PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

GENERALIZED RAMSEY THEORY IX: ISOMORPHIC FACTORIZATIONS IV: ISOMORPHIC RAMSEY NUMBERS

FRANK HARARY AND ROBERT W. ROBINSON

The ramsey number of a graph G with no isolates has been defined as the minimum p such that every 2-coloring of (the lines of) the complete graph K_p contains a monochromatic G. An isomorphic factorization of K_p is a partition of its lines into isomorphic subgraphs. Combining these concepts, we define the isomorphic ramsey number of G as the minimum p such that for all $n \geqq p$, every 2-coloring of K_n which induces an isomorphic factorization contains a monochromatic G. The isomorphic ramsey numbers of all the small graphs (with at most four points) are determined. The extension to $c > 2$ colors is also studied.

1. Introduction. The classical ramsey number, which stems from the pioneering theorem of Ramsey [16], is written $r(K_m, K_n)$ and is defined as the minimum p such that every 2-coloring of (the lines of) K_p contains a red K_m or a green K_n. In the first paper in our series on generalized ramsey theory for graphs [3], the number $r(F, H)$ was defined by analogy for any two graphs with no isolates. We write $r(F)$ for $r(F, F)$ and we define a *small graph* as having at most four points. The numbers $r(F)$ and the numbers $r(F, H)$ were computed for small graphs in [1] and [2], the next two papers in this series. In [10], we considered the minimum possible number of monochromatic copies of F in $K_{r(F)}$. Then the ramsey number of a digraph was introduced in [8] while the ramsey number of a plex (a pure 2-dimensional simplicial complex) was studied in [4]. It was shown in [13] that no further ramsey numbers can arise from the study of the ramsey number of a network, that is, whenever the ramsey number of a network exists, it is equal to that of the underlying graph. For a given F, the smallest number of lines in a graph G such that every 2-coloring of G has a monochromatic F, is the subject of [9].

In our first paper]11] in the second series of the title, we defined an *isomorphic factorization* of a graph G as a partition of its line set $E(G)$ into isomorphic subgraphs $F_1, F_2, \cdots, F_t \cong F$. We then write $F|G$ and $F \in G/t$. Obviously if G/t is not empty, then $t|q(G)$, the number of lines of G. We proved in [11] that the converse of this necessary condition holds for complete graphs.

Divisibility theorem for complete graphs. If $t|p(p-1)/2$, then

K_p/t is not empty.

Analogous considerations for complete multipartite graphs were studied in [12]. The equivalence of isomorphic factorizations of graphs with combinatorial designs of several different varieties was pointed out in [14].

Our present purpose is to combine these two topics, both of which involve partitioning the line set of a graph. The *isomorphic ramsey number* (for two colors) of a given graph G is written $f(G)$ and is defined as the minimum p such that for all $n \geqq p$, every 2-coloring of the lines of K_n which constitutes an isomorphic factorization contains a monochromatic G. That is, every graph $H \in K_n/2$ contains G as a subgraph.

In the next section we find the isomorphic ramsey numbers of all the small graphs. We then consider isomorphic ramsey numbers for c colors with $c > 2$.

2. Isomorphic ramsey numbers. For completeness Fig. 1 shows the ten small graphs (which have no isolated points). The notation is the same as in [6] and [1] except that e is written for an arbitrary line.

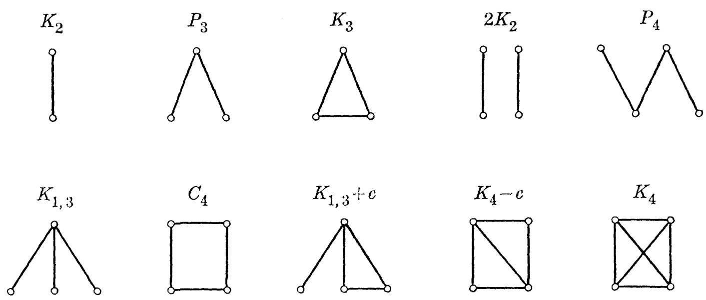

FIGURE 1. The ten small graphs

We state both their ramsey numbers $r(G)$ and their isomorphic ramsey numbers $f(G)$ in Table 1, and then justify them.

Table 1

G	K_2	P_3	K_3	$2K_2$	P_4	$K_{1,3}$	C_4	$K_{1,3}+e$	K_4-e	K_4
$r(G)$	2	3	6	5	5	6	6	7	10	18
$f(G)$	2	2	6	2	2	6	6	6	10	18

The ramsey numbers $r(G)$ in Table 1 were determined in [1]. It is important to note the obvious fact that $f(G)$ is at most $r(G)$. For $p \geqq r(G)$ implies that every 2-coloring of K_p contains a monochromatic G, and so *a fortiori* does every 2-coloring which gives an isomorphic factorization of G. It should also be noted that the residue of $f(G)$ modulo 4 can only be 1 or 2. This is because the number of lines $\binom{p}{2}$ in K_p is odd if p is 2 or 3 modulo 4, and so there are no isomorphic factorizations of K_p into two colors in these cases.

In order to verify some of the values of $f(G)$ in Table 1, we shall need to refer to the unique self-complementary graph of order 4, namely P_4, and the two self-complementary graphs of order 5, namely the cycle C_5 and the graph called A in Fig. 2 because of its typographical appearance.

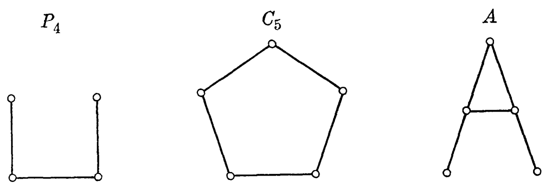

FIGURE 2. Three self-complementary graphs

1. $f(K_2) = 2$. This follows from the fact that $K_2 \not\subset K_1$ and $f(K_2) \leqq r(K_2) = 2$.

2. $f(P_3) = 2$. Similarly we have $P_3 \not\subset K_1$ and $f(P_3) \leqq r(P_3) = 3$, but we have noted that $r(P_3)$ cannot be 3 modulo 4.

3. $f(K_3) = 6$. To see this, note that although $K_3 \subset A$, $K_3 \not\subset C_5$, hence $f(K_3) \geqq 6$. But $r(K_3) = 6$ so $f(K_3) = 6$.

4. $f(2K_2) = 2$. As $2K_2 \subset P_4$ and $r(2K_2) = 5$ and $f(2K_2)$ cannot be 3, it follows that $f(2K_2) = 2$.

5. $f(P_4) = 2$. This verification is parallel to that of $2K_2$.

6. $f(K_{1,3}) = 6$. The reasoning is identical to that for $f(K_3) = 6$.

7. $f(C_4) = 6$. This is similar to both $f(K_3) = 6$ and $f(K_{1,3}) = 6$.

8. $f(K_{1,3} + e) = 6$. As $r(K_{1,3} + e) = 7$ which is a forbidden isomorphic ramsey number, it follows that $f(K_{1,3} + e) \leqq 6$. But $K_{1,3} + e \not\subset C_5$ hence this value is 6.

9. $f(K_4 - e) = 10$. The 9-point graph L of Fig. 3 does not contain a copy of $K_4 - e$. It is straightforward to verify that L is self-complementary. It now follows from $r(K_4 - e) = 10$ that $f(K_4 - e) = 10$.

10. $f(K_4) = 18$. We need to appeal to the ingenious construction by which Greenwood and Gleason [5] established that $r(K_4) > 17$. They took the field Z_{17} as the point set of K_{17}, that is, the numbers $0, 1, 2, \cdots, 16$. They then colored the line joining points i and j red if $i - j$ is a quadratic residue; otherwise the line is colored green. In this 2-coloring of K_{17} there is no monochromatic K_4. In

L:

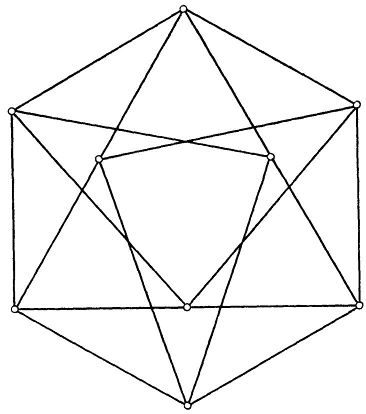

FIGURE 3. The line graph of $K_{3,3}$

addition, one sees that the red and green graphs are isomorphic on multiplying all the elements of Z_{17} by 3. Hence $f(K_4) \geqq 18$, but $r(K_4) = 18$ so we are done.

3. Ramsey numbers involving three colors.

Our object is to establish the values of isomorphic ramsey numbers with three colors for all six of the small graphs having at most three lines. We list the results in tabular form and include the 3-color ramsey numbers of these graphs.

Table 2

G	K_2	P_3	$2K_2$	P_4	$K_{1,3}$	K_3
$r(G;\ 3)$	2	5	5	6	8	17
$f(G;\ 3)$	2	5	5	5	8	17

Much as before, $f(G; 3)$ is not greater than $r(G;\ 3)$. Also $f(G; 3)$ cannot take a value which is congruent to zero modulo 3 as there can be no isomorphic factorization of K_p with three colors if p is congruent to 2 modulo 3.

The numbers $r(G;\ 3)$ in Table 2 are very easy to verify except for $r(K_3;\ 3) = 17$, due to Greenwood and Gleason [5].

$$f(K_2;\ 3) = 2 \quad \text{and} \quad f(P_3;\ 3) = f(2K_2;\ 3) = 5$$

The first of these is trivial. For the remaining two numbers, we show in Fig. 4 the two isomorphic factorizations of K_4 into three parts.

The factorization $2K_2|K_4$ proves that $f(P_3;\ 3) > 4$; on the other hand $(K_1 \cup P_3)|K_4$ implies $f(2K_2;\ 3) > 4$. Since the 3-color ramsey numbers of both P_3 and $2K_2$ are 5, it follows that both their isomorphic ramsey numbers are 5.

FIGURE 4. The graphs in $K_4/3$

$$f(P_4;\ 3) = 5$$

This number is more than 4 by Fig. 4, and is at most 6 by $f(P_4;\ 3) \leqq r(P_4;\ 3) = 6$. But 6 is impossible since it is divisible by 3, so the number is 5.

$$f(K_{1,3};\ 3) = 8$$

The well-known isomorphic factorization of K_7 into three copies of C_7 shows this number to be at least 8. It is at most 8 because $r(K_{1,3};\ 3) = 8$.

$$f(K_3;\ 3) = 17$$

The proof that $r(K_3;\ 3) = 17$ is given in Greenwood and Gleason [5], so this number is at most 17. Their method of showing $r(K_3; 3) > 16$ begins by taking the points of K_{16} as the members of the field of order 16. The nonzero elements of $GF\,[16]$ are partitioned into the three multiplicative cosets of the set of five nonzero cubes in $GF\ [16]$. The edge joining i to j for $i \neq j$ is colored according to the coset containing $i - j$. They verified that there is no monochromatic K_3 in the resulting 3-coloring of K_{16}. It is seen at once that the three factors of K_{16} are isomorphic by the map obtained from multiplying each member of $GF\ [16]$ by a fixed element which is not a cube in this field. Therefore $f(K_3;\ 3)$ is greater than 16, and so must be exactly 17.

It has been shown by Kalbfleisch and Stanton [15] that there are exactly two different isomorphic factorizations of K_{16} into three parts not containing K_3. In the literature of combinatorial designs, these are called proper colorings. Whitehead [18] showed how to obtain the second proper coloring of K_{16} from a sumfree set in $Z_4 \oplus Z_4$. Street gives a complete account of the construction of the proper colorings of K_{16}, including the remarkable fact that the individual factors in the two factorizations are all isomorphic to one another [17, Lemma 8.3] and an elegant drawing of this unique factor graph.

4. Unsolved problems

A. There are many other ramsey numbers which have been determined, including those for stars, paths, cycles, and other graphs. Results and references can be found in [7]. What are the corresponding isomorphic ramsey numbers?

B. It is conceivable that there exists a graph G for which $f(G)$ can be found without knowing $r(G)$. Is there any such graph?

C. It is immediate that if $n < r(G)$ then some 2-coloring of K_n avoids a monochromatic G. We conjecture the corresponding statement for isomorphic ramsey numbers. That is, if $n < f(G)$ and $K_n/2$ is not empty then not every graph in $K_n/2$ contains G.

This would follow at once if it could be shown that if $n < p$ and $K_n/2$ is not empty then every member of $K_p/2$ contains a member of $K_n/2$. This seems highly plausible, and is a well-known fact when $p = n + 1$. However since $K_n/2$ is empty for $n \equiv 2$ or 3 (modulo 4), this fact alone is insufficient to prove the desired result.

D. We also make the conjecture for isomorphic ramsey numbers involving any $c > 2$ colors which is analogous to the preceding conjecture. That is, if $n < f(G; c)$ and K_n/c is not empty, then not every graph in K_n/c contains G.

E. It would be wonderful if a convenient general method could be found for determining $r(G)$ or $f(G)$ for arbitrary G. We doubt it because of the intrinsically intransigent nature of the problem.

Our methods can be applied to evaluate the isomorphic ramsey numbers of several families of graphs, including the paths P_n, the stars $K_{1,n}$ and the bars nK_2. We plan to present this in a forthcoming communication.

References

1. V. Chvátal and F. Harary, *Generalized Ramsey theory for graphs II: Small diagonal numbers*, Proc. Amer. Math. Soc., **32** (1972), 389-394.

2. ———, *Generalized ramsey theory for graphs III: Small off-diagonal numbers*, Pacific J. Math., **41** (1972), 335-345.

3. ———, *Generalized ramsey theory for graphs I: Diagonal numbers*, Periodica Math. Hungar., **3** (1973), 115-124.

4. R. M. Duke and F. Harary, *Generalized ramsey theory VI: Ramsey numbers for small plexes*, J. Austral. Math. Soc., **29** (1976), 400-410.

5. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math., **7** (1955), 1-7.

6. F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass. (1969).

7. ———, *The foremost open problems in generalized ramsey theory*, Proceedings of the Fifth British Combinatorial Conference (C. Nash-Williams and J. Sheehan, eds.) Utilitas Math. Publishing, Winnipeg (1976), 269-282.

8. F. Harary and P. Hell, *Generalized ramsey theory for graphs V: The ramsey number of a digraph*, Bull. London Math. Soc., **6** (1974), 175-182, **7** (1975), 87-88.

9. F. Harary and Z. Miller, *Generalized ramsey theory for graphs VIII: The edge ramsey number*. Acta Math. Hungar. (1980), to appear.

10. F. Harary and G. Prins, *Generalized ramsey theory for graphs IV: The ramsey multiplicity of a graph*, Networks, **4** (1974), 163-173.

11. F. Harary, R. W. Robinson and N. C. Wormald, *Isomorphic factorizations I: Complete graphs*, Trans. Amer. Math. Soc., **242** (1978), 243-260.
12. ———, *Isomorphic factorizations III: Complete multipartite graphs*, Combinatorial Math. IV, Springer Lecture Notes 686, Berlin (1978), 47-54.
13. F. Harary and A. J. Schwenk, *Generalized ramsey theory VII: Multigraphs and networks*, Networks, **8** (1978), 209-216.
14. F. Harary and W. D. Wallis, *Isomorphic factorizations II: Combinatorial designs*, Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing. Utilitas Math. Publishing, Winnipeg, (1978), 13-28.
15. J. G. Kalbfleisch and R. G. Stanton, *On the maximal triangle-free edge-chromatic graphs in three colors*, J. Combinatorial Theory, **5** (1968), 9-20.
16. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc., **30** (1930), 264-286.
17. A. P. Street, *Sum-free sets, Combinatorics*, (by W. D. Wallis, A. P. Street and J. S. Wallis), Springer Lecture Notes 292, Berlin (1972), 123-271.
18. E. G. Whitehead, *Algebraic structure of chromatic graphs associated with the Ramsey number* $N(3, 3, 3; 2)$, Discrete Math., **1** (1971), 113-114.

Received May 23, 1978.

UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109
AND
UNIVERSITY OF NEWCASTLE
NEWCASTLE N.S.W. 2308, AUSTRALIA