z)/\sin 2\pi z & \text{if} \quad z \in \operatorname{supp} \dfrac{\partial G}{\partial \bar{z}} \\ 0 & \text{otherwise} \end{cases}$$

a C^∞-function $H: \boldsymbol{C} \to \boldsymbol{C}$. Because $\operatorname{supp} G \cap I_k = \emptyset$ by our choice of (x_n) we derive from (21) and (18) that $|\partial G/\partial \bar{z}| \leqq \text{const}\, e^v$. Since $|\sin(x + iy)|^2 = |\sin x|^2 + |\sinh y|^2$ the same inequality (with a new constant) holds for H. Consequently the function $He^{-w}(1 + |\cdot|^2)^{-1}$ belongs to $L_2(\boldsymbol{R}^2)$. By Hörmander [7, Theorem 4.4.2] we obtain a solution $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ of the equation

$$(26) \qquad \frac{\partial u}{\partial \bar{z}} = -H$$

such that

$$(27) \qquad u e^{-w}(1 + |\cdot|^2)^{-2} \in L_2(\boldsymbol{R}^2) .$$

We set $F(z): = G(z) + u(z) \sin 2\pi z$, $z \in \boldsymbol{C}$. By (26) and (25) we have $\partial F/\partial \bar{z} = 0$, i.e., F is an entire function. Since $|G|$, too, is majorized by $\text{const}\, e^v$, it follows from (27) that

$$(28) \qquad F \exp(-w - 2\pi |\mathrm{Im}(\cdot)|)(1 + |\cdot|)^{-4} \in L_2(\boldsymbol{R}^2) .$$

Since $v \leqq 0$ (14) is a consequence of (28). (15) follows by noticing that (28) and (17) imply the local square-integrability of $z \mapsto |z - t_k|^{n_k}$ near t_k where $n_k: = \operatorname{ord}(t_k, F) - m_k/2\pi$, and this can only be true if $n_k > -1$. Finally (16) results from the definition

(24) of G and from the choice of (x_n) since $F(x_n) = G(x_n)$ and $\chi(0) = 1$.

REMARK. If we define g by

$$g(z) := \begin{cases} \dfrac{1}{2\pi} \log \left| \dfrac{z}{2i - z} \right| & \text{if} \quad \operatorname{Im} z \leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

and take $\chi_{[0,\infty)}h$ instead of h in the statement and in the proof of Lemma 2 we obtain functions F which are bounded by $(1 + |\cdot|)^4$ on the lower half plane, provided we change the hypothesis (11) into

$$(11)' \qquad \sum_{k=1}^{\infty} m_k/\nu_k < +\infty \, .$$

This means, however, that the intervals J_k become very much larger, in fact so large that no longer all the examples of the following section can be obtained by the modified version of Lemma 2. Nevertheless, the examples needed for the proof of Theorems 4 and 5 can be obtained in this manner.

6. Slowly decreasing entire functions with high order zeros. From Lemma 2 we now derive theorems on the existence of slowly decreasing entire functions with high order zeros. Together with our estimates for the orders of the zeros of slowly decreasing functions they yield proofs of Theorems 4 and 5.

THEOREM 4′. *Let $\omega \in \mathfrak{M}_1$. For any function $m: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that*

$$(29) \qquad m(t) = o(\omega(t)) \quad as \quad t \longrightarrow \infty$$

there exist $f \in \mathscr{E}'$ and $(t_k) \subset \boldsymbol{R}_+$ such that $\hat{f}$ is very slowly decreasing with respect to ω and t_k is a zero of $\hat{f}$ of order greater than $m(t_k)$ for every $k \in \boldsymbol{N}$.

It we set $m := \omega(\max\{1, \log \omega\})^{-1/2}$ we obtain $f \in \mathscr{E}'$ such that $\hat{f}$ is very but (in view of Theorem 7, (ii)) not extremely slowly decreasing with respect to ω. This proves in particular Theorem 4 (if $n > 1$ take tensor products).

Proof of Theorem 4′. Choose $(t_k) \subset \boldsymbol{R}_+$ with $t_{k+1} \geqq 2t_k$ such that $\sum_{k \in N} m_k/l_k$ converges where $m_k := 2\pi(m(t_k) + 1)$ and $l_k := \sqrt{m(t_k)\omega(t_k)}$. Changing m if necessary we may assume that $l_{k+1} \geqq l_k + 1$. Note that with $\nu_k := (2/\pi) \log \omega(t_k)$ (11) is valid. Since by (4) the function $\sqrt{m\omega} \log \omega$ is $o(t)$ as $t \to \infty$, (12) is valid if we choose t_1 large enough. By Lemma 2 there is an entire function $F: \boldsymbol{C} \to \boldsymbol{C}$ such that $\operatorname{ord}(t_k, F) \geqq m(t_k)$ and such that (14) and (16) hold. Since $h \leqq \text{const}\,|\cdot|$ (14) implies by the Paley-Wiener theorem: There is $f \in \mathscr{E}'$ such that $F = \hat{f}$. To conclude from (16) that F is very slowly decreasing with respect to ω it suffices to demonstrate the existence of a constant c and of a sequence (ε_k) converging to zero such that

$$l_k \leqq \varepsilon_k \omega(x) + c\,, \quad x \in I_k,\ k \in \boldsymbol{N}\,, \tag{30}$$

and

$$m_k \leqq \varepsilon_k \omega(x) + c\,, \quad x \in J_k,\ k \in \boldsymbol{N}\,. \tag{31}$$

By (4) there is c' such that $\omega \leqq |\cdot| + c'$. By (29) there is a sequence (η_k) converging to zero such that $l_k \leqq \eta_k \omega(t_k)$. (30) follows since $\omega(t_k) \leqq \omega(x) + \omega(t_k - x) \leqq \omega(x) + l_k + c'$ for $x \in I_k$. To derive (31) we choose by (4) a constant $R > e$ such that $2\omega(y) \leqq |y|/\log|y|$ for every $y \in \boldsymbol{R}$ with $|y| \geqq R$. Hence

$$2\omega(t_k) - 2\omega(x) \leqq 2\omega(t_k - x) \leqq c + \tau_k/\log \tau_k\,, \quad x \in J_k,\ k \in \boldsymbol{N}\,, \tag{32}$$

where $c := \max\{\omega(y);\, |y| \leqq R\}$. Here we have used the fact that the function $t \mapsto t/\log t$ is increasing on $(e, +\infty)$. If $\omega(t_k) \leqq \tau_k$ then $\tau_k/\log\tau_k \leqq \tau_k/\log\omega(t_k) = (2/\pi)l_k \leqq \omega(t_k)$ for sufficiently large k. If on the other hand $\omega(t_k) \geqq \tau_k$, then $\tau_k/\log\tau_k \leqq \omega(t_k)$. So (32) becomes

$$\omega(t_k) \leqq 2\omega(x) + \text{const}\,, \quad x \in J_k,\ k \in \boldsymbol{N}\,, \tag{33}$$

which together with (29) immediately implies (31).

Theorem 4′ shows that the estimate in Theorem 6 is sharp in general. Next we prove a similar result for convolutors in $\mathscr{O}'_C(\mathscr{K}'_p)$.

THEOREM 5′. *Let $\omega \in \mathfrak{M}_1$ and $p \in [1, +\infty)$, and let $m: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a function satisfying (29). Then there exist $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ and $(t_k) \subset \boldsymbol{R}_+$ such that $\hat{f}$ is q-slowly decreasing with respect to ω and t_k is a zero of $\hat{f}$ of order greater than $m(t_k)$. Here $1/p + 1/q = 1$.*

Let $p > 1$. If we again set $m := \omega(\max\{1, \log\omega\})^{-1/2}$ we obtain $f \in \mathscr{O}'_C(\mathscr{K}'_p)$ such that $\hat{f}$ is q-slowly but (in view of Theorem 7, (ii)) not Q-slowly decreasing with respect to ω for every $Q > q$. This proves in particular Theorem 5.

Proof of Theorem 5′: the case $p > 1$. By $\varepsilon: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ we denote the function $\sqrt{m/\max\{1, \omega\}}$. Let us choose a sequence $(t_k) \subset \boldsymbol{R}_+$ with $t_{k+1} \geqq 2t_k$ such that $\sum_{k=1}^{\infty} \varepsilon(t_k)$ converges. We define $m_k := 2\pi(m(t_k) + 1)$ and $l_k := (\varepsilon(t_k)\omega(t_k))^{1/q}$. With $\nu_k := (2/\pi)\log\omega(t_k)$ (11) is valid. Since $q > 1$ the function $(\varepsilon\omega)^{1/q}\log\omega$ is $o(t)$ as $t \to \infty$ by (4); hence if t_1 is large enough (12) holds. Since without loss of generality $m \geqq 1$ we may assume that $l_{k+1} \geqq l_k + 1$ and that $\varepsilon(t_k) \leqq 1 \leqq m(t_k) \leqq \omega(t_k)$. This and the definition of ε imply: $m_k \leqq 4\pi\varepsilon(t_k)l_k^q$. Hence for every $j \in \boldsymbol{N}$ we derive from the definition (13) of h that

$$4h(y) \leqq \left(\sum_{k=1}^{j-1}\frac{m_k}{l_k}\right)|y| + 4\pi\sum_{k=j}^{\infty}\varepsilon(t_k)|y|^q\,, \quad y \in \boldsymbol{R}\,.$$

Since $q > 1$ it follows that

$$h(y) = o(|y|^q) \quad \text{as} \quad y \to \infty\,. \tag{34}$$

An application of Lemma 2 and similar arguments as in the proof of Theorem 4′ lead to the desired assertion. Note that an entire function F satisfying (14) with h fulfilling (34) is of the form $F = \hat{f}$ where $f \in \mathcal{O}'_C(\mathcal{K}'_p)$.

Proof of Theorem 5′: *the case* $p = 1$. Define $\varepsilon: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ by $\varepsilon := m/\max\{\omega, 1\}$. By changing m if necessary we may suppose that $\lim_{t\to\infty}\varepsilon(t)m(t) = +\infty$. Then we can choose a sequence $(t_k) \subset \boldsymbol{R}_+$ with $t_{k+1} \geqq 2t_k$ such that for every $k \in \boldsymbol{N}$

$$1 \leqq \varepsilon(t_k)m(t_k) \leqq \varepsilon(t_{k+1})m(t_{k+1})/2 \tag{35}$$

and

$$\varepsilon(t_{k+1}) < \varepsilon(t_k)^2 \leqq 1\,, \tag{36}$$

and such that with $m_k := 2\pi(m(t_k) + 1)$ and $\nu_k := (2/\pi)\log\omega(t_k)$ (11) holds. Since $\varepsilon(t)^{-1}\log\omega(t) \leqq \omega(t)\log\omega(t)$ for sufficiently large t and since by (4) $\omega\log\omega$ is $o(t)$ as $t \to \infty$, (12) is valid with $l_k := 1/\varepsilon(t_k)$ if t_1 is chosen large enough. By Lemma 2 we obtain an entire function $F: \boldsymbol{C} \to \boldsymbol{C}$ such that $\operatorname{ord}(t_k, F) \geqq m(t_k)$ for every $k \in \boldsymbol{N}$ and such that (14) and (16) hold. Using Cauchy's formula we deduce from (14):

$$|F(x + iy)| \leqq \operatorname{const}(1 + |x|)^4\exp(h(|y| + 1) + 7|y|)\,, \quad x, y \in \boldsymbol{R}\,. \tag{37}$$

This implies: there is $f \in \mathcal{O}'_C(\mathcal{K}'_1)$ such that $F = \hat{f}$ (see for example [5]). To prove that F is extremely slowly decreasing with respect to ω we conclude from (36) that $l_k \leqq l_k^2 < l_{k+1}$ for every k. Hence

$$4h(l_k^2) = \sum_{j=1}^{k}(l_k^2 - l_j)\frac{m_j}{l_j}\,.$$

From the definitions of m_j, l_j and ε and from (35) we see

$$\frac{m_j}{4\pi l_j} \leqq \varepsilon(t_j)m(t_j) \leqq 2^{j-k}\varepsilon(t_k)m(t_k) = 2^{j-k}\omega(t_k)l_k^{-2}\ , \quad j \leqq k\ .$$

Consequently

$$h(l_k^2) \leqq 2\pi\omega(t_k)\ , \quad k \in \boldsymbol{N}\ . \tag{38}$$

Moreover, (35) implies: $\omega(t_k) = \varepsilon(t_k)m(t_k)l_k^2 \geqq l_k^2$. Since for sufficiently large k the inequality $l_k^2 \geqq r_k^2 + 1$ holds where $r_k\colon = (2/\pi)l_k + 1$, it follows from (37) and (38) that for sufficiently large k

$$M_F(x, r_k^2) \leqq \text{const}\,(1 + |x|)^4 \exp(14\omega(t_k))\ , \quad x \in \boldsymbol{R}\ . \tag{39}$$

Since $\tau_k \leqq \omega(t_k)\log\omega(t_k)$ the same arguments as in the proof of Theorem 4′ yield (33). Combining (6) (with $\eta = 1$), (39), (33), (γ) and (16) we conclude that F is extremely slowly decreasing with respect to ω.

The entire functions constructed in the proofs of Theorem 5′ satisfy condition (9). Hence Theorem 5′ shows that for entire functions satisfying (9) the estimate of the orders of the zeros in assertion (i) of Theorem 7 is sharp in general.

As for the question whether or not the estimate in assertion (ii) of Theorem 7 is sharp as well, the method of Lemma 2 seems to yield only the following rather weak result.

THEOREM 8. *For arbitrary $\omega \in \mathfrak{M}_1$ and $s < 1$ there exist $f \in \mathscr{E}'$ and $(t_k) \subset \boldsymbol{R}_+$ such that $\hat{f}$ is extremely slowly decreasing with respect to ω and t_k is a zero of $\hat{f}$ of order greater than $\omega(t_k)^s$ for every $k \in \boldsymbol{N}$.*

Proof. Let $\sigma\colon = (1 - s)/2$. Choose $(t_k) \subset \boldsymbol{R}_+$ with $t_{k+1} \geqq 2t_k$ such that $\sum_{k=1}^{\infty} \omega(t_k)^{-\sigma}$ converges. Define $m_k\colon = 2\pi(\omega(t_k)^s + 1)$ and $l_k\colon = \omega(t_k)^{s+\sigma}$. Then $h \leqq \text{const}\,|\cdot|$. By proceeding similarly as in the proofs of the foregoing theorems we obtain an entire function F, which is the Fourier transform of a distribution $f \in \mathscr{E}'$, such that $\operatorname{ord}(t_k, F) \geqq \omega(t_k)^s$ for every $k \in \boldsymbol{N}$ and such that F is q-slowly decreasing with respect to ω where $q\colon = (s + \sigma)^{-1}$. By Corollary 4 F is even extremely slowly decreasing with respect to ω.

If one wants to obtain better results it seems that one has to use a more refined method than that of Lemma 2 for instance the one employed by Ehrenpreis and Malliavin [4, §2] for the construction of distributions $f \in \mathscr{E}'$ such that $\hat{f}$ is slowly but not very slowly decreasing.

Note added in proof. A slightly more precise version of Lemma 2 leads to an improvement of Theorem 8 showing that the estimate in Theorem 7, (ii), is sharp in general, namely: *For arbitrary* $w \in \mathfrak{M}_1$ *and* $c > 0$ *there exist a distribution* $f \in \mathscr{E}'$ *and a sequence* $(t_k) \subset \boldsymbol{R}_+$ *such that* $\hat{f}$ *is extremely slowly decreasing with respect to* ω *and* $\operatorname{ord}(t_k, \hat{f}) \geqq c\omega(t_k)/\log \omega(t_k)$ *for every* $k \in \boldsymbol{N}$.

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