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EMBEDDING PARTIAL IDEMPOTENT d-ARY QUASIGROUPS

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It is shown that every finite partial idempotent d-quasigroup is embedded in a finite idempotent d-quasigroup.

1. **Introduction.** Evans [3] has proved that every partial Latin square of order n can be embedded in a Latin square of order $2n$. Equivalently, every partial quasigroups of order n can be embedded in a quasigroup of order $2n$. The connection between Latin squares and quasigroups is explained in [2]. Lindner [5] has proved that every idempotent partial quasigroup of order n can be embedded in an idempotent quasigroup of order 2^n, while Hilton [4], using a different technique, reduced this order to $4n$. After Cruse [1] gave a multidimensional analogue of Evans' theorem, Lindner [6] succeeded in proving an embedding theorem for idempotent ternary quasigroups. In the present paper, denoting by $N(p)$ the minimal order of d-quasigroups in which the partial idempotent d-quasigroup (P, p) is embedded, we show that (P, p) is embedded in an idempotent d-quasigroup (Q, q), such that $|Q| \leqq 2N(p)$ if d is odd and $|Q| \leqq 3N(p)$ if d is even.

For $d = 3$ this is an improvement on Lindner's result, but when $d = 2$ our construction gives a higher upper bound than Hilton's. The reason for this is that Hilton's construction relies on the fact that a partial quasigroup can be embedded in a quasigroup with the order doubled. This is not known to be true when $d > 2$ and a direct generalization of Hilton's construction cannot be applied.

2. **Notation and definitions.** If A is a set and $x \in A^d$, then x_i denotes the ith component of $x = (x_1, x_2, \cdots, x_d)$. If $x \in A$, $\bar{x} \in A^d$ is defined as $\bar{x} = (x, x, \cdots, x)$. Similar notation applies to the functions $f: X \to Y^d$ and $g: X \to Y$. For every $x \in X$

$$f(x) = (f_1(x), f_2(x), \cdots, f_d(x))$$

and for every $x \in X^d$, $\bar{g}(x) = (g(x_1), g(x_2), \cdots, g(x_d))$. The function Δ_A: $A \to A^d$ is defined as $\Delta_A(x) = \bar{x}$ for all $x \in A$. The restriction of $f: S \to T$ to $A \subseteqq S$ is denoted by $f|A$. We may take exception when f is a d-ary operation, in which case $f|A$ will often be abbreviated by f. When no danger of ambiguity exists, we do not distinguish between $h: S \to T$ and $g: S \to U$ if $h(x) = g(x)$ for every $x \in S$. The symbol $[x, y]$ denotes the d-tuple

$$((x_1, y_1), (x_2, y_2), \cdots, (x_d, y_d)) ,$$

${}_xU$ stands for $\{[x, y]: y \in U\}$ and S_x denotes the Cartesian product $\{x\} \times S$.

If Q is a nonempty finite set of cardinality n and d is a natural number, we say that $q: U \rightarrow Q$ is a *partial d-quasigroup of order n*, provided $U \subseteqq Q^d$ and the equation $q(x) = q(y)$ implies that either $x = y$ or else x and y differ in at least two of their components. The partial d-quasigroup q may also be denoted by (Q, q) or (Q, U, q). If $U = Q^d$, then q is a *d-quasigroup of order n.*

We observe that if (Q, q) is a finite d-quasigroup, then given $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d$ and y in Q, there exists a unique $x_i \in Q$ such that

$$q(x_1, x_2, \cdots, x_d) = y .$$

A partial d-quasigroup (Q, U, q) is *idempotent* if $x \in Q$ implies $\bar{x} \in U$ and $q(\bar{x}) = x$.

In order to simplify our terminology we refer to ordinary finite quasigroups by calling them binary quasigroups and use the word "quasigroup" to abbreviate the expression "finite d-quasigroup".

(S, T, s) is a *partial subquasigroup* of the partial quasigroup (P, U, q), *if* $S \subseteqq Q$ and $s = q | T$. A partial quasigroup (S, T, s) is *isomorphic* to (Q, U, q), if there exists a bijection $\phi: S \rightarrow Q$ such that $\bar{\phi}(T) = U$ and $q(\bar{\phi}(x)) = \phi(s(x))$ for all $x \in T$. (S, T, s) is *embedded* ("can be embedded") in (Q, U, q) if there exists an injection $\phi: S \rightarrow Q$ such that $\bar{\phi}(T) \subseteqq U$ and $q(\bar{\phi}(x)) = \bar{\phi}(s(x))$ for all $x \in T$. Evidently, (S, T, s) is embedded in (Q, U, q) if and only if the latter has a partial subquasigroup isomorphic to the former.

A function $t: Q \rightarrow Q^d$ is a *transversal* of the quasigroup (Q, q) if

(i) $q(t(x)) = x$ for all $x \in Q$

(ii) $x \neq y$ implies $t_i(x) \neq t_i(y)$ for $i = 1, 2, \cdots, d$. We observe that if (Q, q) is idempotent, then Δ_Q is a transversal of (Q, q). Some quasigroups do not possess transversals. A transversal t of (Q, q) is an *offbeat* transversal if $t(x) \neq \bar{y}$ for all $x, y \in Q$. We say that $f: Q \rightarrow Q^d$ *fixes* P if $P \subseteqq Q$ and $f(x) = \bar{x}$ for all $x \in P$.

3. Transversals and embedding.

LEMMA 1. *Let $n \geqq 2$. Then for every odd $d \geqq 3$ there exists an idempotent d-quasigroup (Q, q) of order n possessing an offbeat transversal.*

Proof. Let $Q = \{0, 1, \cdots, n-1\}$, let

$$q(x) = x_1 + \sum_{i=1}^{(d-1)2} (x_{2i} - x_{2i+1}) \pmod{n}$$

and let

$$t(x) = (x, x + 1, x + 1, \cdots, x + 1) \pmod{n} .$$

Then (Q, q) is an idempotent quasigroup with t as an offbeat transversal.

LEMMA 2. *Let $n \geqq 3$. Then for every $d \geqq 2$ there exists an idempotent d-quasigroup of order n with an offbeat transversal.*

Proof. We may assume that d is even as Lemma 1 covers the case when d is odd. We first deal with the case when $d = 2$. Figure 1 shows an idempotent binary quasigroup of order 6 with an offbeat transversal τ.

	0	1	2	3	4	5	
0	0	2	$\boxed{1}$	5	3	4	$\tau(0) = (1, 4)$
1	4	1	5	2	$\boxed{0}$	3	$\tau(1) = (0, 2)$
2	$\boxed{3}$	4	2	1	5	0	$\tau(2) = (3, 5)$
3	5	0	4	3	1	$\boxed{2}$	$\tau(3) = (2, 0)$
4	2	$\boxed{5}$	3	0	4	1	$\tau(4) = (5, 3)$
5	1	3	0	$\boxed{4}$	2	5	$\tau(5) = (4, 1)$

FIGURE 1

For all other orders $n \geqq 3$ the desired binary quasigroups can be constructed with the help of orthogonnl Latin squares. Now let $d \geqq 4$, d even and $n \geqq 3$. Let $Q = \{0, 1, \cdots, n - 1\}$ and let (Q, l) be an idempotent binary quasigroup (of order n) with an offbeat transversal τ. Let

$$q(x) = l(x_1, x_2) + \sum_{i=2}^{d/2} (x_{2i} - x_{2i-1}) \pmod{n}$$

and let

$$t(x) = (\tau_1(x), \tau_2(x), x, x, \cdots, x) .$$

Then (Q, q) is an idempotent d-quasigroup with t as an offbeat transversal.

LEMMA 3. *Let (Q, q) be a d-quasiguoup with a transversal t and let*

$$q_t(x) = q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) .$$

Then (Q, q_t) *is an idempotent quasigroup.*

Proof. It is clear that q_t maps Q^d into Q. Suppose that $x \neq y$ and $q_t(x) = q_t(y)$. Let i be such that $x_i \neq y_i$. Then $t_i(x_i) \neq t_i(y_i)$. Since
$q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) = q(t_1(y_1), t_2(y_2), \cdots, t_d(y_d))$, $(t_1(x_1), t_2(x_2), \cdots, t_d(x_d))$ and $(t_1(y_1), t_2(y_2), \cdots, t_d(y_d))$ must differ in at least two components. Hence there exists a $j \neq i$ such that $t_j(x_j) \neq t_j(y_j)$ implying $x_j \neq y_j$. Thus x and y differ in at least two components and (Q, q_t) is a quasigroup. If $z \in Q$, then

$$q_t(\bar{z}) = q(t_1(z), t_2(z), \cdots, t_d(z)) = q(t(z)) = z$$

and (Q, q_t) is idempotent.

LEMMA 4. *Let* (P, p) *be an idempotent partial subquasigroup of a (not necessarily idempotent) d-quasigroup* (Q, q) *and let* t *be a transversal of* (Q, q) *fixing* P. *Then* (P, p) *is a partial subquasigroup of* (Q, q_t).

Proof. It suffices to show that q and q_t agree on P^d. Let $x \in P^d$. Then indeed

$$q_t(x) = q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) = q(x_1, x_2, \cdots, x_d) = q(x) .$$

DEFINITION. The *product* (Q, q) of the d-quasigroups (R, r) and (S, s) is defined as follows. $Q = R \times S$ and for every

$$z = [x, y] \in (R \times S)^d$$
$$q(z) = (r(x), s(y)) .$$

If t' and t'' are transversals in (R, r) and (S, s) respectively, their product t is defined by

$$t(x, y) = [t'(x), t''(y)] .$$

LEMMA 5. *The product* (Q, q) *of the quasigroups* (R, r) *and* (S, s) *is a quasigroup. It* t' *and* t'' *are transversal of* (R, r) *and* (S, s) *respectively, then their product* t *is a transversal of* (Q, q). *If* (V, r) *is a subquasigroup of* (R, r), *then* $q|(V \times S)^d$ *is a subquasigroup of* (Q, q). *If* (R, r) *is idempotent and* $x \in R$, *then* (S_x, q) *is isomorphic to* (S, s). *The product of idempotent quasigroups is idempotent.*

Proof, Let (Q, q) be the product of (R, r) and (S, s). Suppose

$q([x, y]) = q([u, v])$ and $[x, y] \neq [u, v]$. Then $r(x) = r(u)$ and $s(y) = s(v)$. If $x \neq u$, then x and u differ in at least two components and so do $[x, y]$ and $[u, v]$. If $x = u$, then $y \neq v$ and again $[x, y]$ and $[u, v]$ differ in at least two components. Thus (Q, q) is a quasigroup. Suppose t' and t'' are transversals of (R, r) and (S, s) respectively and t is their product. Then

$$q(t(x, y)) = q[t'(x), t''(y)] = (r(t'(x)), s(t''(y))) = (x, y) .$$

Suppose $(x, y) \neq (u, v)$. If $x \neq u$, then $t'_i(x) \neq t'_i(u)$ for $i = 1, 2, \cdots, d$; and if $y \neq v$, then $t''_i(y) \neq t''_i(v)$. In any event, if $(x, y) \neq (u, v)$, we have

$$t_i(x, y) = (t'_i(x), t''_i(y)) \neq (t'_i(u), t''_i(v)) = t_i(u, v)$$

for all i. Thus t is a transversal of (Q, q). Suppose (V, r) is a subquasigroup of (R, r). Then the range of $q|(V \times S)^d$ is $V \times S$, so $q|(V \times S)^d$ is a subquasigroup of (Q, q). If (R, r) is idempotent, then $y \mapsto (x, y)$ is an isomorphism from (S, s) to (S_x, q) for every $x \in R$. If (R, r) and (S, s) are both idempotent and $z = (x, y) \in Q$, then

$$q(\bar{z}) = (r(\bar{x}), s(\bar{y})) = (x, y) = z$$

and (Q, q) is idempotent.

LEMMA 6. *Let (R, r) and (S, g) be idempotent quasigroups and let (Q, f) be their product. Let $P \subseteq S$ and let τ be an offbeat transversal of (R, r). For every $z = (x, y) \in Q$ let*

$$t_i(z) = \begin{cases} (x, y) & \text{if } \ y \in P \\ (\tau_i(x), y) & \text{if } \ y \notin P \end{cases}$$

for $i = 1, 2, \cdots, d$. Then t is a transversal of (Q, f), fixing $R \times P$. Furthermore, if $(x, y) \in Q$ and $a \in R$, then $t(x, y) \in S_a^d$ if and only if $x = a$ and $y \in P$.

Proof. Let $(x, y) \in Q$ and $(u, v) \in Q$ be such that $t_i(x, y) = t_i(u, v)$ for some i. Then necessarily $y = v$. If $y \in P$, then

$$(x, y) = t_i(x, y) = t_i(u, v) = t_i(u, y) = (u, y) = (u, v) .$$

If $y \notin P$, then $(\tau_i(x), y) = (\tau_i(u), v)$ implies $(x, y) = (u, v)$. If $y \in P$, then

$$f(t(x, y)) = f([\bar{x}, \bar{y}]) = (r(\bar{x}), g(\bar{y})) = (x, y) .$$

If $y \notin P$, then

$$f(t(x, y)) = f([\tau(x), \bar{y}]) = (r(\tau(x)), g(\bar{y})) = (x, y) .$$

Thus t is a transversal of (Q, f). It is evident from the definition of t, that t fixes $R \times P$. If $a \in R$ and $y \in P$, then of course $t(a, y) \in S_a^d$. On the other hand if $(x, y) \in Q$, $a \in R$ and $y \notin P$, then $t(x) \notin S_a^d$ because $\tau(x) = \bar{a}$ is impossible as τ is an offbeat transversal.

LEMMA 7. *Let (Q, r) be a quasigroup with a subquasigroup (S, r) and let (S, s) be an arbitrary quasigroup (on the set S). For each $x \in Q^d$ let*

$$q(x) = \begin{cases} s(x) & \text{if } x \in S^d \\ r(x) & \text{if } x \notin S^d \end{cases}.$$

Then (Q, q) is a quasigroup.

Proof. Let $x \in Q^d$ and $y \in Q^d$ such that $x \neq y$ and $q(x) = q(y)$. If both x and y belong to S^d, then $s(x) = s(y)$ implies that x and y differ in at least two components. The same is true if neither x nor y belong to S^d. If, say $x \in S^d$ and $y \notin S^d$, assume that x and y differ in exactly one component, say their first. Then $x_1 \neq y_1$ and $x_i = y_i$ if $i \geqq 2$. It follows then, that $y_1 \notin S$. Let $x_1' \in S$ be such that

$$r(x_1', x_2, \cdots, x_d) = s(x_1, x_2, \cdots, x_d) .$$

Then $x_1' \neq y_1$. On the other hand,

$$r(x_1', x_2, \cdots, x_d) = s(x) = r(y) = r(y_1, x_2, \cdots, x_d) ,$$

implying $x_1' = y_1$, a contradiction. Thus (Q, q) is a quasigroup.

DEFINITION. If (Q, r), (S, r), (S, s) and (Q, q) are as in Lemma 7, then (Q, q) is called the *replacement of* (S, r) *by* (S, s) in (Q, r).

THEOREM 1. *Let (P, U, p) be a partial idempotent sub-d-quasigroup of a d-quasigroup (S, s). Then (P, U, p) can be embedded in an idempotent d-quasigroup (Q, q) such that $|Q| \leqq 3|S|$ if d is even and $|Q| \leqq 2|S|$ if d is odd.*

Proof. Let (P, U, p) be a partial idempotent subquasigroup of (S, s). First we deal with the case when $|S| \geqq 3$. Let g be such that (S, g) is an idempotent quasigroup and let (R, r) be an idempotent quasigroup with an offbeat transversal τ. Let (Q, f) be the product of (R, r) and (S, g). Define t as in Lemma 6. Then t is a transversal of (Q, f). Let $a \in R$. Then t fixes $P_a (\subseteqq R \times P)$. Define $s'\colon S_a^d \to S_a$ as follows: $s'([\bar{a}, z]) = (a, s(z))$ for all $z \in S^d$. Then (S, s) is isomorphic to (S_a, s') via $\phi(y) = (a, y)$ for all $y \in S$. Indeed, $\bar{\phi}(S^d) = S_a^d$ and $s'(\bar{\phi}(z)) = s'([\bar{a}, z]) = (a, s(z)) = \phi(s(z))$ for all $z \in S^d$. Let (Q, q) be the replacement of (S_a, f) by (S_a, s') in (Q, f). Then $\phi|P$ establishes an isomorphism from (P, U, p) to $(P_{a'a}U, q)$. Thus (P, U, p) is embedded in (Q, q). Next we will show, that t is a transversal of (Q, q). It suffices to verify that $q(t(x, y)) = (x, y)$ for every $(x, y) \in Q$. Suppose $(x, y) \in Q$. If $t(x, y) \notin S_a^d$, then $q(t(x, y)) = f(t(x, y)) = (x, y)$. If $t(x, y) \in S_a^d$, we must have $x = a$ and $y \in P$ by Lemma 6. But then

$$q(t(x, y)) = q(t(a, y)) = q([\bar{a}, \bar{y}]) = s'([\bar{a}, \bar{y}]) = (a, s(\bar{y})) \\ = (a, p(\bar{y})) = (a, y) = (x, y) .$$

Thus t is a transversal of (Q, q). By Lemma 4 (P, U, p) is embedded in the idempotent (Q, q_t). Clearly, $|Q| = |R|\,|S|$ and the smallest idempotent quasigroup (R, r) with an offbeat transversal is of order 3 or 2, depending on the parity of d.

Now let us look at the case when the order of (S, s) is one or two. Then, if $P = S$, (P, U, p) is embedded in the idempotent (S, s). If $P \neq S$, then (P, U, p) is the unique (idempotent) quasigroup or order one, embedded in itself.

THEOREM 2. *Let (P, p) be a finite partial idempotent d-quasigroup. Then (P, p) can be embedded in a finite idempotent d-quasigroup (Q, q). Furthermore, if $N(p)$ denotes the minimal order of d-quasigroups into which (P, p) can be embedded, then Q can be chosen so that $|Q| \leqq 2N(p)$ if d is odd and $|Q| \leqq 3N(p)$ if d is even.*

Proof. Using Cruse's result [1] that every finite partial d-quasigroup is embedded in a finite d-quasigroup, our theorem immediately follows from Theorem 1.

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