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AN INVERSION FORMULA FOR A DISTRIBUTIONAL FINITE-HANKEL-LAPLACE TRANSFORMATION

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In this paper a Finite-Hankel-Laplace transformation of a certain generalized functions is defined, and an inversion formula is established.

1. Introduction. Schwartz first introduced the Fourier transform of distributions in 1947. Since then, extension of the classical integral transformation to generalized functions has been of continuing interest. Some pertinent references are [1], [2], [3], [4], [5], [7], [8], and [9]. The classical Finite-Hankel-Laplace transform of function f defined on $-\infty < x < \infty$, $0 < y < 1$ is defined as

$$F(s, j_m) = \int_0^1 \int_{-\infty}^{\infty} ye^{-sx} J_n(j_m y) f(x, y) dx dy \tag{1.1}$$

where $J_n(z)$ is the Bessel function of first kind of order $n \geqq -1/2$ and $j_1, j_2, j_3 \cdots$ are positive zeros of $J_n(z)$ arranged in ascending order.

An inversion theorem for the transform (1.1) is as follows.

THEOREM 1.1. *Let $f(x, y)$ satisfy Dirichlet conditions in the interval $-\infty < x < \infty$, $0 < y < 1$ and $ye^{-cx} f(x, y)$ be absolutely integrable on $-\infty < x < \infty$, $0 < y < 1$ for some positive value of c. If its Finite-Hankel-Laplace transform in that range is defined as* (1.1), *then at any point (x, y), $-\infty < x < \infty$, $0 < y < 1$ at which the function $f(x, y)$ is continuous,*

$$f(x, y) = \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{2J_n(j_m y)}{J_{n+1}^2(j_m)} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} F(s, j_m) ds$$

where $\mathrm{Re}\,(s) = \sigma > c$.

It is natural to try to extend the classical Finite-Hankel-Laplace transform to generalized functions. In this paper above Theorem 1.1 is extended to generalized functions.

2. The notation and terminology. In notation and terminology we follow [2] and [10]. The open set $(-\infty, \infty) \times (0, 1)$ will be denoted by I. We will use the following operators,

$$D_x^k \Omega_{n,y}^{k'} = D_x^k \left(D_y^2 + \frac{1}{y} Dy - \frac{n^2}{y^2} \right)^{k'}, \; k, k' = 0, 1, 2, 3, \cdots \tag{2.1}$$

where $n \geqq -1/2$ and the expression,

$$T_N(y, \tau) = \sum_{m=1}^{N} \frac{2J_n(j_m y)J_n(j_m \tau)}{J_{n+1}^2(j_m)} . \tag{2.2}$$

3. **The spaces $LU_{a,b,c,n}$, $LU(w, z, c, n)$ and their duals.** Let a, b, c, n be real numbers such that $c \geqq 1/2$, $n \geqq -1/2$, and let $k_{a,b}^{(x)}$ be the function defined as

$$k_{a,b}^{(x)} = \begin{cases} e^{ax} & 0 \leqq x < \infty \\ e^{bx} & -\infty < x < 0 \end{cases}$$

then $LU_{a,b,c,n}$ is defined as the linear space of all complex valued smooth functions $\phi(x, y)$ on $-\infty < x < \infty$, $0 < y < 1$ such that for each $k, k' = 0, 1, 2, 3, \cdots$

$$\rho_{a,b,k,k'}^{c,n}[\phi(x, y)] = \sup_{\substack{-\infty<x<\infty \\ 0<y<1}} |k_{a,b}^{(x)} y^c D_x^k \Omega_{n,y}^{k'}[y^{-1}\phi(x, y)]| < \infty .$$

We assign to $LU_{a,b,c,n}$ the topology generated by the semi-norms $\{\rho_{a,b,k,k'}^{c,n}\}_{k,k'=0}$. Hence $LU_{a,b,c,n}$ is countably multinormed space which is complete. The dual space $LU'_{a,b,c,n}$ consists of all continuous linear functionals on $LU_{a,b,c,n}$. By the Theorem 1.8.3 [10, p. 21] $LU'_{a,b,c,n}$ is also complete. If $a \leqq d$ and $e \leqq b$, then $LU_{d,e,c,n} \subset LU_{a,b,c,n}$, and the topology of $LU_{d,e,c,n}$ is stronger than the topology induced on it by $LU_{a,b,c,n}$. Consequently the restriction of any member $f \in LU'_{a,b,c,n}$ to $LU_{d,e,c,n}$ is in $LU'_{d,e,c,n}$.

We turn now to a certain countable union space $LU(w, z, c, n)$. Let w denote either a finite real number or $-\infty$ and z denote either a finite real number or $+\infty$. Choose two monotonic sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ such that $a_k \rightarrow w_+$ and $b_k \rightarrow z_-$. Then $LU(w, z, c, n)$ is defined as countable union space of all $LU_{a_k, b_k, c, n}$ spaces; thus $LU(w, z, c, n) = \bigcup_{k=1}^{\infty} LU_{a_k, b_k, c, n}$. A sequence $\{\phi_k\}_{k=1}^{\infty}$ converges in $LU(w, z, c, n)$ if and only if it converges in $LU_{a_k, b_k, c, n}$ for some k. Since for each k, $LU_{a_k, b_k, c, n}$ is complete and hence a countable-union space, $LU(w, z, c, n)$ is complete. $LU'(w, z, c, n)$ denotes the dual space of $LU(w, z, c, n)$. Hence $LU'(w, z, c, n)$ is also complete [10, p. 25].

Now we note several facts to which we will refer later.

(I) Clearly, $D(I)$ is sub-space of $LU_{a,b,c,n}$ as well as of $LU(w, z, c, n)$, whatever be the value of a, b, w or z; the convergence in $D(I)$ implies the convergence in $LU_{a,b,c,n}$ and also convergence in $LU(w, z, c, n)$. Consequently, the restriction of any member of $LU'_{a,b,c,n}$ or $LU'(w, z, c, n)$ to $D(I)$ is a member of $D'(I)$. Hence the member of $LU'_{a,b,c,n}$ and $LU'(w, z, c, n)$ are distributions in the sense of Zemanian [10, p. 39].

(II) Since $D(I)$ is dense in $LU(w, z, c, n)$ for every w, z therefore by Theorem 1.9.1 [10, p. 24] $LU'(w, z, c, n)$ is a subspace of $D'(I)$.

(III) Let $w \leqq x$ and $y \leqq z$, then $LU(x, y, c, n) \subset LU(w, z, c, n)$ and convergence in $LU(x, y, c, n)$ implies the convergence in $LU(w, z, c, n)$. Since $D(I) \subset LU(x, y, c, n)$ and $D(I)$ is dense in $LU(w, z, c, n)$, $LU(x, y, c, n)$ is also dense in $LU(w, z, c, n)$. Hence by Theorem 1.9.1 [10, p. 24] $LU'(w, z, c, n)$ is a subspace of $LU'(x, y, c, n)$.

(IV) If $f(x, y)$ is locally integrable function defined on $-\infty < x < \infty$, $0 < y < 1$ and if $\int_0^1 \int_{-\infty}^{\infty} |[k_{a,b}^{(x)}]^{-1} y^{1-c} f(x, y)| dxdy$ exists, then $f(x, y)$ generates a regular generalized function on $Lu'_{a,b,c,n}$ through the definition

$$\langle f, \phi \rangle = \int_0^1 \int_{-\infty}^{\infty} f(x, y) \phi(x, y) dxdy, \ \phi \in LU_{a,b,c,n} .$$

Similarly, if $w < a$ and $b < z$, then f generates a regular member of $LU'(w, z, c, n)$ through the definition

$$\langle f, \phi \rangle = \int_0^1 \int_{-\infty}^{\infty} f(x, y) \phi(x, y) dxdy, \ \phi \in LU(w, z, c, n) .$$

(V) For each $m = 1, 2, 3, \cdots$ and $n \geqq -1/2$, $c \geqq 1/2$ and $a \leqq \operatorname{Re}(s) \leqq b$, the function $e^{-sx} y J_n(j_m y)$ is a member of $LU_{a,b,c,n}$.

For all $a > w$ and $z > b$, $e^{-sx} y J_n(j_m y)$ is a member of $LU(w, z, c, n)$.

4. **The generalized Finite-Hankel-Laplace transformation.** Let c, n satisfy $n \geqq -1/2$, $c \geqq 1/2$. We shall call a generalized function f as Finite-Hankel-Laplace-transformable if it belongs to $LU'(w, z, c, n)$ for some real number w, z. Let σ_f and ρ_f defined as follows:

$$\sigma_f = \inf \{w / f \in LU'(w, z, c, n)\}$$
$$\rho_f = \sup \{z / f \in LU'(w, z, c, n)\} .$$

We are now in a position to define the generalized Finite-Hankel-Laplace transformation, which we denoted by $\mathscr{L}\mathscr{H}_n$. For given Finite-Laplace-transformable generalized function f, let Ω_f denote the strip $\{s / \sigma_f < \operatorname{Re}(s) < \rho_f\}$ and let $\{j_m\}$ be the positive zeros of $J_n(z)$ arranged in ascending order. Then, the Finite-Hankel-Laplace transform $F(s, j_m)$ of f is defined as the application of f to the kernel $e^{-sx} y J_n(j_m y)$, i.e.,

$$\mathscr{L}\mathscr{H}_n(f)(s, j_m) = F(s, j_m) = \langle f(x, y), e^{-sx} y J_n(j_m y) \rangle \tag{4.1}$$

where $s \in \Omega_f$ and $\{j_m\}$ are the positive zeros of $J_n(z)$. For any $s \in \Omega_f$ and all j_m, the right-hand-side of (4.1) has meaning as the application of $f \in LU'(\sigma_f, \rho_f, c, n)$ to $ye^{-sx} J_n(j_m y) \in LU(\sigma_f, \rho_f, c, n)$ (or equivalently the application of $f \in LU'_{a,b,c,n}$ to $ye^{-sx}J_n(j_m y) \in LU_{a,b,c,n}$ for any $\sigma_f < a \leqq \mathrm{Re}\,(s) \leqq b < \rho_f$).

If $f(x, y)$ is a locally integrable function such that $ye^{-sx}J_n(j_m y)$ is absolutely integrable and $\sigma_f < a < b < \rho_f$ then its conventional Finite-Hankel-Laplace transform

$$\int_0^1 \int_{-\infty}^{\infty} ye^{-sx}J_n(j_m y)f(x, y)dxdy$$

exists for at least one $s \in \Omega_f$ and for all $j_{m'}$ where $\{j_m\}$ are positive zeros of $J_n(z)$ and can be identified with our generalized Finite-Hankel-Laplace transform (4.1).

5. **Inversion and uniqueness.** We shall now derive an inversion formula for Finite-Hankel-Laplace-tranformation. The proof of the inversion formula requires some lemmas.

LEMMA 5.1. *Let* $\mathscr{L}\mathscr{H}_n(f) = F(s, j_m)$ *for* $s \in \Omega_f$ *and* j_m, *let* $\phi(x, y) \in D(I)$, *and set for* $0 < a' < b' < 1$

$$\Phi(s, j_m) = \int_{a'}^{b'} \int_{-\infty}^{\infty} ye^{+sx}J_n(j_m y)\phi(x, y)dxdy\ .$$

Then for any fixed real number R, $0 < R < \infty$

$$\begin{aligned}&\int_{-R}^{R} \langle f(t, \tau), e^{-st}\tau J_n(j_m\tau)\rangle \Phi(s, j_m)dw \\ &= \left\langle f(t, \tau), \int_{-R}^{R} e^{-st}\tau J_n(j_m\tau)\Phi(s, j_m)dw \right\rangle\end{aligned} \tag{5.1}$$

where $s = \sigma + iw$ *and* σ *is fixed with* $\sigma_f < \sigma < \rho_f$.

Proof. As it is obvious that the functions $e^{-st}\tau J_n(j_m\tau)$ and $\int_{-R}^{R} e^{-st}\tau J_n(j_m\tau)\Phi(s, j_m)\, dw$ are members of the space $LU_{a,b,c,n}$ with t, τ as the variables of testing functions, the expressions on both the sides of (5.1) have sense. Using the technique of Riemann sums as used in [10, Lemma 3.5.1. p. 64], (5.1) can be easily established.

LEMMA 5.2. *Let* a', b' *be any two real numbers satisfying* $0 < a' < b' < 1$. *Then*

$$\lim_{R,N\to\infty} \int_{a'}^{b'} \int_{-\infty}^{\infty} yT_N(y, \tau)\frac{\mathrm{Sin}\, R(x-t)}{(x-t)}dxdy = \pi \tag{5.2}$$

when $-\infty < t < \infty$, $a' < \tau < b'$.

The proof can be left to the reader.

LEMMA 5.3. *Let* a, b, c *and* R *be real numbers such that* $a < \sigma < b$ *and* $c \geqq 1/2$, *let* $\psi(x, y) \in D(I)$. *Then for fixed* y, $\psi(x, y) \in D(I_y) I_y = \{(x, y)/ -\infty < x < \infty, y \text{ is fixed}\}$

$$(5.3) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} \psi(x, y) e^{\sigma(x-t)} \frac{\operatorname{Sin} R(x-t)}{(x-t)} dx$$

converges in $LU_{a,b,c,n}$ *to* $\psi(t, y)$ *as* $R \to \infty$.

The proof is similar to that of Lemma 3.5.2 [10, p. 66].

LEMMA 5.4. *If* $\psi(x, y) \in D(I)$, *then*

$$(5.4) \qquad \begin{aligned} &\frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} \int_{-\infty}^{\infty} [e^{\sigma(x-t)} \psi(x, y) \\ &\qquad - \psi(t, \tau)] y T_N(y, \tau) \frac{\operatorname{Sin} R(x-t)}{(x-t)} dx dy \end{aligned}$$

converges to zero uniformly as $R, N \to \infty$ *for all* $(t, \tau) \in (-\infty, \infty) \times (0, 1)$, *where the support of* $\psi(x, y)$ *is contained in* $[A, B] \times [a', b']$; *where* $-\infty < A < B < \infty$, $0 < a' < b' < 1$.

Proof. Let us divide the interval $(-\infty, \infty) \times (0, 1)$ into four disjoint sets $[(-\infty, A) U (B, \infty)] \times (0, 1)$, $(A, B) \times (b', 1)$, $(A, B) \times (0, a')$ and $[A, B] \times [a', b']$. For $(t, \tau) \in [(-\infty, A) U (B, \infty)] \times (0, 1)$, $\psi(t, \tau) = 0$ since $\psi(t, \tau)$ is supported by $[A, B] \times [a', b']$.

Therefore

$$\begin{aligned} &\frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} \int_{-\infty}^{\infty} [e^{\sigma(x-t)} \psi(x, y) - \psi(t, \tau)] y T_N(y, \tau) \frac{\operatorname{Sin} R(x-t)}{(x-t)} dx dy \\ &\quad = \frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} \int_{-\infty}^{\infty} e^{\sigma(x-t)} \psi(x, y) y T_N(y, \tau) \frac{\operatorname{Sin} R(x-t)}{(x-t)} dx dy \,. \end{aligned}$$

In view of Lemma 5.3 as $R \to \infty$, this integral reduces to

$$\frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} y \psi(t, y) T_N(y, \tau) dy \,.$$

Thus we want to show that for fixed t and $0 < y < 1$

$$(5.5) \qquad \lim_{N \to \infty} \frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} y \psi(t, y) T_N(y, \tau) dy = 0$$

uniformly for all (t, τ). Since $\psi(t, y) \in D(I_t)$,

$$I_t = \{(t, y)/t \text{ is fixed} \,, \quad 0 < y < 1\}$$

then $\psi(t, y)$ is bounded say by K

$$\left|k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}[\psi(t, y)yT_N(y, \tau)]dy\right|$$
$$\leqq K\tau^c k_{a,b}^{(t)}\int_{a'}^{b'}|yT_N(y, \tau)|dy .$$

In view of the analogue of Riemann Lebesque lemma [6, p. 589], for given $\varepsilon > 0$ there exists a positive integer N_0 such that

$$\left|\int_{a'}^{b'}yT_N(y, \tau)dy\right| \leqq \frac{8c_1^2\varepsilon}{\pi c_2^2(2-\tau-b')\sqrt{\tau}}$$

for all $N \geqq N_0$, which is bounded by $c'\varepsilon/(1-b')\sqrt{\tau}.$ Therefore, for all $N \geqq N_0$ and for all $(t, \tau) \in [(-\infty, A)U(B, \infty)] \times (0, 1)$

$$k_{a,b}^{(t)}\tau^c\left|\int_{a'}^{b'}y\psi(t, y)T_N(y, \tau)dy\right| \leqq \frac{c' \in \tau^{c-1\backslash 2}}{(1-b')} \leqq \frac{c'\varepsilon}{(1-b')}$$

since $c \geqq 1/2$. Hence as ε is arbitrary, we have (5.5) as stated above.

Thus

$$\begin{aligned}\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}\int_{-\infty}^{\infty}&[e^{\sigma(x-t)}\psi(x, y)\\ &-\psi(t, \tau)]yT_N(y, \tau)\frac{\operatorname{Sin} R(x-t)}{(x-t)}dxdy \longrightarrow 0\end{aligned} \tag{5.6}$$

as $R, N \rightarrow \infty$, uniformly for all $(t, \tau) \in [(-\infty, A)U(B, \infty)] \times (0, 1)$.

In a similar manner we can prove that for all $(t, \tau) \in (A, B) \times (b', 1)$ and $(t, \tau) \in (A, B) \times (0, a')$

$$\begin{aligned}\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}\int_{-\infty}^{\infty}&[e^{\sigma(x-t)}\psi(x, y) - \psi(t, \tau)]\\ &\times yT_N(y, \tau)\frac{\operatorname{Sin} R(x-t)}{(x-t)}dxdy \longrightarrow 0\end{aligned} \tag{5.7}$$

uniformly as $R, N \rightarrow \infty$.

Next we want to show that

$$\begin{aligned}\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}\int_{-\infty}^{\infty}&[e^{\sigma(x-t)}\psi(x, y) - \psi(t, \tau)]\frac{\operatorname{Sin} R(x-t)}{(x-t)}\\ &\times yT_N(y, \tau)dxdy \longrightarrow 0 \quad \text{as} \quad R, N \longrightarrow \infty\end{aligned}$$

uniformly for all $(t, \tau) \in [A, B] \times [a', b']$.

Now

$$\begin{aligned}\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}\int_{-\infty}^{\infty}&e^{\sigma(x-t)}\psi(x, y) - \psi(t, \tau)]\\ &\times\frac{\operatorname{Sin} R(x-t)}{(x-t)}yT_N(y, \tau)dxdy\end{aligned}$$

$$\longrightarrow \frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} [\psi(t, y) - \psi(t, \tau)] y T_N(y, \tau) dy ,$$

$$\text{as} \quad R \longrightarrow \infty .$$

Hence we need to show that for fixed t and $a' < \tau < b'$

$$\frac{1}{\pi} k_{a,b}^{(t)} \tau^c \int_{a'}^{b'} [\psi(t, y) - \psi(t, \tau)] y T_N(y, \tau) dy \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

uniformly for all (t, τ). Let $F(y, t, \tau)(y^2 - \tau^2) = y^{-n}[\psi(t, y) - \psi(t, \tau)]$ for $0 < y < 1$, $0 < \tau < 1$, and t is fixed. Now define function

$$\begin{aligned} G(y, t, \tau) &= F(y, t, \tau),\ y = \tau \\ &= \frac{y^{-n} D\psi(t, y)}{2y},\ y \neq \tau,\ D = \frac{\partial}{\partial y} , \end{aligned}$$

$G(y, t, \tau)$ is continuous of y, t, and τ in the domain $\{t$ is fixed, $0 < y < 1, 0 < \tau < 1\}$. Now

$$\begin{aligned} &\int_{a'}^{b'} y[\psi(t, y) - \psi(t, \tau)] T_N(y, \tau) dy \\ &\quad = \int_{a'}^{b'} y^{n+1} F(y, t, \tau)(y^2 - \tau^2) T_N(y, \tau) dy \\ &\quad = \int_{a'}^{b'} y^{n+1} G(y, t, \tau)(y^2 - \tau^2) T_N(y, \tau) dy \end{aligned}$$

as the value of the integral remains unchanged by replacing expression $f(y, t, \tau)(y^2 - \tau^2)$ by $G(y, t, \tau)(y^2 - \tau^2)$.

Let us now divide the interval $a' \leqq \tau \leqq b'$ in to p equal parts by the points $a' = y_0, y_1, \cdots, y_p = b'$ and after choosing positive number ε, p so large that

$$\sum_{m=1}^{p} (U_m - L_m)(y_m - y_{m-1}) < \varepsilon$$

where U_m and L_m are upper and lower bounds of $G(t, y, \tau)$ in $y_{m-1} \leqq y \leqq y_m$, $a' \leqq \tau \leqq b'$. Let $G(t, y, \tau) = G(t, y_{m-1}, \tau) + w_m(t, y, \tau)$ for $y_{m-1} \leqq y \leqq y_m$, $a' \leqq \tau \leqq b'$ so that $|w_m(t, y, \tau)| \leqq U_m - L_m$.

Using uniform continuity of the function $G(t, y, \tau)$ over the region $\{a' \leqq y \leqq b', a' \leqq \tau \leqq b'\}$ and following the lines in the proof of the analogue of Riemann Lebesgue lemma [6, p. 589], for an arbitrary $\varepsilon > 0$ we get a positive integer N_1 such that

$$\left| \int_{a'}^{b'} y^{n+1} G(t, y, \tau)(y^2 - \tau^2) T_N(y, \tau) dy \right| \leqq \frac{c' \cdot \varepsilon}{(1 - b')\sqrt{\tau}}$$

for all $N \geqq N_1$. Hence for all (t, τ), t is fixed and $a' < \tau < b'$

$$k_{a,b}^{(t)}\tau^c\int_{a'}^{b'} y[\psi(t, y) - \psi(t, \tau)]T_N(y, \tau)dy$$
$$\leqq \frac{c' \in \tau^{c-1\backslash 2}}{(1-b')} \leqq \frac{c'\varepsilon}{(1-b')} \quad \text{where} \quad c \geqq \frac{1}{2} .$$

Since ε is arbitrary,

$$\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'} y[\psi(t, y) - \varphi(t, \tau)]T_N(y, \tau)dy \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

uniformly for all fixed t and $\tau \in [a', b']$.

Thus

$$\frac{1}{\pi}k_{a,b}^{(t)}\tau^c\int_{a'}^{b'}\int_{-\infty}^{\infty}[e^{a(x-t)}\psi(x, y) - \psi(t, \tau)] \tag{5.8}$$
$$\frac{\text{Sin } R(x-t)}{(x-t)} y T_N(y, \tau)dxdy \longrightarrow 0$$

uniformly for all $(t, \tau) \in [A, B] \times [a', b']$ as $R, N \to \infty$. Combining (5.6), (5.7) and (5.8), the lemma yields.

LEMMA 5.5. *For* $\phi(x, y) \in D(I)$, $0 < a' < b' < 1$

$$\Phi(s, j_m) = \int_{a'}^{b'}\int_{-\infty}^{\infty} y e^{+sx} J_n(j_m y)\phi(x, y)dxdy . \tag{5.9}$$

$$\tau M_{R,N}(t, \tau) = \tau\left[\sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\frac{1}{2\pi}\int_{-R}^{R}[e^{-st}J_n(j_m\tau)\Phi(s, j_m)]dw\right] \tag{5.10}$$

converges in $LU_{a,b,c,n}$ *to* $\tau\phi(t, \tau)$ *as* $R, N \to \infty$ *for all* $(t, \tau) \in (-\infty, \infty) \times (0, 1)$.

Proof. Since the integrand in (5.10) is a smooth function and ϕ is of bounded support, we may differentiate under the integral sign, and obtain

$$D_t^k \Omega_{n,\tau}^{k'}[\tau^{-1}\cdot\tau M_{R,N}(t, \tau)]$$
$$= \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\left[\frac{1}{2\pi}\int_{-R}^{R} D_t^k \Omega_{n,\tau}^{k'}[e^{-st}J_n(j_m\tau)]\Phi(s, j_m)dw\right]$$
$$= \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\left\{\frac{1}{2\pi}\int_{-R}^{R}(-1)^{k+k'}s^k j_m^{2k'}e^{-st}J_n(j_m\tau)\right.$$
$$\left.\times\left[\int_{a'}^{b'}\int_{-\infty}^{\infty} e^{sx}yJ_n(j_m y)\phi(x, y)dxdy\right]dw\right\}$$
$$= \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\left\{\frac{1}{2\pi}\int_{-R}^{R}(-1)^k e^{-st}J_n(j_m\tau)\right.$$
$$\left.\times\left[\int_{a'}^{b'}\int_{-\infty}^{\infty}(-1)^{k'}s^k j_m^{2k'} y e^{sx}J_n(j_m y)\phi(x, y)\, dxdy\right]dw\right\}$$

$$= \sum_{m=1}^{N} \frac{2}{J_{n+1}^2(j_m)} \left\{ \frac{1}{2\pi} \int_{-R}^{R} (-1)^k e^{-st} J_n(j_m \tau) \right.$$

$$\left. \times \left[\int_{a'}^{b'} \int_{-\infty}^{\infty} D_x^k \Omega_{n,y}^{k'} [e^{sx} J_n(j_m y)] y \phi(x, y)^{-} dx dy \right] dw \right\} .$$

Now we consider

$$(-1)^k \int_{a'}^{b'} \int_{-\infty}^{\infty} D_x^k \Omega_{n,y}^{k'} [e^{sx} J_n(j_m y)] y \phi(x, y) dx dy .$$

Upon integrating by parts the inner integral k times and since $\phi(x, y)$ is of compact support this integral reduces to

$$\int_{a'}^{b'} y \Omega_{n,y}^{k'} J_n(j_m y) \left[\int_{-\infty}^{\infty} D_x^k [\phi(x, y)] e^{sx} dx \right] dy .$$

Again integrating by parts $2k'$ times we get

$$\int_{a'}^{b'} \int_{-\infty}^{\infty} D_x^k \Omega_{n,y}^{k'} [\phi(x, y)] y e^{sx} J_n(j_m y) dx dy .$$

Therefore

$$D_t^k \Omega_{n,\tau}^{k'} [\tau^{-1} \cdot \tau M_{R,N}(t, \tau)] = \frac{1}{2\pi} \sum_{m=1}^{N} \frac{2}{J_{n+1}^2(j_m)} \left\{ \int_{-R}^{R} e^{-st} J_n(j_m \tau) \right.$$

$$\left. \times \left[\int_{a'}^{b'} \int_{-\infty}^{\infty} D_x^k \Omega_{n,y}^{k'} [\phi(x, y)] y e^{sx} J_n(j_m y) dx dy \right] dw \right\} .$$

Changing the order of integration, we obtain

$$D_t^k \Omega_{n,\tau}^{k'} [\tau^{-1} \tau M_{R,N}(t, \tau)]$$

$$= \frac{1}{2\pi} \int_{a'}^{b'} \int_{-\infty}^{\infty} \left[D_x^k \Omega_{n,y}^{k'} [\phi(x, y)] y T_N(y, \tau) \left[\int_{-R}^{R} e^{s(x-t)} dw \right] dx dy \right]$$

$$= \frac{2}{2\pi} \int_{a'}^{b'} \int_{-\infty}^{\infty} D_x^k \Omega_{n\ y}^{k'} [\phi(x, y)] y T_N(y, \tau) \frac{\operatorname{Sin} R(x-t)}{(x-t)} e^{\sigma(x-t)} dx dy .$$

Hence in light of Lemma 5.2 we have as $R, N \to \infty$

$$D_t^k \Omega_{n,\tau}^{k'} \tau^{-1} [\tau M_{R,N}(t, \tau) - \tau \phi(t, \tau)]$$

$$= \frac{1}{\pi} \int_{a'}^{b'} \int_{-\infty}^{\infty} [e^{\sigma(x-t)} D_x^k \Omega_{n,y}^{k'} [\phi(x, y)] - D_t^k \Omega_{n,\tau}^{k'} [\phi(t, \tau)]]$$

$$\times \frac{\operatorname{Sin} R(x-t)}{(x-t)} y T_N(y, \tau) dx dy$$

$$= \frac{1}{\pi} \int_{a'}^{b'} \int_{-\infty}^{\infty} [e^{\sigma(x-t)} \psi(x, y) - \psi(t, \tau)] \frac{\operatorname{Sin} R(x-t)}{(x-t)} y T_N(y, \tau) dx dy$$

where $\psi(x, y) = D_x^k \Omega_{n,y}^{k'} [\phi(x, y)]$ which is again a member of $D(I)$ with support contained in $[A, B] \times [a', b']$. Hence it suffices to show that

$$\frac{1}{\pi}k_{a,b}^{(t)}\tau^{c}\left[\int_{a'}^{b'}\int_{-\infty}^{\infty}[e^{\sigma(x-t)}\psi(x, y) - \psi(t, \tau)]\right]$$
$$\times\frac{\operatorname{Sin} R(x-t)}{(x-t)}yT_N(y, \tau)dxdy$$

converges to zero as $R, N \to \infty$ uniformly for all $(t, \tau) \in (-\infty, \infty) \times (0, 1)$. This is true in view of Lemma 5.4, and thus the proof of Lemma 5.5 is complete.

6. The main theorem. *Let $f(x, y)$ be a Finite-Hankel-Laplace-transformable function and $F(s, j_m)$ the generalized Finite-Hankel-Laplace-transform of f as defined by*

$$F(s, j_m) = \langle f(x, y), e^{-sx}yJ_n(j_m y)\rangle$$

for $s \in \Omega_f$ and $\{j_m\}$, the positive zeros of $J_n(z)$.

Then in the sense of convergence in $D'(I)$

$$f(x, y) = \lim_{R,N\to\infty}\frac{1}{2\pi i}\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\int_{\sigma-iR}^{\sigma+iR}e^{sx}F(s, j_m)ds \tag{6.1}$$

where σ is any fixed number $\sigma_f < \sigma < \rho_f$.

Proof. Let $\phi(x, y)$ be an arbitrary member of $D(I)$. We wish to show that

$$\begin{aligned}&\left\langle\frac{1}{2\pi i}\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\int_{\sigma-iR}^{\sigma+iR}e^{sx}F(s, j_m)ds, \phi(x, y)\right\rangle\\ &\quad= \langle f(t, \tau), \phi(t, \tau)\rangle \quad \text{as} \quad R, N \longrightarrow \infty .\end{aligned} \tag{6.2}$$

Since $\phi(x, y) \in D(I)$ iff $y\phi(x, y) \in D(I)$, then (6.2) will be equivalent to showing that

$$\begin{aligned}&\left\langle\frac{1}{2\pi i}\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\int_{\sigma-iR}^{\sigma+iR}e^{sx}F(s, j_m)ds, y\phi(x, y)\right\rangle\\ &\quad= \langle f(t, \tau), \tau\phi(t, \tau)\rangle \quad \text{as} \quad R, N \longrightarrow \infty .\end{aligned} \tag{6.3}$$

As $\phi(x, y) \in D(I)$, let us assume that the support of $\phi(x, y)$ is contained in $[A, B] \times [a', b']$, where $-\infty < A < B < \infty$, $0 < a' < b' < 1$. As

$$\frac{1}{2\pi i}\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\int_{\sigma-iR}^{\sigma+iR}e^{sx}F(s, j_m)ds$$

is locally integrable and since $y\phi(x, y) \in D(I)$ then without limit notation (6.3) can be written as

$$\int_{a'}^{b'}\int_{-\infty}^{\infty}y\phi(x, y)\left[\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}e^{sx}F(s, j_m)ds\right]dxdy .$$

Put $s = \sigma + iw$. We get

$$\int_{a'}^{b'}\int_{-\infty}^{\infty} y\phi(x, y)\Big[\sum_{m=1}^{N}\frac{2J_n(j_m y)}{J_{n+1}^2(j_m)}\frac{1}{2\pi}\int_{-R}^{R} e^{sx}F(s, j_m)dw\Big]dxdy .$$

Since $\phi(x, y)$ has a compact support and the integrand is a continuous function of (x, y, w) we can interchange the order of integration.

$$\sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\Big\{\frac{1}{2\pi}\int_{-R}^{R}\langle f(t, \tau), e^{-st}\tau J_n(j_m\tau)\rangle \times\Big[\int_{a'}^{b'}\int_{-\infty}^{\infty} e^{sx}J_n(j_m y)y\phi(x, y)dxdy\Big]dw\Big\}$$
$$= \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\Big\{\frac{1}{2\pi}\int_{-R}^{R}\langle f(t, \tau), e^{-st}\tau J_n(j_m\tau)\rangle\Phi(s, j_m)dw\Big\}$$

where

$$\Phi(s, j_m) = \int_{a'}^{b'}\int_{\infty}^{\infty} e^{sx}yJ_n(j_m y)\phi(x, y)dxdy .$$

Now by Lemma 5.1 we have

$$\sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\Big\{\frac{1}{2\pi}\int_{-R}^{R}[\langle f(t, \tau), e^{-st}\tau J_n(j_m\tau)\rangle\Phi(s, j_m)]dw\Big\}$$
$$= \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\Big\{\Big\langle f(t, \tau), \frac{1}{2\pi}\int_{-R}^{R} e^{-st}\tau J_n(j_m\tau)\Phi(s, j_m)dw\Big\rangle\Big\} .$$

Since f is a continuous linear functional, we have

$$\sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\Big\langle f(t, \tau), \frac{1}{2\pi}\int_{-R}^{R} e^{-st}\tau J_n(j_m\tau)\Phi(s, j_m)dw\Big\rangle$$
$$= \Big\langle f(t, \tau), \sum_{m=1}^{N}\frac{2}{J_{n+1}^2(j_m)}\frac{1}{2\pi}\int_{-R}^{R} e^{-st}\tau J_n(j_m\tau)\Phi(s, j_m)dw\Big\rangle .$$

Because $f \in LU'_{a,b,c\ n}$ and in view of Lemma 5.5, the last expression tends to $\langle f(t, \tau\rangle, \tau\phi(t, \tau)\rangle$ as $R, N \to \infty$. This completes our proof of the main theorem.

Uniqueness Theorem 6.1. *If* $\mathscr{L}\mathscr{H}_n(f) = F(s, j_m)$ *and* $\mathscr{L}\mathscr{H}_n(g) = G(s, j_m)$ *for all* $s \in \Omega_f = \{s/\sigma_f < \mathrm{Re}(s) < \rho_f\}$ *and* $s \in \Omega_g = \{s/\sigma g < \mathrm{Re}(s) < \rho_g\}$ *and* $\{j_m\}$, *positive zeros of* $J_n(z)$, *if* $\Omega_f \cap \Omega_g \neq 0$, *and if* $F(s, j_m) = G(s, j_m)$ *for* $s \in \Omega_f \cap \Omega_g$, *then* $f = g$ *in the sense of equality in* $D'(I)$.

Proof. By above theorem, in the sense of convergence in $D'(I)$

$$f = \lim_{R,N\to\infty} \left\{\frac{1}{2\pi i} \sum_{m=1}^{N} \frac{2J_n(j_m y)}{J_{n+1}^2(j_m)} \left[\int_{\sigma-iR}^{\sigma+iR} e^{sx} F(s, j_m) ds\right]\right\}$$
$$= \lim_{R,N\to\infty} \left\{\frac{1}{2\pi i} \sum_{m=1}^{N} \frac{2J_n(j_m y)}{J_{n+1}^2(j_m)} \left[\int_{\sigma-iR}^{\sigma+iR} e^{sx} G(s, j_m) ds\right]\right\}$$
$$= g(x, y).$$

Hence $f = g$, in the sense of equality in $D'(I)$.

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