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ON MAPPING AN n-BALL INTO AN $(n+1)$-BALL IN COMPLEX SPACE

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This paper is concerned with proper holomorphic mappings from the unit n-ball B^n in complex n-space C^n into the unit $(n+1)$-ball in C^{n+1}. It will be shown that if such a mapping f is sufficiently regular at the boundary then the image of f lies in a complex hyperplane, provided $n \geqq 3$.

The main aspect of the problem dealt with here is the minimum regularity assumptions required of f at the boundary. The main result gives the extension past almost all real analytic boundary points of a proper bolomorphic mapping f of a domain in C^n into B^{n+1} assuming f continues to a C^3 immersion of the boundary.

Biholomorphic mappings between strongly pseudo-convex domains with C^∞ or real analytic boundaries have received much more attention. It has recently become clear that the boundary smoothness of such mappings follows rather easily if one assumes initially some small amount of regularity. See [2], [4], and [5] for the real analytic case, and [3] for the C^∞ case.

We shall show that a similar situation holds in the present case. More precisely, we prove the following theorem, which is of a local nature.

THEOREM. *Let $D \subset C^n$, $n \geqq 3$, be a domain which contains a strongly pseudo-convex analytic real hypersurface M in its boundary. Let f be a holomorphic mapping of D into the unit ball $B^{n+1} \subset C^{n+1}$ which extends to a three times continuously differentiable immersion of M into the unit sphere S^{2n+1}. Then f extends holomorphically to a neighborhood of every point in some dense open subset of M.*

In order to apply this theorem to mappings of B^n into B^{n+1}, we recall the following result, which is Theorem 3.1 of [6]:

PROPOSITION. *Let V be a nonsingular portion of a complex hypersurface in C^{n+1} and let $N = V \cap S^{2n+1}$. Suppose $n \geqq 3$ and N is locally equivalent to S^{2n-1} as C-R manifolds. Then V is an open subset of a complex hyperplane.*

This is proved using the Chern-Moser theory and a "pseudo-conformal" analogue of the Gauss equations. See [6] for more details.

The theorem and proposition give immediately the following corollary.

COROLLARY. *In addition to the hypothesis of the theorem suppose that D is the unit ball $B^n \subset C^n$, $n \geqq 3$. Then the image of f lies in a complex hyperplane.*

The corollary is false for $n = 2$, as the simple example $(z, w) \to (z^2, \sqrt{2}zw, w^2)$ of H. Alexander shows.

For the proof of the theorem let $z = (z_1, \cdots, z_n)$ be coordinates on C^n and $r(z, \bar{z})$, $dr \neq 0$, be a real analytic function vanishing on M. Restricting to a neighborhood of a point of M, we may assume $r_n \neq 0$ and put

$$(1) \qquad L_\alpha = r_n(\partial/\partial z_\alpha) - r_\alpha(\partial/\partial z_n), \quad 1 \leqq \alpha < n ,$$

where

$$r_j = \partial r/\partial z_j , \quad r_{\bar{j}} = \partial r/\partial \bar{z}_j , \quad \text{etc.}$$

Also, let $z' = (z'_1, \cdots, z'_{n+1})$ be coordinates on C^{n+1} and put $r'(z', \bar{z}') = z'\cdot\bar{z}' - 1$, where $z'\cdot\bar{z}' = \Sigma z'_j\bar{z}'_j$. The map f is given by $z' = f(z)$.

We begin with a lemma, the proof of which is an easy extension of the argument given by H. Lewy in [2] for the equidimensional case. See also [5] and S. I. Pinčuk [4]. Lemma I.3 of [4] applies to our case also and allows us to drop the requirement that f be an immersion along M.

LEMMA. *Let z_0 be a fixed point of M. Suppose that for some choice of α and β the vectors f, $L_\gamma f$, $1 \leqq \gamma < n$, and $L_\alpha L_\beta f$ are linearly independent over the complex numbers at the point z_0. Then under the assumptions of the theorem f continues holomorphically to a neighborhood of z_0 in C^n.*

Proof. Under our assumptions we have $r'\cdot f = ur$ where $u > 0$ is real analytic in D and C^3 on $D \cup M$. Since $L_\alpha r \equiv L_\alpha \bar{f} \equiv 0$ we also have

$$L_\alpha(r'\cdot f) = L_\alpha f\cdot\bar{f} = (L_\alpha u)r ,$$

$$L_\alpha L_\beta(r'\cdot f) = L_\alpha L_\beta f\cdot\bar{f} = (L_\alpha L_\beta u)r .$$

Thus, these functions are continuous on $D \cup M$ and vanish on M:

$$(2) \qquad r = 0 \Longrightarrow f\cdot\bar{f} - 1 = L_\alpha f\cdot\bar{f} = L_\alpha L_\beta f\cdot\bar{f} = 0 .$$

Now we fix a complex line l which intersects M transversely near z_0 and let w be a point of l near z_0 but outside $D \cup M$. Viewing $r(z, \bar{z})$ as a power series in z and $\bar{z}$, we can define a unique z in $l \cap D$ by the equation $r(z, \bar{w}) = 0$, since $r_n \neq 0$. This point $z = z(\bar{w})$ depends anti-holomorphically on w. With z thus fixed we define w' implicitly by the following equations

$$(3) \qquad f(z)\cdot\bar{w}' = 1 \,, \quad L_\alpha f(z, \bar{w})\cdot\bar{w}' = L_\alpha L_\beta f(z, \bar{w})\cdot\bar{w}' = 0 \,.$$

These equations are linear in $\bar{w}'$ with coefficients which are anti-holomorphic in w for w in l and outside $D \cup M$. The implicit function theorem (or Cramer's rule) yields a unique solution $w' = \tilde{f}(w)$ which is holomorphic for w in l and outside $D \cup M$. As w approaches a point in M its image $z(\bar{w})$ approaches the same point. The continuity assumptions, the uniqueness of solutions to (3), and equation (2) guarantee that f and $\tilde{f}$ agree on $M \cap l$. Hence, f extends holomorphically to l. We now vary the line l parallel to itself as in [2] obtaining a continuous extension of f which is holomorphic on either side of each $l \cap M$. By Morera's theorem f is seen to be holomorphic in the parameters on which l depends and hence holomorphic. For further details we refer to [2].

We now consider the case in which $f, L_\gamma f, 1 \leqq \gamma < n$, and all $L_\alpha L_\beta f$ are linearly dependent at all points of some open subset of M. By (2) and the fact that f is an immersion on M it follows that the $L_\alpha L_\beta f$ are linear combinations of the $L_\gamma f$. This says that part of the "second fundamental form" of M vanishes.

To study the behavior of M immersed in S^{2n+1} we consider a unitary frame field $e_0, e_\alpha, 1 \leqq \alpha < n, e_n$ adapted to M as follows. Let $e_0 = if = iz'$, the e_α span the holomorphic tangent space $H(M)$ of M, and e_n be in the holomorphic tangent space $H(S^{2n+1})$ and orthogonal to $H(M)$. We define differential one-forms θ_i, ω_{ij}, by

$$(4) \qquad df = dz' = \sum_{i=0}^{n} \theta_i e_i \,,$$

$$(5) \qquad de_i = \sum_{j=0}^{n} \omega_{ij} e_j \,.$$

Since the frame is unitary and $e_0 = iz'$ we have

$$(6) \qquad \begin{aligned} &\omega_{ij} + \bar{\omega}_{ji} = 0 \,, \\ &\omega_{0j} = i\theta_j,\ \omega_{j0} = i\bar{\theta}_j \,. \end{aligned}$$

Now let $e = e_0 + ae_n$, where the C^2 function a is chosen so that e is tangent to M. From (4) we get

$$(7) \qquad dz' = \theta_0 e + \theta_\alpha e_\alpha + (\theta_n - a\theta_0)e_n \,;$$

repeated Greek indices are summed from one to $n-1$. Hence, $\theta_n =$

$a\theta_0$ when restricted to M. Substituting into (5) gives

$$(8) \qquad de = i(1 + a\bar{a})\theta_0 e + (i\theta_\alpha + a\omega_{n\alpha})e_\alpha + (da + a\omega_{nn} - ia^2\bar{a}\theta_0)e_n \, ,$$

and

$$(9) \qquad de_\alpha = i\bar{\theta}_\alpha e + \omega_{\alpha\beta}e_\beta + (\omega_{\alpha n} - ia\bar{\theta}_\alpha)e_n \, .$$

We shall show that

$$(10) \qquad \begin{aligned} \omega_{\alpha n} - ia\bar{\theta}_\alpha &\equiv 0 \, , \quad \bmod \theta_0 \, , \\ da + a\omega_{nn} &\equiv 0 \, , \quad \bmod \theta_0 \, . \end{aligned}$$

It follows from (4) and (6) that $\theta_0 = -i\omega_{00}$ is a nonzero real one-form annihilating the holomorphic tangent spaces $H(S)$ and $H(M)$, and that the θ_i span the (1, 0)-forms restricted to S^{2n+1}. Since f is C^3 the exterior derivatives of (4) and (5) exist and $ddf = dde_i = 0$. Substituting (5) and (6) into these exterior derivatives gives

$$(11) \qquad \begin{aligned} d\theta_0 &= i\theta_\alpha \wedge \bar{\theta}_\alpha \, , \\ d\theta_\alpha &= \theta_\beta \wedge (\omega_{\beta\alpha} - i\delta_{\beta\alpha}\theta_0) + a\theta_0 \wedge \omega_{n\alpha} \, , \\ d\omega_{\alpha n} &= (\omega_{\alpha\beta} - \delta_{\alpha\beta}\omega_{nn}) \wedge \omega_{\beta n} + a\theta_0 \wedge \bar{\theta}_\alpha \, . \end{aligned}$$

It follows from the first of these equations that the Levi form of M relative to this coframe is the identity matrix $\delta_{\alpha\beta}$.

Let X_α, $\bar{X}_\alpha$, and $X = \bar{X}$ be the vector fields on M dual to θ_α, $\bar{\theta}_\alpha$, θ_0. It follows that the X_α are linear combinations of the operators (1). From (4) we have $X_\alpha f = e_\alpha$ and $Xf = e$. By definition of the Levi form.

$$\bar{X}_\beta e_\alpha = \bar{X}_\beta X_\alpha f = [\bar{X}_\beta, X_\alpha]f = i\delta_{\beta\alpha}e + B_{\beta\alpha\gamma}e_\gamma \, ,$$

for some functions B, since $\bar{X}_\alpha f = 0$. Our present assumption implies that $X_\beta e_\alpha$ is a linear combination of the e_γ,

$$X_\beta e_\alpha = A_{\beta\alpha\gamma}e_\gamma \, .$$

Thus,

$$\begin{aligned} de_\alpha &= X_\beta e_\alpha \theta_\beta + \bar{X}_\beta e_\alpha \bar{\theta}_\beta + Xe_\alpha \theta_0 \\ &= (A_{\beta\alpha\gamma}\theta_\beta + B_{\beta\alpha\gamma}\bar{\theta}_\beta)e_\gamma + i\bar{\theta}_\alpha e + Xe_\alpha\theta_0 \, . \end{aligned}$$

From this and equation (9) we see that $\theta_0 = 0$ implies $\omega_{\alpha n} = ia\bar{\theta}_\alpha$; hence we put

$$(12) \qquad \omega_{\alpha n} = ia\bar{\theta}_\alpha + b_\alpha\theta_0 \, ,$$

for some functions b_α of class C^1. The first equation of (10) is proved.

Now we take the exterior derivative of (12), use the equations (11), and compute mod θ_0. This yields

$$[(da + a\omega_{nn})\delta_{\alpha\beta} + b_\alpha\theta_\beta] \wedge \bar{\theta}_\beta \equiv 0 \ , \quad \mathrm{mod}\, \theta_0 \ .$$

Hence,

$$(da + a\omega_{nn})\delta_{\alpha\beta} + b_\alpha\theta_\beta \equiv 0 \ , \quad \mathrm{mod}\, \theta_0 \ .$$

It is here that we must assume $n \geqq 3$. We can then take $\beta \neq \alpha$ and get $b_\alpha = 0$. Putting $\beta = \alpha$ gives the second equation of (10).

Now let $t \to z(t)$ be a smooth curve in M which is always tangent to the holomorphic tangent planes of M. From (8), (9), and (10) we have a homogeneous system of equations of the form

$$\frac{df}{dt} = \xi_0 e + \xi_\alpha e_\alpha \ ,$$

$$\frac{de}{dt} = \eta e + \eta_\alpha e_\alpha \ ,$$

$$\frac{de_\alpha}{dt} = i\bar{\xi}_\alpha e + \eta_{\alpha\beta} e_\beta \ ,$$

where the ξ's and η's are C^1 functions of t. It follows that the complex hyperplane spanned by e, e_α, $1 \leqq \alpha < n$, is a constant plane P, and that $f(z(t))$ remains in $P \cap S^{2n+1}$.

Because of the strong pseudo-convexity of M, which is reflected in the first equation of (11), the bundle $H(M)$ of holomorphic tangent planes forms a contact structure. By a classical theorem (see [1]) this contact structure is locally equivalent to the standard one given by the contact form $dz - y_1 dx_1 - \cdots - y_{n-1} dx_{n-1}$ on R^{2n-1}. For this contact structure it is clear that the set of points which may be connected to the origin by piecewise smooth (or even piecewise-linear) curves which are tangent to the distribution of planes contains an open set.

From these considerations it follows that an open subset of $f(M)$ lies in $P \cap S^{2n+1}$; hence $f(D \cup M) \subset P$. By the Lewy-Pinčuk theorem f extends holomorphically past M.

The above reasoning shows that the closed set B of points at which all $L_\alpha L_\beta f$ are dependent on the $L_\gamma f$ is either all of M, and f extends holomorphically past every point of M; or B has no interior and f extends to a neighborhood of every point of M not in B. This finishes the proof.

It seems reasonable to conjecture that the theorem is true under the weaker hypothesis that f be C^2 to the boundary, which is all that is required in the lemma. Also, the conclusion may hold for all points of M.

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