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CHARACTERIZING REDUCED WITT RINGS II

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We have recently given a recursive construction of all reduced Witt rings of fields with finitely many places into the real numbers. In this paper we extend the construction to include all reduced Witt rings of fields. We then demonstrate how this recursion process can be used to prove facts about these rings.

1. Introduction. Given a field F, we denote its Witt ring of nondegenerate symmetric bilinear forms by $W(F)$. Modulo the nilradical, we obtain the reduced Witt ring $W_{\text{red}}(F)$. We begin with some notation and basic facts about these rings from [11] and [12] which will be used throughout this paper. Our interest is only in fields F for which the Witt ring is not entirely torsion. This is equivalent to the field being formally real [13]. In this case, $W_{\text{red}}(F)$ can be naturally embedded in $\mathscr{C}(X, \boldsymbol{Z})$, the ring of continuous functions from the topological space $X = X(F)$ of minimal prime ideals of $W(F)$ with the Zariski topology to the ring of integers with the discrete topology. One can also identify X with the set of orderings of F and with the set of ring homomorphisms from $W(F)$ to $\boldsymbol{Z}$. In dealing with elements of X, we shall usually think of them as orderings, represented by a set of positive elements P. More generally, one can consider "abstract Witt rings" (defined in [12]) and obtain a similar embedding. We will generally identify such rings with their canonical embeddings in rings of continuous functions, hence considering an element f in $W_{\text{red}}(F)$ as a continuous function $f: X(F) \to \boldsymbol{Z}$.

Our goal in this paper is to separate the reduced Witt rings of fields from among the class of all abstract Witt rings. This will be done by extending results in [6] to obtain a ring theoretic construction of a category of rings whose objects represent all isomorphism classes of reduced Witt rings of fields. The recursive nature of this construction provides a strong method of proof for questions concerning the ring structure of the reduced Witt ring. This will be demonstrated in §4 where we obtain a new proof of a powerful recent theorem due to Becker and Bröcker [1, Theorem 5.3]. In §3 we state and prove the characterization theorem for reduced Witt rings, generalizing [6, Theorem 2.1].

Section 2 is devoted to defining and briefly studying the category of rings in which we are interested. As a matter of notation, we shall let R^* denote the multiplicative group of units in any ring R.

For any set S, we denote the cardinality of S by $|S|$. Further notation will be developed in the next section.

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2. Construction of categories. Our objective in this section is to construct a category of rings which contains a ring isomorphic to $W_{\text{red}}(F)$ for any formally real field F.

DEFINITION 2.1. Let $\mathscr{R}_0$ be the full subcategory of the category of rings with 1 whose objects are the rings constructed in the following recursive definitions:

(a) $\boldsymbol{Z}$ is in $\mathscr{R}_0$.

(b) If R_1 and R_2 are in $\mathscr{R}_0$ and M_i is the unique maximal ideal of R_i containing 2, then $R = \boldsymbol{Z} + M_1 \times M_2$ is in $\mathscr{R}_0$, where $\boldsymbol{Z}$ has the diagonal embedding in $R_1 \times R_2$.

(c) If R_0 is in $\mathscr{R}_0$, so is the group ring $R = R_0[\varDelta]$ where $\varDelta$ is a group of order 2.

REMARK 2.2. For further details, see [6] where this construction is studied in depth. In particular, it is shown that R is in $\mathscr{R}_0$ if and only if there exists a field F with a finite number of orderings such that R is isomorphic to $W_{\text{red}}(F)$ [6, Theorem 2.1].

We now wish to embed this category in a category $\mathscr{R}$, ensuring that every reduced Witt ring will be isomorphic to an object in $\mathscr{R}$.

DEFINITION 2.3. The category $\mathscr{R}$ is defined to be the full subcategory of the category of rings with 1 whose objects consist of the objects in $\mathscr{R}_0$ together with rings of the form

$$R = (\varprojlim R_\alpha) \cap \mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$$

where $(R_\alpha, f_{\alpha\beta})$ is an inverse system of rings R_α in $\mathscr{R}_0$ and surjective homomorphisms $f_{\alpha\beta}\colon R_\beta \to R_\alpha$; and X_α is the associated topological space of minimal prime ideals of R_α. Note that $f_{\alpha\beta}$, being surjective, induces on injection $f^*_{\alpha\beta}\colon X_\alpha \to X_\beta$. Thus $(X_\alpha, f^*_{\alpha\beta})$ is a direct system of topological spaces. Since we identify each R_α with its natural embedding in $\mathscr{C}(X_\alpha, \boldsymbol{Z})$, the elements of $\varprojlim R_\alpha$ can be considered as (not necessarily continuous) functions from $\varinjlim X_\alpha$ to $\boldsymbol{Z}$, and so the intersection takes place in the set of all functions from $\varinjlim X_\alpha$ to $\boldsymbol{Z}$. We further require that $X = \varinjlim X_\alpha$ be a Boolean space and R be a Witt subring of $\mathscr{C}(X, \boldsymbol{Z})$ as defined in [11, § 3].

In view of [11, Corollary 3.7], this is equivalent to to requiring that R be additively generated by its units, which are the continuous elements of $\varprojlim R_\alpha^*$.

REMARK 2.4. The conditions that X be Boolean and that R be an abstract Witt ring are certainly necessary. We do not know if the latter condition is automatic in this situation. It does guarantee that X is the same as the space of homomorphisms of R into $\boldsymbol{Z}$ (or equivalently, minimal prime ideals of R), which we shall denote by $X(R)$.

THEOREM 2.5. *The inverse limit in the category $\mathscr{R}$ of a surjective inverse system $(R_\alpha, f_{\alpha\beta})$ with all R_α in $\mathscr{R}_0$ is $(\varprojlim R_\alpha) \cap \mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$ when this ring is ano bject in $\mathscr{R}$; that is, when $\varinjlim X_\alpha$ is a Boolean space and the ring is additively generated by its units.*

Proof. Given such an inverse system, let R be in $\mathscr{R}$ with a compatible family of homomorphisms $\varphi_\alpha: R \to R_\alpha$. Then, as in the category of rings, there exists a unique homomorphism $\varphi: R \to \varprojlim R_\alpha$ such that $f_\beta \varphi = \varphi_\beta$ for each β, where f_β is the homomorphism from $\varprojlim R_\alpha$ to R_β. Since $R \in \mathscr{R}$, the ring R is contained in $\mathscr{C}(X(R), \boldsymbol{Z})$. Each φ_α induces a continuous map $\varphi_\alpha^*: X_\alpha \to X(R)$, so we obtain a unique continuous map $\psi: \varinjlim X_\alpha \to X(R)$ extending each φ_α^*. This induces a homomorphism of R into $\mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$. Thus the image of φ is contained in $(\varprojlim R_\alpha) \cap \mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$ and so by definition this is the inverse limit in $\mathscr{R}$.

DEFINITION 2.6. In view of the previous theorem, the ring R defined in Definition 2.3 will be denoted by $\varprojlim_{\mathscr{R}} R_\alpha$. We continue to use $\varprojlim R_\alpha$ to denote the inverse limit in the category of all rings with identity.

3. The characterization theorem. The purpose of this section is to prove the following theorem.

THEOREM 3.1 (*Characterization Theorem*). *Let F be a formally real field. Then the reduced Witt ring of F is isomorphic to one of the rings in $\mathscr{R}$ (cf. Definitions* 2.1 *and* 2.3).

REMARK 3.2. We conjecture that the converse is also true. In fact, given any R in $\mathscr{R}$, we conjecture that there exists a pythagorean field whose Witt ring is isomorphic to R. This has been proved for $\mathscr{R}_0$ in [6].

We shall begin by relating some definitions and results of several authors [1, 2, 10, 14, 15] which we will need in order to prove the theorem. Throughout this section, F will denote a formally real field.

DEFINITION 3.3 (cf. [1, 2]). A subset T of F is called a *preordering* if it is closed under addition and multiplication, contains the squares of F and does not contain -1. A preordering is called a *fan* if it satisfies $T + aT = T \cup aT$ for all $a \in F$, $a \notin -T$. Note that T^*, the set of nonzero elements of T, forms a subgroup of F^*.

DEFINITION 3.4 (cf. [10, 11]). For any subset T of F, let $V(T) = \{P \in X(F) \mid t \in P$ for all $0 \neq t \in T\}$. For any subset $Y \subset X(F)$, let $\Gamma(Y) = \{a \in F^* \mid a \in P$ for all $P \in Y\} = \bigcap_{P \in Y} P$. A subset Y of $X(F)$ is said to be *saturated* if $Y = V(\Gamma(Y))$. (An equivalent concept is that of *subspace* in [14].)

LEMMA 3.5. *Any finite subset of $X(F)$ is contained in a finite saturated subset.*

Proof. For any subset S of $X(F)$, the set $V(\Gamma(S))$ is saturated. The fact that it is finite follows immediately from [14, Lemma 2.1].

Saturated sets have been used by Kleinstein and Rosenberg in [10] to generalize results for fields using preorderings [1, 2] to Witt rings of semilocal rings and abstract Witt rings. The following lemma gives the relationship between these concepts.

LEMMA 3.6. (a) *If T is a preordering of F with $[F^*: T^*] < \infty$, then $V(T)$ is a finite saturated subset of $X(F)$.*

(b) *If Y is a finite saturated subset of $X(F)$, then $\Gamma(Y) \cup \{0\}$ is a preordering of F with $[F^*: \Gamma(Y)] < \infty$.*

Proof. Let T be a preordering of F with T^* of finite index in F^*. Then $V(\Gamma(V(T))) = V(T)$ since $\Gamma(V(T)) = T$ [1, Satz 1] and thus $V(T)$ is saturated. It is finite by [15].

For part (b), let Y be a finite saturated subset and consider $\Gamma(Y)$. Since $\Gamma(Y)$ is an intersection of orderings, it clearly satisfies the definition of preordering with the addition of 0, and has finite index in F^* since Y is finite.

The final lemma we need gives the relationship between the above concepts and the rings we have been considering.

LEMMA 3.7. *Let R be the quotient ring of $W_{\text{red}}(F)$ obtained by restricting the elements to a finite saturated subset Y of $X(F)$. Then R is isomorphic to an object in $\mathscr{R}_0$.*

Proof. By [15, Theorem 2.2], the ring R satisfies the hypotheses of [14, Theorem 4.11]. The conclusion is that R is isomorphic to the Witt ring of some field with Y corresponding to its set of orderings, and so R is isomorphic to an object in $\mathscr{R}_0$.

REMARKS 3.8. (a) In the situation of the previous lemma, it is possible for R to be isomorphic to an object in $\mathscr{R}_0$ even though the set Y is not saturated. For example, we may have $W_{\text{red}}(F)$ isomorphic to the integral group ring $\boldsymbol{Z}[\Lambda]$ where Λ is a group of exponent 2 and order 4. Take Y to be any three of the four elements in $X(F)$. One can then check that Y is not saturated but the restriction of the functions in $\boldsymbol{Z}[\Lambda] \subset \mathscr{C}(X, \boldsymbol{Z})$ to the subset Y gives a ring isomorphic to one in $\mathscr{R}_0$.

(b) It is even possible for the ring R in (a) to be the homomorphic image of $\boldsymbol{Z}[\Lambda]$ under the canonical homomorphism induced by inclusion of fields. For example, it is well known that the Witt ring of the field of iterated Laurent series $F = \boldsymbol{R}((x))((y))$ over the real numbers has Witt ring isomorphic to $\boldsymbol{Z}[\Lambda]$. By [7] we can embed F in a field K to which each ordering of F extends uniquely and which satisfies SAP (that is, K contains an element a which is positive in the three orderings in Y and negative in the fourth ordering). We can then form $L = K(a^{1/2}, a^{1/4}, a^{1/8}, \cdots)$, so that the homomorphism $W(F) \to W_{\text{red}}(L)$ is canonically the same as $\boldsymbol{Z}[\Lambda] \to R$. Using the techniques of [5, §4], we can enlarge L to a pythagorean field L' so that the homomorphism becomes $W(F) \to W(L')$.

(c) If one approaches this subject by looking instead at subrings of $W_{\text{red}}(F)$ and quotient spaces of $X(F)$, he finds that a similar situation exists. There are subrings which are abstract Witt rings in the sense of [11] but which are not in $\mathscr{R}_0$. On the other hand, there also exist abstract Witt rings which never occur as a subring of the reduced Witt ring of a field.

Proof of Theorem 3.1. Let F be a formally real field and $R = W_{\text{red}}(F)$. For each finite saturated subset X_α of $X = X(F)$, let R_α be the corresponding quotient ring of R. By Lemma 3.7 each ring R_α is isomorphic to a ring in $\mathscr{R}_0$, hence in $\mathscr{R}$. Thus we are done if X is finite. We now assume that X is infinite and show that R is isomorphic to $\varprojlim_{\mathscr{R}} R_\alpha$ (where, for reasons of notation, we identify the rings R_α with the objects in the category $\mathscr{R}$ isomorphic to them).

Note first that the rings R_α form an inverse system with all of the induced homomorphisms surjective since, by Lemma 3.5, the sets X_α form a direct system under inclusion with $\varinjlim X_\alpha = \cup X_\alpha = X$. Furthermore, we have a natural homomorphism $\varphi: R \to \varprojlim R_\alpha \cap \mathscr{C}(X, \boldsymbol{Z})$ as in the proof of Theorem 2.5 since the only properties of R used in that proof are well known properties of the reduced Witt ring. Each singleton set $\{x\} \subset X$ is clearly saturated, so the corresponding quotient ring of R is among the rings R_α. If $f \in R$ is in the kernel of φ, the image of f upon restriction to each singleton $\{x\}$ is zero, and so f is zero as an element of $\mathscr{C}(X, \boldsymbol{Z})$, hence as an element of R. Therefore φ is injective.

Now let $f \in \varprojlim R_\alpha \cap \mathscr{C}(X, \boldsymbol{Z})$. By Lemma 3.6, if we have any preordering T with $[F^*: T^*] < \infty$, the set $X_T = \{P \in X \mid P \supset T^*\}$ is saturated. Thus the restriction of f to X_T has a preimage in R, since $R \to R_T$ is surjective where R_T is the quotient ring corresponding to X_T. By [1, Corollary 5.2], this is sufficient to guarantee that f lies in the image of φ. Therefore φ is an isomorphism. Since R, being the reduced Witt ring of a field, is additively generated by its units, the ring $\varprojlim R_\alpha \cap \mathscr{C}(X, \boldsymbol{Z})$ is an object in $\mathscr{R}$. Thus R is isomorphic to $\varprojlim_{\mathscr{R}} R_\alpha$ and the theorem is proved.

4. Applications.

In [4], Brown attempted to characterize, for a formally real field F, the elements of $\mathscr{C}(X, \boldsymbol{Z})$ which lie in the image of $W_{\mathrm{red}}(F)$ using valuation theory. He was led to define the concept of an "exact" field. In [1], Becker and Bröcker have not only shown that all formally real fields are exact, but have obtained the characterization in a much more pleasing form as follows.

THEOREM 4.1 [1, Theorem 5.3]. *Let F be any formally real field. The function f in $\mathscr{C}(X(W_{\mathrm{red}}(F)), \boldsymbol{Z})$ lies in the image of $W_{\mathrm{red}}(F)$ if and only if*

$$\sum_{P \supset T} f(P) \equiv 0 \quad \left(\bmod \frac{1}{2}[F^*: T^*]\right)$$

for all fans T with $[F^*: T^*] < \infty$.

We shall give a new proof of this theorem which will demonstrate the use of our recursive construction of reduced Witt rings. Using an idea first presented in [8], we will cast the theorem in ring theoretic terms.

DEFINITION 4.2. For any ring $R \in \mathscr{R}$, we call a subspace Y of

$X(R)$ a 2^n-*box* if $|Y| = 2^n$ and the quotient ring obtained by restricting the functions in R to the subspace Y is an integral group ring. (This quotient ring is studied carefully in [8, Theorem 3.8].)

REMARK 4.3. The original work on fans done by Becker and Köpping [2] shows that fans with finite index in the field are precisely the intersections of orderings in a 2^n-box for some n. Thus a fan has the same relationship with a 2^n-box as a preordering has with a saturated set. Furthermore, it is shown that $[F^*: T^*]$ equals twice the number of orderings containing T.

THEOREM 4.4. *Let $R \in \mathscr{R}$. The function f in $\mathscr{C}(X(R), \boldsymbol{Z})$ lies in R if and only if for every integer $n \geqq 1$ and every 2^n-box B in $X(R)$,*

$$\sum_{P \in B} f(P) \equiv 0 \pmod{2^n} .$$

Proof. The result is clearly true if $R = \boldsymbol{Z}$. We proceed inductively, verifying the theorem for all of the rings constructed in Definitions 2.1 and 2.3. First assume the theorem holds for two rings R_1 and R_2 in $\mathscr{R}_0$, and let $R = \boldsymbol{Z} + M_1 \times M_2$ be as in (2.1b). It is shown in [6] that $X(R)$ is the disjoint union of $X(R_1)$ and $X(R_2)$ and that any 2^n-box, $n > 1$, is entirely contained in either $X(R_1)$ or $X(R_2)$. Since M_i is contained in $\mathscr{C}(X(R_i), 2\boldsymbol{Z})$, it follows from [11, Proposition 3.8] that $f \in R$ if and only if f restricted to $X(R_i)$ lies in R_i for $i = 1, 2$ and parity is maintained ($P_i \in X(R_i)$, $i = 1, 2$ implies $f(P_1) \equiv f(P_2) \pmod 2$). Thus the theorem holds for R.

Next assume the theorem holds for R_0 in $\mathscr{R}_0$ and let $R = R_0[\varDelta]$ as in (2.1c). It is shown in [6] that $X(R)$ can be written as $X(R_0) \times \{1, 2\}$ where the generator λ of $\varDelta$, as an element of $\mathscr{C}(X(R), \boldsymbol{Z})$, is 1 on $X(R_0) \times \{1\}$ and -1 on $X(R_0) \times \{2\}$. The ring R is additively generated by the constant functions together with functions $2\chi_U$ where χ_U is the characteristic function of a set U of the form $\{\sigma \in X(R) \mid \sigma(g) = 1\}$ or its complement for some unit $g \in R$ [11, §3]. So it will suffice to show that $2\chi_U$ satisfies the congruences for any 2^n-box B in $X(R)$. If B is contained in one copy of $X(R_0)$, this is clear since $2\chi_U$ restricts to an element of R_0 on that copy. Otherwise, B must be a union of two 2^{n-1}-boxes, $B_i \subset X(R_0) \times \{i\}$, $i = 1, 2$ [8, Theorem 3.8]. Since B is a 2^n-box, we have $|U \cap B_i| = 0$ or 2^{n-1} [8], and thus

$$\sum_{P \in B} 2\chi_U(P) = 0 , \quad 2^n \quad \text{or} \quad 2^{n+1} ,$$

and so is congruent to 0 modulo 2^n. Conversely, assume $f \in$

$\mathscr{C}(X(R_0) \times \{1, 2\}, \boldsymbol{Z})$ satisfies all of the congruences. Define $f_1, f_2 \in \mathscr{C}(X(R_0), \boldsymbol{Z})$ by $f_1(x) = 1/2(f(x, 1) + f(x, 2))$ and $f_2(x) = 1/2(f(x, 1) - f(x, 2))$. These take integral values since $\{(x, 1), (x, 2)\}$ is a 2-box. Since $f(x, t) = f_1(x) + f_2(x)\lambda(t)$ for (x, t) in $X(R_0) \times \{1, 2\}$, we need only show that f_1 and f_2 lie in R_0. Let B be any 2^n-box in $X(R_0)$. Then

$$\sum_{P \in B} f_1(P) = \frac{1}{2} \sum_{P \in B} f(P, 1) + f(P, 2) \equiv 0 \pmod{2^n}$$

since $B \times \{1, 2\}$ is a 2^{n+1}-box. Similarly, f_2 satisfies the congruences and so $f_1, f_2 \in R_0$.

Finally, let $R = (\varprojlim R_\alpha) \cap \mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$ as in Definition 2.3 where each R_α satisfies the theorem. If $f \in R$, the congruences hold for f since each 2^n-box in $X(R)$ gives rise to a quotient ring in $\mathscr{R}_0$, and thus the box is contained in some X_α corresponding to one of the rings R_α. Conversely, assume the congruences all hold for $f \in \mathscr{C}(\varinjlim X_\alpha, \boldsymbol{Z})$. Then the image f_α of f in $\mathscr{C}(X_\alpha, \boldsymbol{Z})$ lies in R_α for each α by hypothesis, and so $f \in \varprojlim R_\alpha$. Since f is continuous, f is in R.

This concludes the proof of the theorem.

REMARK 4.5. It is interesting to note that converse of Theorem 4.4 does not hold. That is, there exist abstract Witt rings whose elements are characterized by the system of congruences, but which do not lie in $\mathscr{R}$. The smallest such ring is the quotient ring of $R = \boldsymbol{Z}(\Lambda)$, where Λ is a group of exponent 2 and order 8, obtained by deleting one point from the 2^3-box $X(R)$.

Results such as the equivalence of WAP and SAP [9, Theorem 5.3] and generalizations to stability [3, Satz 3.17], [8, Theorem 4.3], [16] can also be proved easily from Theorem 3.1, using a recursive method as in the above proof. The main difficulty is in formulating the results so that they can be proved for inverse limits. For example, to get the equivalence of the Hasse-Minkowski Property with WAP and SAP, it is better to use Definition 1.22 of [10] than the original definition in [9].

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