A CUSHIONING-TYPE WEAK COVERING PROPERTY

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We discuss certain covering properties which are based
on Michael's notion of a cushioned collection. In particular,
we discuss property L of Bacon and certain variations on
property L in connection with isocompactness and the rela-
tionship between $\aleph_1$-compactness and the Lindelöf property.
We then introduce property θL which is a common gener-
alization of property L and weak $\delta\theta$-refinability, and consider
uses of this property in similar connections.

0. Introduction. There are a large number of covering
properties generalizing paracompactness which have been the subjects
of investigation in the last thirty years. Most of these involve
some sort of generalization of the notion of locally finite refinements.
For example, the metacompact, meta-Lindelöf, θ-refinable, weakly
θ-refinable, $\delta\theta$-refinable and weakly $\delta\theta$-refinable spaces are all classes
defined by covering properties of this type.

In 1970, Bacon presented a covering property, property L, which
is based on a generalization of Michael's characterization of para-
compactness in terms of cushioned refinements. We present certain
variations on property L and discuss how these may be used as
hypotheses in place of weak $\delta\theta$-refinability in a number of theorems.

We present a new covering property, property θL, which is a
common generalization of property L and weak $\delta\theta$-refinability and
still implies isocompactness [every closed countably compact subset is
compact]. We discuss certain variations on property θL, and find
that many of the results obtained using property L or weak $\delta\theta$-
refinability remain true when one of these variations is used in place
of property L or weak $\delta\theta$-refinability in the hypothesis. In particular,
we establish a number of results relating $\aleph_1$-compactness with the
Lindelöf property and with closed completeness.

We now list certain conventions which will be used in this
paper. A perfect mapping is a closed continuous function with
the property that the inverse image of each point in the range is
compact. We indicate a function f whose domain is the set A and
whose range is contained in the set B by $f: A \to B$. For a collection
$\mathscr{A}$ of sets, $\cup \mathscr{A} = \cup \{A: A \in \mathscr{A}\}$ and $\operatorname{ord}(x, \mathscr{A}) = |\{A: x \in A \in \mathscr{A}\}|$.
If $f: X \to Y$ and $A \subset X$, we write $f(A)$ to indicate the set $\{f(x): x \in A\}$.

We include for the benefit of the reader the following defini-
tions. Original sources are listed in [16].

DEFINITION 0.1. Suppose X is a space and $\mathscr{W}$ is an open cover

of X of the form $\mathscr{W} = \bigcup_{n \in \omega} \mathscr{V}_n$. Consider the following conditions on $\mathscr{W}$:

(i) For each $n \in \omega$, $\mathscr{V}_n$ covers X.

(ii) For each $x \in X$, there exists $n_x \in \omega$ such that $0 < \operatorname{ord}(x, \mathscr{V}_{n_x}) < \aleph_0$.

(iii) For each $x \in X$, there exists $n_x \in \omega$ such that $0 < \operatorname{ord}(x, \mathscr{V}_{n_x}) \leqq \aleph_0$.

(A) If every open cover of X has a refinement $\mathscr{W}$ satisfying (iii), then X is called *weakly $\delta\theta$-refinable*.

(B) If every open cover of X has a refinement $\mathscr{W}$ satisfying (i) and (iii), then X is called *$\delta\theta$-refinable*.

(C) If every open cover of X has a refinement $\mathscr{W}$ satisfying (ii), then X is called *weakly θ-refinable*.

(D) If every open cover of X has a refinement $\mathscr{W}$ satisfying (i) and (ii), then X is called *θ-refinable*.

1. Property L. In [12], Michael characterized paracompactness in terms of open covers having cushioned refinements. In this section, we present a family of covering properties which generalize paracompactness using a generalization of the notion of a cushioned refinement. We denote by Card the class of all infinite cardinal numbers.

DEFINITION 1.1. For $\kappa \in$ Card, and $\mathscr{U}$ and $\mathscr{V}$ collections of subsets of a space X, we say $\mathscr{V}$ is *κ-weakly cushioned* in $\mathscr{U}$ if and only if there exists a function $f \colon \mathscr{V} \to \mathscr{U}$ such that if $\mathscr{G} \subset \mathscr{V}$ with $|\mathscr{G}| \leqq \kappa$ and $x \colon \mathscr{G} \to \cup \mathscr{G}$ with $x(G) \in G$ for each $G \in \mathscr{G}$, then $\overline{\{x(G) \colon G \in \mathscr{G}\}} \subset \cup f(\mathscr{G})$.

We say a space X satisfies *property κL* if and only if for every open cover $\mathscr{U}$ of X there is a sequence $\langle \mathscr{D}_n \colon n \in \omega \rangle$ of collections of subsets of X such that $\bigcup_{n \in \omega} \mathscr{D}_n$ is a covering of X, and for each $n \in \omega$, $\mathscr{D}_n$ is κ-weakly cushioned in $\omega \mathscr{U}$. ($\omega \mathscr{U}$ is the set of unions of countable subcollections of $\mathscr{U}$.)

When $\kappa = \aleph_0$, we have *property L* as defined by Bacon in [2].

THEOREM 1.2. [2] *Every space which satisfies property L is isocompact.*

REMARK. If $\alpha \leqq \beta \in$ Card and X satisfies property βL, then X satisfies property αL.

The following statement is clear from the definition, but we state it here since it implies that within the class of spaces having countable tightness, and thus the class of sequential spaces, we need

only consider property L in attempting to establish relationships between classes. Recall that a space X is said to have countable tightness provided that whenever $A \subset X$ and $x \in \bar{A}$, then there is a countable set $C \subset A$ with $x \in \bar{C}$

THEOREM 1.3. *If X is a space with countable tightness, then for each $\kappa \in \mathrm{Card}$, X satisfies property κL if and only if X satisfies property L.*

In many cases, the members of a class of spaces satisfy property κL for every $\kappa \in \mathrm{Card}$. We demonstrate this fact for certain well known classes in the next few paragraphs. We see in 1.4 that symmetrizable spaces satisfy property κL for every $\kappa \in \mathrm{Card}$. This is not true of certain weakenings of symmetrizability, see [5], [6].

THEOREM 1.4. [5] *Every symmetrizable space satisfies property κL for every $\kappa \in \mathrm{Card}$.*

THEOREM 1.5. *Every $\delta\theta$-refinable space satisfies property κL for every $\kappa \in \mathrm{Card}$.*

Proof. Suppose $\mathscr{U}$ is an open cover of the $\delta\theta$-refinable space X. Let $\bigcup_{n \in \omega} \mathscr{V}_n$ be a $\delta\theta$-refinement of $\mathscr{U}$, i.e., for each $n \in \omega$, $\mathscr{V}_n$ is an open refinement of $\mathscr{U}$ and covers X, and for each $x \in X$, there is $n_x \in \omega$ such that $\mathrm{ord}\,(x, \mathscr{V}_{n_x}) \leq \aleph_0$. For each $V \in \bigcup_{n \in \omega} \mathscr{V}_n$, choose $U_V \in \mathscr{U}$ with $V \subset U_V$. For each $n \in \omega$, we define $\mathscr{D}_n = \{\{x\}: \mathrm{ord}\,(x, \mathscr{V}_n) \leq \aleph_0\}$ and we define $f_n: \mathscr{D}_n \to \omega\mathscr{U}$ by $f_n(\{x\}) = \cup\{U_V: x \in V \in \mathscr{V}_n\}$. Suppose $\mathscr{G} \subset \mathscr{D}_n$, and $x: \mathscr{G} \to \cup\mathscr{G}$ with $x(G) \in G$ for every $G \in \mathscr{G}$, and let $y \in \overline{\{x(G): G \in \mathscr{G}\}}$. Choose $V \in \mathscr{V}_n$ with $y \in V$. Then there exists $G \in \mathscr{G}$ with $x(G) \in V$. Thus $y \in U_V \subset f_n(G)$, and so we have $\overline{\{x(G): G \in \mathscr{G}\}} \subset \cup f_n(\mathscr{G})$. Since no reference is made to $|\mathscr{G}|$, this completes the proof.

It follows from 1.5 that all semistratifiable [4] and all regular σ-spaces [14] satisfy property κL for every $\kappa \in \mathrm{Card}$. We mention these since they are classes which are of interest to many topologists. On the other hand, the ordinal space ω_1 does not satisfy property κL for any $\kappa \in \mathrm{Card}$, thus we cannot get an analogue to 1.5 for the M-spaces [13], or p-spaces [1], for instance.

We see in the next theorem that each of the properties κL for $\kappa \in \mathrm{Card}$ is preserved by closed continuous functions.

THEOREM 1.6. *If $f: X \to Y$ is a closed continuous surjection, then for each $\kappa \in \mathrm{Card}$, if X satisfies property κL, then so does Y.*

Proof. Suppose $\mathscr{U}$ is an open cover of Y. Then $\mathscr{U}^* = \{f^{-1}(U):$ $U \in \mathscr{U}\}$ is an open cover of X. Since X satisfies property κL, we may choose a sequence $\langle \mathscr{D}_n^*: n \in \omega \rangle$ of collections of subsets of X such that $\bigcup_{n \in \omega} \mathscr{D}_n^*$ covers X and, for each $n \in \omega$, $\mathscr{D}_n^*$ is κ-weakly cushioned in $\omega \mathscr{U}^*$. For each $n \in \omega$, we let $\mathscr{D}_n = \{f(D^*): D^* \in \mathscr{D}_n^*\}$. Since f is surjective, $\bigcup_{n \in \omega} \mathscr{D}_n$ covers Y. Suppose $n \in \omega$. For each $D \in \mathscr{D}_n$, pick $A_D \in \mathscr{D}_n^*$ such that $D = f(A_D)$. Suppose $\phi_n^*: \mathscr{D}_n^* \to \omega \mathscr{U}^*$ is the function by which $\mathscr{D}_n^*$ is κ-weakly cushioned in $\omega \mathscr{U}^*$. We define $\phi_n: \mathscr{D}_n \to \omega \mathscr{U}$ by $\phi_n(D) = f(\phi_n^*(A_D))$ for each $D \in \mathscr{D}_n$. Suppose $n \in \omega$, $\mathscr{G} \subset \mathscr{D}_n$ with $|\mathscr{G}| \leq \kappa$, and $y: \mathscr{G} \to \bigcup \mathscr{G}$ such that $y(G) \in G$ for every $G \in \mathscr{G}$. For each $G \in \mathscr{G}$ pick $x(G) \in A_G$ such that $f(x(G)) = y(G)$. Then

$$
\begin{aligned}
\overline{\{y(G): G \in \mathscr{G}\}} &= \overline{\{f(x(G)): G \in \mathscr{G}\}} \\
&= \overline{f(\{x(G): G \in \mathscr{G}\})} \\
&= f(\overline{\{x(G): G \in \mathscr{G}\}}) \\
&\subset f(\bigcup \phi_n^*(\{A_G: G \in \mathscr{G}\})) \\
&= f(\bigcup \{\phi_n^*(A_G): G \in \mathscr{G}\}) \\
&= \bigcup \{f(\phi_n^*(A_G)): G \in \mathscr{G}\} \\
&= \bigcup \{\phi_n(G): G \in \mathscr{G}\} .
\end{aligned}
$$

The third equality holds since f is a closed continuous function. The inclusion holds since $|\{A_G: G \in \mathscr{G}\}| \leq |\mathscr{G}| \leq \kappa$. The other relations are obvious, and the theorem is proved.

We see from the next theorem that each of the properties κL for $\kappa \in \text{Card}$ is preserved by perfect preimages. Thus, for perfect mappings, we have property κL preserved in both directions, in view of 1.6.

THEOREM 1.7. *If* $f: X \to Y$ *is a closed continuous surjection and, for each* $y \in Y$, $f^{-1}(\{y\})$ *is Lindelöf, then for each* $\kappa \in \text{Card}$, *if* Y *satisfies property* κL, *then so does* X.

Proof. Let $\mathscr{U}$ be an open cover of X. For each $y \in Y$, there is an element U of $\omega \mathscr{U}$ which contains $f^{-1}(\{y\})$. Since f is a closed continuous function, there exists an open subset V of Y containing y such that $f^{-1}(V) \subset U$. Thus we have an open refinement $\mathscr{W}$ of $\omega \mathscr{U}$ of the form $\mathscr{W} = \{f^{-1}(V): V \in \mathscr{V}\}$, where $\mathscr{V}$ is an open cover of Y. Apply property κL to $\mathscr{V}$ to obtain sequences $\langle \mathscr{D}_n: n \in \omega \rangle$ and $\langle \phi_n: n \in \omega \rangle$ with $\mathscr{D}_n$ κ-weakly cushioned in $\omega \mathscr{V}$ by ϕ_n for each $n \in \omega$. For each $n \in \omega$, we let $\mathscr{H}_n = \{f^{-1}(D): D \in \mathscr{D}_n\}$ and define $\psi_n: \mathscr{H}_n \to \omega \mathscr{W}$ by $\psi_n(f^{-1}(D)) = f^{-1}(\phi_n(D))$. Clearly, $\bigcup_{n \in \omega} \mathscr{H}_n$ covers X. We shall show that for each $n \in \omega$, $\mathscr{H}_n$ is κ-weakly cushioned in $\omega \mathscr{W}$ and

thus in $\omega\mathscr{U}$. Suppose $\mathscr{G} \subset \mathscr{H}_n$ with $|\mathscr{G}| \leq \kappa$. Then there exists $\mathscr{E} \subset \mathscr{D}_n$ with $|\mathscr{E}| \leq \kappa$ such that $\mathscr{G} = \{f^{-1}(D): D \in \mathscr{E}\}$. Suppose $x: \mathscr{G} \to \cup \mathscr{G}$ with $x(G) \in G$ for each $G \in \mathscr{G}$ and $z \in \overline{\{x(G): G \in \mathscr{G}\}}$. Since $z \in \overline{\{x(f^{-1}(D)): D \in \mathscr{E}\}}$, we have that $f(z) \in \overline{\{f(x(f^{-1}(D))): D \in \mathscr{E}\}} \subset \cup \phi_n(\mathscr{E})$. Thus $z \in f^{-1}(\cup \phi_n(\mathscr{E})) = \cup\{f^{-1}(\phi_n(D)): D \in \mathscr{E}\} = \cup \psi_n(\mathscr{G})$. This completes the proof.

COROLLARY 1.7.1. *For every $\kappa \in Card$, property κL is preserved by perfect preimages.*

2. Property θL. The similarity between the type of results which one may obtain using property L and weak $\delta\theta$-refinability (see [5], [17]) leads one to believe that there must be some relationship between these properties. However, Examples 3.3 and 3.4 of [6] are weakly $\delta\theta$-refinable (in fact, weakly $\bar{\theta}$-refinable [15]) spaces which do not satisfy property L. Further, for each $\kappa \in Card$, the set κ^+ with the topology generated by the subsets whose compliment's cardinality does not exceed κ, i.e., the co-κ topology, is a space which satisfies property κL and is not $\delta\theta$-refinable. Gary Gruenhage has recently constructed an example, assuming the continuum hypothesis, of a space with a point countable base (hence, satisfying property κL for every $\kappa \in Card$) which is not weakly θ-refinable. At this time, no example is known to the author of a space which satisfies property L and is not weakly $\delta\theta$-refinable. It seems that such a space must surely exist however.

We now define and discuss a property which generalizes both property L and weak $\delta\theta$-refinability.

DEFINITION 2.1. For $\kappa \in Card$, we say a space X satisfies *property $\theta\kappa L$* if and only if for every open cover $\mathscr{U}$ of X there exists a sequence $\langle \mathscr{D}_n: n \in \omega \rangle$ of collections of subsets of X and a sequence $\langle \mathscr{V}_n: n \in \omega \rangle$ of open refinements of $\mathscr{U}$ such that $\bigcup_{n \in \omega} \mathscr{D}_n$ covers X and for each $n \in \omega$, $\cup \mathscr{D}_n \subset \cup \mathscr{V}_n$ and $\mathscr{D}_n$ is κ-weakly cushioned in $\omega \mathscr{V}_n$ *in the space* $\cup \mathscr{V}_n$.

In analogy with the treatment of property $\aleph_0 L$, we shall refer to property $\theta \aleph_0 L$ by property θL. No confusion will arise between the notations θL and κL since we shall never use the symbol θ to describe the cardinality of a collection.

Below we list a few results which are immediate from the definitions of these properties.

PROPOSITION 2.1.1.

(a) *For each $\kappa \in Card$, if X satisfies property $\theta\kappa L$ and A is an F_σ-subset of X, then A satisfies property $\theta\kappa L$.*

(b) *For each $\kappa \in Card$, if X has countable tightness, then X satisfies property $\theta\kappa L$ if and only if X satisfies property θL.*

(c) *If $\alpha, \beta \in Card$ and $\alpha \leq \beta$, then any space which satisfies property $\theta\beta L$ also satisfies property $\theta\alpha L$.*

THEOREM 2.2. *Every weakly $\delta\theta$-refinable space satisfies property $\theta\kappa L$, for every $\kappa \in Card$.*

Proof. Suppose X is a weakly $\delta\theta$-refinable space and $\mathscr{U}$ is an open cover of X. Let $\bigcup_{n \in \omega} \mathscr{V}_n$ be a weak $\delta\theta$-refinement of $\mathscr{U}$, i.e., $\bigcup_{n \in \omega} \mathscr{V}_n$ is an open refinement of $\mathscr{U}$ which covers X and for each $x \in X$, there is $n_x \in \omega$ such that $0 < \text{ord}(x, \mathscr{V}_{n_x}) \leq \aleph_0$. Let $\mathscr{D}_n = \{\{x\}: 0 < \text{ord}(x, \mathscr{V}_n) \leq \aleph_0\}$, for each $n \in \omega$. Define $f_n: \mathscr{D}_n \to \omega \mathscr{V}_n$ by $f_n(\{x\}) = \bigcup\{V: x \in V \in \mathscr{V}_n\}$, for each $n \in \omega$. Suppose $\mathscr{G} \subset \mathscr{D}_n$, $x: \mathscr{G} \to \bigcup \mathscr{G}$ with $x(G) \in G$ for each $G \in \mathscr{G}$, and $z \in \overline{\{x(G): G \in \mathscr{G}\}} \cap \bigcup \mathscr{V}_n$. Since $z \in \bigcup \mathscr{V}_n$, we may choose $V \in \mathscr{V}_n$ with $z \in V$. Then there exists $G \in \mathscr{G}$ such that $x(G) \in V$. Thus $z \in V \subset f_n(\{x(G)\})$. Since $|\mathscr{G}|$ is not considered in the above argument, we have that $\mathscr{D}_n$ is κ-weakly cushioned in $\omega \mathscr{V}_n$ in the space $\bigcup \mathscr{V}_n$, for every $\kappa \in Card$.

THEOREM 2.3. *For each $\kappa \in Card$, every space which satisfies property κL also satisfies property $\theta\kappa L$.*

Proof. If X satisfies property κL and $\mathscr{U}$ is an open cover of X, then we let the $\mathscr{V}_n$ in the definition of property $\theta\kappa L$ be $\mathscr{U}$ for each $n \in \omega$. The theorem clearly follows.

THEOREM 2.4. *Every space which satisfies property θL is isocompact.*

Proof. Since property θL is clearly closed hereditary, we need only show that every countably compact space which satisfies property θL is compact.

Suppose X is a space which satisfies property θL and $\mathscr{U}$ is an open cover of X. Let $C(\mathscr{U})$ be the collection of all subsets of X which are covered by a countable subcollection of $\mathscr{U}$. Apply the definition of property θL to $\mathscr{U}$ to obtain sequences $\langle \mathscr{D}_n: n \in \omega \rangle$, $\langle \mathscr{V}_n: n \in \omega \rangle$ and $\langle f_n: n \in \omega \rangle$ such that, for each $n \in \omega$, $\mathscr{V}_n$ is an open refinement of $\mathscr{U}$, $\bigcup \mathscr{D}_n \subset \bigcup \mathscr{V}_n$, and $\mathscr{D}_n$ is a collection of subsets of X which is $\aleph_0$-weakly cushioned in $\omega \mathscr{V}_n$ is the space $\bigcup \mathscr{V}_n$ by the function $f_n: \mathscr{D}_n \to \omega \mathscr{V}_n$, and $\bigcup_{n \in \omega} \mathscr{D}_n$ covers X. Let $C_n = \bigcup \mathscr{D}_n$ for each $n \in \omega$. If $X \in C(\mathscr{U})$, then $\mathscr{U}$ contains a finite subcover. Therefore, we assume $X \notin C(\mathscr{U})$. We define n_0 to be the first element of the set $\{m: C_m \notin C(\mathscr{U})\}$. Let $E_0 = X \setminus \bigcup \mathscr{M}$, where $\mathscr{M} \subset \mathscr{U}$ is countable

and covers C_j for each $j < n_0$. Suppose $k < \omega$ and E_j, n_j are chosen for $j \leq k$ such that:

(1) E_j is closed.

(2) $n_j = $ first element of the set $\{m: E_j \cap C_m \notin C(\mathcal{U})\}$.

(3) $E_{j+1} \subset E_j$, whenever E_{j+1} is defined.

(4) $E_j \cap C_m = \varnothing$, for $m < n_j$.

Let $B = E_k \backslash \cup \mathscr{V}_{n_k}$. We claim that $B \notin C(\mathcal{U})$. Suppose $\mathscr{H} \subset \mathcal{U}$ is countable and $B \subset \cup \mathscr{H}$. Since $E_k \cap C_{n_k} \notin C(\mathcal{U})$, it follows that there exists $a_0 \in (E_k \cap C_{n_k}) \backslash \cup \mathscr{H}$. Choose $D_0 \in \mathscr{D}_{n_k}$ such that $a_0 \in D_0$. Let $U_0 = f_{n_k}(D_0)$. Suppose $m < \omega$ and a_j, D_j, U_j are chosen for $j \leq m$ such that:

(i) $a_j \in D_j \in \mathscr{D}_{n_k}$

(ii) $a_j \in (E_k \cap C_{n_k}) \backslash (\bigcup_{s<j} f_{n_k}(D_s) \cup \cup \mathscr{H})$.

(iii) $U_j = f_{n_k}(D_j) \backslash \overline{\{a_s: s < j\}}$.

The set $\bigcup_{j \leq m} f_{n_k}(D_j)$ is the union of a countable subcollection of $\mathscr{V}_{n_k}$. Hence there exists a countable subcollection $\mathscr{W}$ of $\mathcal{U}$ such that $\bigcup_{j \leq m} f_{n_k}(D_j) \cup \cup \mathscr{H} \subset \cup \mathscr{W}$. By (2) above, there exists $a_{m+1} \in (E_k \cap C_{n_k}) \backslash U \mathscr{W}$. Choose $D_{m+1} \in \mathscr{D}_{n_k}$ such that $a_{m+1} \in D_{m+1}$ and let $U_{m+1} = f_{n_k}(D_{m+1}) \backslash \overline{\{a_j: j \leq m\}}$. In this way we construct $\{a_m: m \in \omega\}$, $\{U_m: m \in \omega\}$ and $\{D_m: m \in \omega\}$. Suppose $x \in \overline{\{a_m: m \in \omega\}}$. By the construction, $x \notin \cup \mathscr{H}$, but $x \in E_k$ since E_k is closed. Thus $x \in (E_k \backslash \cup \mathscr{H}) \subset \cup \mathscr{V}_{n_k}$, and so $x \in \overline{\{a_m: m \in \omega\}} \cap \cup \mathscr{V}_{n_k}$. Now $\mathscr{D}_{n_k}$ is $\aleph_0$-weakly cushioned in $\omega \mathscr{V}_{n_k}$, in the space $\cup \mathscr{V}_{n_k}$; thus, $x \in \bigcup_{m \in \omega} f_{n_k}(D_m)$. It follows from the $\aleph_0$-weak cushioning of $\mathscr{D}_{n_k}$ in $\omega \mathscr{V}_{n_k}$ that $\{U_m: m \in \omega\}$ is a countable open cover of $\overline{\{a_m: m \in \omega\}}$ with no finite subcover. Of course, $\overline{\{a_m: m \in \omega\}}$ is a closed subset of X; hence, it is countably compact. This contradiction establishes that $B \notin C(\mathcal{U})$. There exists $m \in \omega$ such that $B \cap C_m \notin C(\mathcal{U})$. Let n_{k+1} be the first such m. Let $\mathscr{W}$ be a countable subcollection of $\mathcal{U}$ covering $B \cap C_j$ for each $j < n_{k+1}$. Let $E_{k+1} = B \backslash \cup \mathscr{W}$. The set $E_{k+1} = B \backslash \cup \mathscr{W} = (E_k \backslash \cup \mathscr{V}_{n_k}) \backslash \cup \mathscr{W}$ is clearly closed. By choice of n_{k+1} and by the observation above, (2) and (3) are satisfied. Since $(\bigcup_{j < n_{k+1}} C_j) \cap B \subset \cup \mathscr{W}$, condition (4) is satisfied. Note that $n_{k+1} > n_k$, by construction, and thus, by (4), $\cap \{E_k: k \in \omega\} = \varnothing$. However, $\langle E_k: k \in \omega \rangle$ is decreasing sequence of nonempty closed sets. This contradiction to the countable compactness of X establishes the result.

The above proof combines the ideas of the proof of the lemma in [5] with the proof in [16] of the isocompactness of weakly $\delta\theta$-refinable spaces. This style of proof can be used to obtain certain strengthenings of theorems of Blair [3] as we see in the following three theorems.

Recall that a space is called *closed-complete* [9] (respectively, *realcompact* [10]) in case every closed (respectively, z-) ultrafilter with the countable intersection property is fixed. (In addition, real-

compact spaces are required to be completely regular and T_2.)

THEOREM 2.5. *If X is a T_1 $\aleph_1$-compact space which satisfies property $\theta\aleph_1 L$, then X is closed-complete.*

Proof. Suppose $\mathscr{F}$ is a free closed ultrafilter on X which has the countable intersection property. Let $\mathscr{U} = \{X\backslash F: F \in \mathscr{F}\}$. The collection $\mathscr{U}$ is an open cover of X with no countable subcover. As we did in the proof of 2.4, we shall construct a sequence $\langle E_k: k \in \omega\rangle$ satisfying (1)-(4) of the proof of 2.4. Our proof here will differ (in the notation of the proof of 2.4) in the way we show that $B \in C(\mathscr{U})$. To see that this is true, we continue the construction used in the proof of 2.4 to obtain sets $\{a_\alpha: \alpha < \omega_1\}$, $\{U_\alpha: \alpha < \omega_1\}$, and $\{D_\alpha: \alpha < \omega_1\}$ such that:

    (i)   For each $\alpha < \omega_1$, $a_\alpha \in D_\alpha \in \mathscr{D}_{n_k}$.

    (ii) For each $\alpha < \omega_1$, $a_\alpha \in (E_k \cap C_{n_k})\backslash(\bigcup_{\beta<\alpha} f_{n_k}(D_\beta) \cup \cup \mathscr{H})$.

    (iii) For each $\alpha < \omega_1$, $U_\alpha = f_{n_k}(D_\alpha)\backslash\overline{\{a_\beta: \beta < \alpha\}}$.

Reasoning similar to that used in the proof of 2.4 shows that $\{U_\alpha: \alpha < \omega_1\}$ is an open cover of $\overline{\{a_\alpha: \alpha < \omega_1\}}$, but for each $\beta < \omega_1$, $U_\beta \cap \{a_\alpha: \alpha < \omega_1\} = \{a_\beta\}$ which contradicts the $\aleph_1$-compactness of X. For each $k \in \omega$, no countable subcollection of $\mathscr{U}$ covers E_k, and thus $E_k \cap F \neq \varnothing$ for each $F \in \mathscr{F}$. However, $\mathscr{F}$ is a closed ultrafilter, and so $E_k \in \mathscr{F}$ for every $k \in \omega$. But $\cap\{E_k: k \in \omega\} = \varnothing$ contradicting the countable intersection property and the result is established.

From the proof above and the proof of 2.2, we have the following analogue of Corollary 3.2 of [3].

COROLLARY 2.5.1. *If every uncountable discrete subset of X has a 2-limit point in X, then the following are equivalent:*

    (a) *X is closed complete.*

    (b) *If $\mathscr{F}$ is any free closed ultrafilter on X, then we may apply $\theta\aleph_1 L$ to $\{X\backslash F: F \in \mathscr{F}\}$. (In saying "we may apply $\theta\aleph_1 L$ to $\{X\backslash F: F \in \mathscr{F}\}$", we mean that if we let $\mathscr{U} = \{X\backslash F: F \in \mathscr{F}\}$ then we may obtain sequences $\langle \mathscr{D}_n: n \in \omega\rangle$ and $\langle \mathscr{V}_n: n \in \omega\rangle$ as in 2.1 with $\kappa = \aleph_1$.)*

THEOREM 2.6. *If X is an $\aleph_1$-compact T_1 closed subspace of a product of regular spaces which satisfy property $\theta\aleph_1 L$, then X is closed-complete.*

Proof. This can be established exactly as 3.4 of [3], since inverse projection mappings preserve weak cushioning.

A space X is a *cb-space* [11] provided for each decreasing sequence $\langle F_n: n \in \omega\rangle$ of closed subsets of X with $\bigcap_{n \in \omega} F_n = \varnothing$ there

is a sequence $\langle Z_n : n \in \omega \rangle$ of zero-sets of X such that $Z_n \supset F_n$ for each $n \in \omega$ and $\bigcap_{n \in \omega} Z_n = \varnothing$.

THEOREM 2.7. *If X is a T_2 completely regular $\aleph_1$-compact cb-space, then the following are equivalent.*

(a) *X is realcompact.*

(b) *X can be embedded as a closed subspace of a product of regular weakly $\delta\theta$-refinable spaces.*

(c) *If $\mathscr{F}$ is any free closed ultrafilter on X, then $\{X \backslash F : F \in \mathscr{F}\}$ has a weak $\delta\theta$-refinement.*

(d) *X can be embedded as a closed subspace of a product of regular spaces which satisfy property $\theta \aleph_1 L$.*

(e) *If $\mathscr{F}$ is any free closed ultrafilter on X, then we may apply $\theta \aleph_1 L$ to $\{X \backslash F : F \in \mathscr{F}\}$.*

If, in addition, X is normal, then (a), (b), (c), (d), (e) are also equivalent to (f) and (g).

(f) *If $\mathscr{A}$ is any free z-ultrafilter on X, then $\{X \backslash Z : Z \in \mathscr{A}\}$ has a weak $\delta\theta$-refinement.*

(g) *If $\mathscr{A}$ is any free z-ultrafilter on X, then we may apply $\theta \aleph_1 L$ to $\{X \backslash Z : Z \in \mathscr{A}\}$.*

Proof. The equivalence of (a), (b), (c) and, for normal X, (f) is due to Blair [3]. It is clear that (c) implies (e), (b) implies (d) and (f) implies (g). The fact that (d) implies (a) and (e) implies (a) follows from 2.5, 2.6 and the result of Dykes [9] that a T_2 completely regular closed-complete cb-space is realcompact. The fact that, for normal spaces, (g) implies (e) can be proved in exactly the same manner as Blair's proof that (f) implies (c). Hence the theorem is established.

As we saw in 2.1.1, we may replace "$\theta \aleph_1 L$" in the above with "θL" under the additional assumption that the spaces with which we are working have countable tightness.

Within the framework of isocompact spaces, it has been a question of considerable interest whether $\aleph_1$-compactness will imply the Lindelöf property, [15], [8], [5]. We see in [5] that the answer is "yes" for T_1 spaces satisfying property $\aleph_1 L$, and hence for T_1 spaces with countable tightness satisfying property L. Let us now consider similar results for spaces satisfying property $\theta \kappa L$. We will not have so nice a theorem as was obtained for property L, of course, since the example given in [8] is weakly θ-refinable (hence, satisfies property $\theta \kappa L$ for every $\kappa \in \mathrm{Card}$ by 2.2), $\aleph_1$-compact and not Lindelöf. We do obtain several results by assuming that certain of the conditions are satisfied hereditarily, or by assuming that closed subsets are G_δ-sets. Analogues of 2.8 and 2.10 have previously been obtained

for weakly $\delta\theta$-refinable spaces. These results were communicated to the author by H. H. Wicke.

THEOREM 2.8. *If X satisfies property θL and is hereditarily $\aleph_1$-compact, then X is Lindelöf.*

Proof. Suppose $\mathscr{U}$ is an open cover of X with no countable subcover. Apply the definition of property θL to obtain $\langle \mathscr{D}_n : n \in \omega \rangle$ and $\langle \mathscr{V}_n : n \in \omega \rangle$ as in 2.1 with $\kappa = \aleph_0$. There exists $n \in \omega$ such that no countable subcollection of $\mathscr{V}_n$ covers $\cup \mathscr{D}_n$. We apply the lemma in [5] in the space $\cup \mathscr{V}_n$ to obtain functions $a: \omega_1 \to \cup \mathscr{D}_n$, $D: \omega_1 \to \mathscr{D}_n$ and $U: \omega_1 \to \{$the open subsets of $\cup \mathscr{V}_n\}$ such that the following are true:

(1) For each $\alpha \in \omega_1$, $a_\alpha \in \cup \mathscr{D}_n \backslash \bigcup_{\beta < \alpha} f_n(D_\beta)$.

(2) For each $\alpha \in \omega_1$, $a_\alpha \in D_\alpha$.

(3) For each $\alpha \in \omega_1$, $U_\alpha \cap \{a_\beta : \beta \in \omega_1\} = \{a_\alpha\}$.

(4) $\bigcup_{\alpha \in \omega_1} U_\alpha = U_{\alpha \in \omega_1} f_n(D_\alpha)$.

(5) $|\{a_\alpha : \alpha < \omega_1\}| = \aleph_1$.

(f_n is the cushioning function for the nth level.) The set $\{a_\alpha : \alpha < \omega_1\}$ is uncountable and relatively discrete. This contradiction establishes the result.

COROLLARY 2.8.1. *A space is hereditarily Lindelöf if and only if it is hereditarily $\aleph_1$-compact and hereditarily satisfies property θL.*

THEOREM 2.9. *For each $\kappa \in Card$, if X is perfect and satisfies property $\theta \kappa L$, then X satisfies property κL.*

Proof. Suppose $\mathscr{U}$ is an open cover of X. Apply the definition of property $\theta \kappa L$ to obtain sequences $\langle \mathscr{D}_n : n \in \omega \rangle$ and $\langle \mathscr{V}_n : n \in \omega \rangle$ as in 2.1. Note that we may assume the collection $\mathscr{D}_n$ to be a disjoint collection, for each $n \in \omega$. For each $n \in \omega$, choose a sequence $\langle F(n, k) : k \in \omega \rangle$ of closed sets such that $\cup \mathscr{V}_n = \bigcup_{k \in \omega} F(n, k)$. We define $\mathscr{D}^*_{(n,k)} = \{D \cap F(n, k) : D \in \mathscr{D}_n\}$. For each $(n, k) \in \omega \times \omega$, $F(n, k)$ is a closed set which is contained in $\cup \mathscr{V}_n$. Since $\mathscr{D}_n$ is κ-weakly cushioned in $\omega \mathscr{V}_n$ in the space $\cup \mathscr{V}_n$ and $\mathscr{V}_n$ refines $\mathscr{U}$ for each $n \in \omega$, and $\cup \mathscr{D}^*_{(n,k)} \subset \cup \mathscr{D}_n \cap F(n, k)$ for each $(n, k) \in \omega \times \omega$, we easily see that $\mathscr{D}^*_{(n, k)}$ is κ-weakly cushioned in $\omega \mathscr{U}$ for each $(n, k) \in \omega \times \omega$ by the cushioning function $f^*_{(n,k)}$ defined in the obvious way from f_n, and the theorem follows.

We can use 2.9 together with 3.6 of [5] to obtain the following results. These could also be obtained from 2.8 using 2.1.1a.

THEOREM 2.10. *If X is a perfect, T_1, $\aleph_1$-compact space which satisfies property θL, then X is hereditarily Lindelöf.*

COROLLARY 2.10.1. *If X is a T_3 space, then X is hereditarily Lindelöf if and only if X is perfect, $\aleph_1$-compact and satisfies property θL.*

The following can be proved in the same way that we proved 1.7. Unfortunately, we do not know if the analogue of 1.6 holds for property $\theta \kappa L$.

THEOREM 2.11. *If $f: X \to Y$ is a closed continuous surjection and, for each $y \in Y$, $f^{-1}(\{y\})$ is Lindelöf, then for each $\kappa \in Card$, if Y satisfies property $\theta \kappa L$, then so does X.*

COROLLARY 2.11.1. *For each $\kappa \in Card$, property $\theta \kappa L$ is preserved under perfect preimages.*

Examples are readily available to demonstrate that property θL is strictly weaker than property L. In fact, the examples in [8] and Σ in [7] both have property $\theta \kappa L$ for every $\kappa \in Card$ but do not satisfy property L. It is inconceivable to the author that property θL could imply weak $\delta \theta$-refinability; however, at this writing no example is known. Of course, an example of a space which satisfies property L and is not weakly $\delta \theta$-refinable would supply the needed example here as well.

3. **Questions.** The following is a list of some open questions regarding the properties discussed in this paper.

Question 3.1. Is the closed continuous image of a weakly $\delta \theta$-refinable space necessarily weakly $\delta \theta$-refinable?

Question 3.2. Does the closed continuous image of a space which satisfies property θL necessarily satisfy property θL?

Question 3.3. Is there a T_1, $\aleph_1$-compact space which satisfies property L and is not Lindelöf?

Of course, such a space could not satisfy property $\aleph_1 L$, and therefore could not have countable tightness. Also, such a space could not be hereditarily $\aleph_1$-compact, and therefore could not be perfect.

Question 3.4. Is there a T_1, $\aleph_1$-compact space which satisfies property θL (or L) and is not closed-complete?

Such a space could not have countable tightness or satisfy property $\theta\aleph_1 L$.

Question 3.5. Is there a space which satisfies property L (preferably with countable tightness) and is not weakly $\delta\theta$-refinable?

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