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COMMON FIXED POINTS AND ITERATION OF COMMUTING NONEXPANSIVE MAPPINGS

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The following result is shown. Let $T_i (i = 1, 2, \cdots, \nu)$ be commuting nonexpansive self-mappings on a compact convex subset D of a Banach space and let x be any point in D. Then the sequence

$$\left\{\left[\prod_{n_{\nu-1}=1}^{n_\nu}\left[S_\nu \prod_{n_{\nu-2}=1}^{n_{\nu-1}}\left[\cdots\left[S_3 \prod_{n_1=1}^{n_2}\left[S_2 \prod_{n_0=1}^{n_1} S_1\right]\right]\cdots\right]\right]\right]x\right\}_{n_\nu=1}^{\infty}$$

converges to a common fixed point of $\{T\}_{i=1}^{\nu}$, where $S_i = (1-\alpha_i)I + \alpha_i T_i$, $0 < \alpha_i < 1$, I is the identity mapping.

In [2], DeMarr proved that if $T_i (i \in J, J$ is an index set) are commuting nonexpansive self-mappings on a compact convex subset D of a Banach space (i.e., $\|Tx - Ty\| \leqq \|x - y\|$ for all x, y in D, and $T_i T_j = T_j T_i$ for all $i, j \in J$), then $T_i (i \in J)$ have a common fixed point in D.

The problem we shall consider in this paper is that of constructing a sequence of points $\{x_n\}_{n=1}^{\infty}$ in D that converges to the common fixed point of T_i ($i \in J$, J is a finite index set).

If a Banach space is strictly convex (i.e., $\|\alpha x + (1-\alpha)y\| < \max\{\|x\|, \|y\|\}$ for $x \neq y$, $0 < \alpha < 1$), the problem was solved in [5].

Throughout this paper, we denote an identity mapping by I and the set of fixed points of T by $F[T]$. And we define $\prod_{i=1}^{n+1} T_i = T_{n+1}(\prod_{i=1}^{n} T_i)$ for any positive integer n and $\prod_{i=1}^{1} T_i = T_1$.

We have the following main theorem.

THEOREM. *Let $T_i (i = 1, 2, \cdots \nu)$ be commuting nonexpansive mappings from a compact convex subset D of a Banach space into itself, and let x be any point in D.*

Then $\bigcap_{i=1}^{\nu} F[T_i]$ is nonempty and the sequence $\{x_{n_\nu}^{\infty}\}$ converges to a point in $\bigcap_{i=1}^{\nu} F[T_i]$, where x_{n_ν} is defined for each positive integer n_ι by

$$\left[\prod_{n_{\iota-1}=1}^{n_\iota}\left[S_\iota \prod_{n_{\iota-2}=1}^{n_{\iota-1}}\left[S_{\iota-1}\cdots\left[S_3 \prod_{n_1=1}^{n_2}\left[S_2 \prod_{n_0=1}^{n_1} S_1\right]\right]\cdots\right]\right]\right]x$$

where $S_i = (1-\alpha_i)I + \alpha_i T_i$, $0 < \alpha_i < 1 (i = 1, 2, \cdots, \nu)$.

Before proving the theorem, we first prove the following lemmas on which the proof of theorem is based.

LEMMA 1. *Let T and P be nonexpansive mappings from a* *bounded convex subset D of a Banach space into itself that satisfy the conditions*

$$(1) \qquad P(D) = F[P] \quad \textit{and} \quad T(P(D)) \subset P(D) .$$

Let x_0 be any point in D and let α be any number such that $0 < \alpha < 1$. Then the sequences $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{x_n - Px_n\}_{n=0}^{\infty}$ respectively converge to zero, where x_n is defined for each positive integer n by

$$(2) \qquad x_n = (1-\alpha)y_n + \alpha T y_n , \quad y_n = Px_{n-1} ,$$

that, is $x_n = (SP)^n x_0$, where $S = (1-\alpha)I + \alpha T$.

Proof. We see from (1) that for all $n \geqq 1$

$$(3) \qquad y_n = Py_n \quad \text{and} \quad Ty_n = PTy_n .$$

Since T and P are nonexpansive mappings, we have, from (2) and (3), for all $n \geqq 0$

$$||y_{n+1} - Ty_{n+1}|| = ||Px_n - PTy_{n+1}|| \leqq ||x_n - Ty_{n+1}||$$

and, from (2) and (3), for all $n \geqq 1$

$$\begin{aligned} ||x_n - Ty_{n+1}|| &\leqq ||x_n - Ty_n|| + ||Ty_n - Ty_{n+1}|| \\ &\leqq (1-\alpha)||y - Ty_n|| + ||y_n - y_{n+1}|| \\ &\leqq (1-\alpha)||y_n - Ty_n|| + ||Py_n - Px_n|| \\ &\leqq (1-\alpha)||y_n - Ty_n|| + ||y_n - x_n|| \\ &\leqq (1-\alpha)||y_n - Ty_n|| + \alpha||y_n - Ty_n|| = ||y_n - Ty_n|| \end{aligned}$$

from which, we obtain

$$||y_{n+1} - Ty_{n+1}|| \leqq ||x_n - Ty_{n+1}|| \leqq ||y_n - Ty_n|| \qquad \text{for all} \quad n \geqq 1 .$$

Hence the sequence $\{||y_n - Ty_n||\}_{n=1}^{\infty}$, which is nonincreasing and bounded below, has a limit.

Suppose that $\lim ||y_n - Ty_n|| = r > 0$, that is, for any $\varepsilon > 0$, there is an integer m such that

$$(4) \qquad r \leqq ||y_n - Ty_n|| \leqq (1+\varepsilon)r \qquad \text{for all} \quad n \geqq m .$$

Also, from the boundedness of D, we can choose M such that

$$(5) \qquad L \leqq (M-m)r < 2L , \quad \text{where} \quad L \text{ is a diameter of } D .$$

We have from (3), (2) and (4) that for any $n \geqq m$ and $k \geqq 0$

$$\begin{aligned} ||y_n - y_{n+k+1}|| &\leqq ||y_n - y_{n+1}|| + ||y_{n+1} - y_{n+2}|| + \cdots + ||y_{n+k} - y_{n+k+1}|| \\ &\leqq ||Py_n - Px_n|| + ||Py_{n+1} - Px_{n+1}|| + \cdots + ||Py_{n+k} - Px_{n+k}|| \\ &\leqq ||y_n - x_n|| + ||y_{n+1} - x_{n+1}|| + \cdots + ||y_{n+k} - x_{n+k}|| \end{aligned}$$

$$(6) \qquad \leqq \alpha(k+1)(1+\varepsilon)r\ .$$

Now we shall prove by induction that

$$(7) \qquad (1+\alpha k)(1+\varepsilon)r - (1-\alpha)^{-k}\varepsilon r \leqq \|Ty_M - y_{M-k}\|$$

for any k such that $0 \leqq k \leqq M - m$.

When $k = 0$, the result is trivial. Now we assume that (7) is true for some k such that $0 \leqq k \leqq M - m - 1$. We see, from (3) and (2), that

$$\begin{aligned}\|Ty_M - y_{M-k}\| &= \|PTy_M - Px_{M-(k+1)}\| \leqq \|Ty_M - x_{M-(k+1)}\| \\ &= \|(1-\alpha)(Ty_M - y_{M-(k+1)}) + \alpha(Ty_M - Ty_{M-(k+1)})\| \\ &\leqq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha\|y_M - y_{M-(k+1)}\|\end{aligned}$$

from which and (6), it follows that

$$\|Ty_M - y_{M-k}\| \leqq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha^2(k+1)(1+\varepsilon)r\ .$$

From this and the assumption by induction, we have

$$\begin{aligned}&(1+\alpha k)(1+\varepsilon)r - (1-\alpha)^{-k}\varepsilon r \\ &\quad \leqq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha^2(k+1)(1+\varepsilon)r\end{aligned}$$

and it is clear that this inequality is equal to (7) with $k+1$ for k. Hence, by induction, it follows that (7) is true for any k such that $0 \leqq k \leqq M - m$.

Since $\log(1+t) \leqq t$ for all $t \in (-1, \infty)$, we have from (5) that

$$\begin{aligned}(1-\alpha)^{-(M-m)} &= \exp\left[(M-m)\log\left(1+\frac{\alpha}{1-\alpha}\right)\right] \\ &\leqq \exp\left[(M-m)\frac{\alpha}{1-\alpha}\right] \leqq \exp\left(\frac{2L}{(1-\alpha)r}\right)\ .\end{aligned}$$

Thus it follows from (7) with $M - m$ for k that

$$\begin{aligned}\|Ty_M - y_m\| &\geqq (1+\alpha(M-m))(1+\varepsilon)r - \varepsilon r \exp\left(\frac{2L}{(1-\alpha)r}\right) \\ &\geqq (r+L) - \varepsilon r \exp\left(\frac{2L}{(1-\alpha)r}\right)\ .\end{aligned}$$

Since ε is any positive number, this inequality is imcompatible with the definition of L. Hence we obtain that $r = 0$, that is,

$$(8) \qquad \lim_{n\to\infty} \|y_n - Ty_n\| = 0\ .$$

Now since T and P are nonexpansive mappings, we have from (2) and (3) that, for all $n \geqq 1$,

$$\begin{aligned}||x_n - Tx_n|| &= ||(1 + \alpha)y_n + \alpha Ty_n - T((1 - \alpha)y_n + \alpha Ty_n)|| \\ &= ||(1 - \alpha)y_n - (1 - \alpha)Ty_n + Ty_n - T((1 - \alpha)y_n + \alpha Ty_n)|| \\ &\leqq (1 - \alpha)||y_n - Ty_n|| + \alpha||y_n - Ty_n|| \\ &= ||y_n - Ty_n||\end{aligned}$$

and

$$\begin{aligned}||x_n - Px_n|| &= ||(1 - \alpha)y_n + \alpha Ty_n - P((1 - \alpha)y_n + \alpha Ty_n)|| \\ &= ||(1 - \alpha)[Py_n - P((1 - \alpha)y_n + \alpha Ty_n)] \\ &\quad + \alpha[PTy_n - P((1 - \alpha)y_n + \alpha Ty_n)]|| \\ &\leqq 2\alpha(1 - \alpha)||y_n - Ty_n|| .\end{aligned}$$

Therefore we obtain that from (8) that

$$\lim_{n\to\infty}||x_n - Tx_n|| = \lim_{n\to\infty}||x_n - Px_n|| = 0 .$$

LEMMA 2. *Let T and P be nonexpansive mappings from a compact convex subset D of a Banach space into itself such that*

(9) $$P(D) = F[P] \quad and \quad T(P(D)) \subset P(D) .$$

Let x_0 be any point in D. Define $x_n = \bar{P}_n x_0$ for each positive integer n, where $\bar{P}_n = (SP)^n$, $S = (1 - \alpha)I + \alpha T$, $0 < \alpha < 1$. Then it follows that

(10) *for any x_0 in D, $\lim_{n\to\infty}(SP)^n x_0 = Px_0$ exists, which is, denoted by $\bar{P}x_0$,*

(11) $$\bar{P}(D) = F[\bar{P}] = F[T] \cap F[P]$$

and

(12) $$\{\bar{P}_n\}_{n=1}^{\infty} \text{ converges uniformly to } \bar{P} .$$

Proof. Since D is compact, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$ that converges to a point u in D. From the boundedness of D, Lemma 1 is applicable, so we have,

$$\begin{aligned}||u - Tu|| &\leqq \lim_{i\to\infty}\{||u - x_{n_i}|| + ||x_{n_i} - Tx_{n_i}|| + ||Tx_{n_i} - Tu||\} \\ &\leqq \lim_{i\to\infty}\{2||x_{n_i} - n|| + ||x_{n_i} - Tx_{n_i}||\} = 0 ,\end{aligned}$$

and similarly $||u - Pu|| = 0$.

From this, it follows that

(13) $$u \in F[T] \cap F[P] .$$

Since (9) implies (3), we see from (13) and (3) that for all $n \geqq 0$,

$$\begin{aligned}\|u - x_{n+1}\| &= \|u - ((1-\alpha)y_{n+1} + \alpha T y_{n+1})\| \\ &\leqq (1-\alpha)\|u - y_{n+1}\| + \alpha \|Tu - Ty_{n+1}\| \\ &\leqq \|u - y_{n+1}\| = \|Pu - Px_n\| \leqq \|u - x_n\| .\end{aligned}$$

From this, we obtain that $\lim_{n\to\infty}\|u-x_n\|=\lim_{n_i\to\infty}\|u-x_{n_i}\|=0$. Hence we have proved that (10) is true, that is, for any x_0 in D, $\bar{P}(x_0) = \lim_{n\to\infty}(SP)^n x_0$ is well-defined. From (13), we see that $\bar{P}(x_0) \in F[T] \cap F[P]$ for all x_0 in D, that is,

$$\bar{P}(D) \subset F(T) \cap F[P] . \tag{14}$$

And we have that, for any v in $F[T] \cap F[P]$,

$$v = (SP)^n v = \lim_{n\to\infty}(SP)^n v = \bar{P}v ,$$

so we see that

$$F[T] \cap F[P] \subset F[\bar{P}] . \tag{15}$$

Also, clearly $w = \bar{P}w \in \bar{P}(D)$ for all w in $F[\bar{P}]$. From this, (14) and (15), we get (11).

Finally we shall prove (12). Let ε be any positive number. Since D is compact, there are finite points $\{x_0^1, x_0^2, \cdots, x_0^k\}$ such that, for any x in D,

$$\min\{\|x - x_0^i\| : 1 \leqq i \leqq k\} < \frac{\varepsilon}{3} . \tag{16}$$

From (10), we can choose N such that

$$\|(SP)^n x_0^i - \bar{P}x_0^i\| < \frac{\varepsilon}{3} \quad \text{for all} \quad n \geqq N \quad \text{and} \quad 1 \leqq i \leqq k . \tag{17}$$

Let x_0 be any point in D. From (16), we can take x_0^j such that

$$\|x_0 - x_0^j\| < \frac{\varepsilon}{3} . \tag{18}$$

Since SP is nonexpansive, clearly $\bar{P}$ is also nonexpansive. Hence we obtain from (17) and (18) that, for all $n \geqq N$,

$$\begin{aligned}&\|(SP)^n x_0 - \bar{P}x_0\| \\ &\quad \leqq \|(SP)^n x_0 - (SP)^n x_0^j\| + \|(SP)^n x_0^j - \bar{P}x_0^j\| + \|\bar{P}x_0^j - \bar{P}x_0\| \\ &\quad \leqq 2\|x_0 - x_0^j\| + \|(SP)^n x_0^j - \bar{P}x_0^j\| \leqq \varepsilon\end{aligned}$$

which implies (12).

LEMMA 3. *Let T and $P_n (n = 1, 2, \cdots)$ be nonexpansive mappings from a compact convex subset D of a Banach space into itself. Assume* *that the following conditions are satisfied:*

(19) *for any x in D,* $\lim_{n\to\infty} P_n x = Px$ *exists* ,

(20) $P(D) = F[P] \subset F[P_n]$ *for all* $n \geqq 1$

(21) P_n *converges uniformly to* P

and

(22) $T(P(D)) \subset P(D)$.

Then it follows that

(23) *for any x in D,* $\lim_{n\to\infty} \hat{P}_n x = \hat{P}x$ *exists, where* $\hat{P}_n = \prod_{i=1}^n (SP_i)$, $S = (1-\alpha)I + \alpha T$, $0 < \alpha < 1$,

(24) $\hat{P}(D) = F[\hat{P}] = F[T] \cap F[P] \subset F[\hat{P}_n]$ *for all* $n \geqq 1$

and

(25) $\hat{P}_n$ *converges uniformly to* $\hat{P}$.

Proof. Let ε be any positive number. Since P satisfies the conditions of P in Lemma 2, from (12), we can choose N such that

(26) $$\|(SP)^N y - \bar{P}y\| < \frac{\varepsilon}{2} \quad \text{for all } y \text{ in } D ,$$

where $\bar{P}$ is defined as in Lemma 2.

From (21), there exists M such that

$$\|SPx - SP_n x\| \leqq \|Px - P_n x\| \leqq \frac{\varepsilon}{2N}$$

for all $n \geqq M$ and all x in D.

This implies that, for all n such that $n \geqq M$

$$\begin{aligned} &\|\hat{P}_n x - (SP)^N \hat{P}_{n-N} x\| \\ &\quad \leqq \|(SP_n)\hat{P}_{n-1}x - (SP)\hat{P}_{n-1}x\| + \|(SP)\hat{P}_{n-1}x - (SP)(S)^{N-1}\hat{P}_{n-N}x\| \\ &\quad \leqq \frac{\varepsilon}{2N} + \|\hat{P}_{n-1}x - (SP)^{N-1}\hat{P}_{n-N}x\| \\ &\quad \leqq 2\frac{\varepsilon}{2N} + \|\hat{P}_{n-2}x - (SP)^{N-2}\hat{P}_{n-N}x\| \leqq \cdots \leqq \frac{\varepsilon}{2} . \end{aligned}$$

From this and (26), we have that, for all n such that $n \geqq \max\{N, M\}$,

$$\begin{aligned} \|\hat{P}_n x - \bar{P}(\hat{P}_{n-N}x)\| &\leqq \|\hat{P}_n x - (SP)^N \hat{P}_{n-N}x\| + \|(SP)^N \hat{P}_{n-N}x - \bar{P}(\hat{P}_{n-N}x)\| \\ &\leqq \varepsilon . \end{aligned}$$

Since Lemma 2 says that $\bar{P}(D) = F[T] \cap F[P]$, this implies that there exists a subsequence $\{\hat{P}_{n_i}x\}_{i=1}^{\infty}$ that converges to a point u in $F[T] \cap F[P]$. Also we see, from (20), for all $n \geqq 1$,

$$||\hat{P}_{n+1}x - u|| = ||SP_{n+1}\hat{P}_n x - SP_{n+1}u|| \leqq ||\hat{P}_n x - u|| .$$

Hence we get that $\lim_{n\to\infty}||\hat{P}_n x - u|| = \lim_{i\to\infty}||\hat{P}_{n_i}x - u|| = 0$, that is, $\hat{P}_n x$ converges to a point in $F[T] \cap F[P]$ for any x in D. This implies (23), and

$$\hat{P}(D) \subset F[T] \cap F[P] . \tag{27}$$

If $v \in F[T] \cap F[P]$, then $v = \hat{P}_n v = \lim_{n\to\infty}\hat{P}_n v = \hat{P}v$, so we see

$$F[T] \cap F[P] \subset F[\hat{P}] . \tag{28}$$

Since clearly $F[\hat{P}] \subset \hat{P}(D)$ and $F[T] \cap F[P] \subset F[\hat{P}_n]$ for all $n \geqq 1$, (24) follows from (27) and (28).

Now we shall prove (25). Let ε be any positive number. As in the proof of Lemma 2, we can choose finite points $\{x_0^1, x_0^2, \cdots, x_0^k\}$ from D satisfying (16). From (23), we can choose N' such that

$$||\hat{P}_n x_0^i - \hat{P}x_0^i|| \leqq \frac{\varepsilon}{3} \qquad \text{for all} \quad n \geqq N' \quad \text{and} \quad 1 \leqq i \leqq k . \tag{29}$$

Let x_0 be any point in D. By (16), we can take x_0^j that satisfies (18).

Since $\hat{P}$ is nonexpansive, we obtain from (18) and (29) that, for all $n \geqq N'$,

$$\begin{aligned} &||\hat{P}_n x_0 - \hat{P}x_0|| \\ &\quad \leqq ||\hat{P}_n x_0 - \hat{P}_n x_0^j|| + ||\hat{P}_n x_0^j - \hat{P}x_0^j|| + ||\hat{P}x_0^j - \hat{P}x_0|| \\ &\quad \leqq 2||x_0 - x_0^j|| + ||\hat{P}_n x_0^j - \hat{P}x_0^j|| \leqq \varepsilon . \end{aligned}$$

This implies (25).

LEMMA 4. *Let $T_i (i = 1, 2, \cdots, k)$ be a commuting family of mappings. Then it follows that*

$$T_k\Big(\bigcap_{i=1}^{k-1} F[T_i]\Big) \subset \bigcap_{i=1}^{k-1} F[T_i] .$$

Proof. Let x be any point in $\bigcap_{i=1}^{k-1} F[T_i]$. We see that $T_k x = T_k T_i x = T_i T_k x$ for all i such that $1 \leqq i \leqq k-1$, which implies that $T_k x$ belongs to $F[T_i]$ for all $1 \leqq i \leqq k-1$.

Proof of theorem. For all i such that $1 \leqq i \leqq \nu$, put

$$\Big[\prod_{n_{i-1}=1}^{n_i}\Big[S_i \prod_{n_{i-2}=1}^{n_{i-1}}\Big[S_{i-1} \cdots \Big[S_2 \prod_{n_0=1}^{n_1} S_1\Big] \cdots \Big]\Big]\Big]x = P_{n_i}^{(i)}x .$$

We shall prove the theorem by induction. Let us assume that the following conditions are true for some integer j such that $1 \leqq j \leqq \nu - 1$:

$$\text{(30)} \qquad \text{for any } x \text{ in } D,\ \lim_{n_j\to\infty} P^{(j)}_{n_j} x = P^{(j)} x \text{ exists,}$$

$$\text{(31)} \qquad P^{(j)}(D) = F[P^{(j)}] = \bigcap_{i=1}^{j} F[T_i] \subset F[P^{(j)}_{n_j}] \text{ for all integers } n_j \geqq 1 ,$$

$$\text{(32)} \qquad \{P^{(j)}_{n_j}\}_{n_j=1}^{\infty} \text{ converges uniformly to } P^{(j)}$$

and

$$\text{(33)} \qquad T(P^{(j)}(D)) \subset P^{(j)}(D) .$$

Since $P^{(j+1)}_{n_{j+1}} x = [\prod_{n_j=1}^{n_j+1} (S_{j+1} P^{(j)}_{n_j})]x$, we can apply Lemma 3 by regarding $T_{j+1}, S_{j+1}, P^{(j)}, P^{(j)}_{n_j}, P^{(j+1)}_{n_{j+1}}, P^{(j+1)}$ and conditions (30)-(33) as T, S, P, P_n, P_n, P and conditions (19)-(22). Hence we have,

$$\text{(34)} \qquad \text{for any } x \text{ in } D,\ \lim_{n_{j+1}\to\infty} P^{(j+1)}_{n_{j+1}} x = P^{(j+1)} x \text{ exists ,}$$

$$\text{(35)} \qquad P^{(j+1)}(D) = F[P^{(j+1)}] = \bigcap_{i=1}^{j+1} F[T_i] \subset F[P^{(j+1)}_{j+1}] \text{ for all } n_{j+1} \geqq 1$$

and

$$\text{(36)} \qquad \{P^{(j+1)}_{n_{j+1}}\}_{n_{j+1}=1}^{\infty} \text{ converges uniformly to } P^{(j+1)} .$$

Moreover, if $j + 2 \leqq \nu$, Lemma 4 shows from (35) that

$$\text{(37)} \qquad T_{j+1}(P^{(j+1)})(D) \subset P^{(j+1)}(D) .$$

When $j = 1$, conditions (30)-(32) immediately follow by regarding P in Lemma 2 as an identity mapping. Also from (31) and Lemma 4, we get (33).

Therefore, by induction, it follows that $\lim_{n_\nu\to\infty} P^{(\nu)}_{n_\nu} x = P^{(\nu)} x \in P^{(\nu)}(D) = \bigcap_{i=1}^{\nu} F[T_i]$. This completes the proof of the theorem.

From the finite intersection property, we have the following result due to DeMarr [2]. And note that we do not assume Zorn's lemma in our proof.

COROLLARY 1. *Let $T_i (i \in J, J$ is an index set) be commuting nonexpansive mapping from a compact convex subset of a Banach space into itself. Then there exists a point u in D such that $T_i u = u$ for all $i \in J$.*

When $\nu = 1$ and $\alpha_1 = 1/2$, we have the following corollary, which is essentially equal to the result we have obtained as a Corollary 2 in [3].

COROLLARY 2. *Let T be a nonexpansive mapping from a compact convex subset D of a Banach space into itself. Then* $\{((I + T)/2)^n x\}_{n=1}^{\infty}$ *converges to a fixed point of T.*

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