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ON Y-CLOSED SUBSPACES OF X, FOR BANACH SPACES $X \subset Y$; EXISTENCE OF ALTERNATING ELEMENTS IN SUBSPACES OF $C(J)$

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If $X \subset Y$ are Banach spaces, with continuous embedding, we consider property (P3): If $L \subset X$ is a closed subspace of Y, then L is finite dimensional. If the embedding $X \hookrightarrow Y$ is compact (property (P1)), then (P3) follows. It is shown that (P1) implies also (P2): In (P3) the dimension of L can be estimated from above in terms of the norm of the mapping id: $(L, \|\cdot\|_Y) \to (L, \|\cdot\|_X)$. For some examples which are known to satisfy (P3) but not (P1), we show that also (P2) is valid.

The main tool for the proof of (P1) $\Rightarrow$ (P2) is the existence of "alternating" elements in subspaces of R^k and $C[0, 1]$. In order to obtain such elements we investigate the structure of certain subsets of the unit cube in R^k.

Introduction. Let $(Y, \|\cdot\|_Y)$ be a Banach space, $X \subset Y$ a linear subspace which is also a Banach space $(X, \|\cdot\|_X)$, with continuous embedding id: $(X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$. If additionally

(P1) $\qquad$ id: $(X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$ *is compact* ,

then

(P3) *any $\|\cdot\|_Y$-closed subspace $L \subset Y$, which is contained in X, is finite dimensional.*

(The closed graph theorem shows that $\|\cdot\|_Y$ and $\|\cdot\|_X$ are equivalent on L, and then (P1) implies that the unit ball in L is relatively compact.) There are, however, examples of pairs (X, Y) which satisfy (P3) but not (P1) ([3], [6], [7]).

In order to state a quantitative version of (P3) we define the function $\varphi: [0, \infty) \to N_0 \cup \{\infty\}$,

$$\begin{aligned}\varphi(K)(=\varphi(K; X, Y))\\ := \sup\{\dim L;\ L \subset X \text{ linear subspace},\ \|x\|_X \leqq K\|x\|_Y (x \in L)\}\\ = \sup\{\dim L;\ L \subset X \text{ linear subspace},\ N_L \leqq K\},\end{aligned}$$

where

$$N_L := \sup\{\|x\|_X;\ x \in L,\ \|x\|_Y \leqq 1\}$$

is the norm of the mapping id: $(L, \|\cdot\|_Y) \to (L, \|\cdot\|_X)$. Then

(P2) $\qquad \varphi(K) < \infty \quad$ *for all* $\quad K \in [0, \infty)$

implies (P3): If $L \subset X$ is $\|\cdot\|_Y$-closed in Y, then id: $(L, \|\cdot\|_Y) \to (L, \|\cdot\|_X)$ is continuous by the closed graph theorem, i.e., $N_L < \infty$. Therefore $\dim L \leqq \varphi(N_L) < \infty$.

With the function $\psi: N_0 \to [0, \infty]$,

$$\psi(n)(=\psi(n; X, Y)) := \inf\{N_L;\ L \subset X \ \text{linear subspace},\ \dim L \geqq n\}$$

($\psi(n) = \infty$ if $n > \dim X$), it is easy to see that (P2) is equivalent to

(P2′) $$\psi(n) \longrightarrow \infty \quad \textit{for} \quad n \longrightarrow \infty .$$

The function φ was defined in [6], [7] in the context considered there. For the same case Pajor defined the function ψ and noticed the equivalence of (P2) and (P2′) (private communication of M. Rogalski to R. Tandler, 1977).

For $Y := C[0, 1]$, $X := W_p^1(0, 1)$, $1 < p \leqq \infty$, property (P1) is satisfied. In [6], [7] it is shown that also (P2) is satisfied. The function φ is not calculated explicitly, its finiteness being proved by a compactness argument. The search for an explicit expression of φ for this case was the starting point of this paper. We obtain it not only for the cases $1 < p \leqq \infty$ but also for $p = 1$, thereby giving a new proof of (P3) for this latter case (§4).

For the computation of φ in the cases just mentioned we use the existence of alternating elements in the $\|\cdot\|_\infty$-unit ball of subspaces of $C[0, 1]$ (§2). These elements are also used to obtain (P1) $\Rightarrow$ (P2) for the special case $X \subset Y = C[0, 1]$. It turns out that this case is already the general case for a pair (X, Y) satisfying (P1)(§3).

1. Subsets of alternating elements in the unit cube in $\boldsymbol{R}^k$. We are going to use the following notations: For $k \in \boldsymbol{N}_0$ we denote by

$$E^k := [-1, 1]^k = \{x \in \boldsymbol{R}^k;\ \|x\|_\infty \leqq 1\},\ E^0 = \{0\} ,$$

the unit cube in $\boldsymbol{R}^k$(=unit ball in $(\boldsymbol{R}^k, \|\cdot\|_\infty)$).

For $n \in \boldsymbol{N}$ we define

$$F_{k,n}^{\pm} := \{x \in E^k;\ \text{there exist } 1 \leqq j_1 < j_2 < \cdots < j_n \leqq k \text{ such that } x_{j_r} = \pm(-1)^{n-r} (r = 1, \cdots, n)\} ,$$
$$F_{k,n} := F_{k,n}^- \cup F_{k,n}^+ .$$

For convenience of notation we also denote $F_{k,0} = F_{k,0}^- = F_{k,0}^+ = E^k$. Let us note that $F_{k,1} = \{x \in E^k;\ \|x\|_\infty = 1\}$ is just the boundary of E^k. Also the following identities are easily proved:

$$F_{k,n}^- = -F_{k,n}^+,\ F_{k,n} = -F_{k,n} ,$$
$$F_{k,n}^+ \cap F_{k,n}^- = F_{k,n+1} (n \in \boldsymbol{N}),\ F_{k,n} = \emptyset \quad \text{for} \quad n > k .$$

THEOREM 1.1.[1] *For all $k \in N_0$ there exists a homeomorphism*

$$g_k: E^k \longrightarrow F^+_{k+1,1} ,$$

with the following additional properties:

(a) $g_k(-x) = -g_k(x)$ *for all* $x \in F_{k,1}$,

(b) *for all* $n \in N$, *the mapping* g_k *induces homeomorphisms*

$$g_k|_{F^{(\pm)}_{k,n}}: F^{(\pm)}_{k,n} \longrightarrow F^{(\pm)}_{k+1,n+1} .$$

Proof. We proceed by induction on k. For $k = 0$ we have the unique map $g_0: E^0 = \{0\} \to F^+_{1,1} = \{(1)\}$, which has properties (a), (b) because the sets considered there are empty.

Let us assume that the statement is true for $k-1$. Then we define a mapping

$$\dot{g}_k: F_{k,1} \longrightarrow F_{k+1,2} ,$$

$$\dot{g}_k(x): = \begin{cases} (g_{k-1}(x_1, \cdots, x_{k-1}), \ -1) & \text{if} \quad x_k = -1 , \\ (g_{k-1}(x_1, \cdots, x_{k-1}), x_k) & \text{if} \quad -1 < x_k < 1 , \\ (-g_{k-1}(-x_1, \cdots, -x_{k-1}), 1) & \text{if} \quad x_k = 1 , \end{cases}$$

for $x = (x_1, \cdots, x_k) \in F_{k,1}$. At the end of proof we are going to extend $\dot{g}_k$ to the desired homeomorphism g_k. Before we do this we want to show some properties of $\dot{g}_k$.

(i) We show that $\dot{g}_k$ is a homeomorphism. It is easy to see that $F_{k,1}$ is the union of the three closed sets

$$G^{\pm}: = \{x \in E^k; \ x_k = \pm 1\} ,$$
$$G': = \{x \in E^k; \ (x_1, \cdots, x_{k-1}) \in F_{k-1,1}\} ,$$

and that $F_{k+1,2}$ is the union of three closed sets

$$H^{\pm}: = \{y \in E^{k+1}; \ (y_1, \cdots, y_k) \in F^{\mp}_{k,1}, \ y_{k+1} = \pm 1\} ,$$
$$H': = \{y \in E^{k+1}; \ (y_1, \cdots, y_k) \in F_{k,2}\} .$$

Now, $g^-_k: = \dot{g}_k|_{G^-}: G^- \to H^-$ is a homeomorphism, by the assumption that $g_{k-1}: E^{k-1} \to F^+_{k,1}$ is a homeomorphism. Taking into account $F^-_{k,1} = -F^+_{k,1}$, we obtain by the same argument that $g^+_k: = \dot{g}_k|_{G^+}: G^+ \to H^+$ is a homeomorphism. If $x \in G'$ is such that $x_k = 1$, then from $(x_1, \cdots, x_{k-1}) \in F_{k-1,1}$ we obtain $\dot{g}_k(x) = (-g_{k-1}(-x_1, \cdots, -x_{k-1}), 1) = (g_{k-1}(x_1, \cdots, x_{k-1}), 1)$ by property (a) of Theorem 1.1. This shows that for all $x \in G'$ we have $\dot{g}_k(x) = (g_{k-1}(x_1, \cdots, x_{k-1}), x_k)$. So we obtain that $g'_k: = \dot{g}_k|_{G'}: G' \to H'$ is a homeomorphism, by the assumption that $g_{k-1}|_{F_{k-1,1}}: F_{k-1,1} \to F_{k,2}$ is a homeomorphism (property (b) of Theorem 1.1).

From what we have shown it follows that $\dot{g}_k$ is continuous. To show that $\dot{g}_k$ is a homeomorphism it suffices to show the equalities

[1] Concerning the results of this section cf. "Added in proof, 1."

$$\dot{g}_k(G^- \cap G^+) = H^- \cap H^+ ,$$
$$\dot{g}_k(G^\pm \cap G') = H^\pm \cap H' ,$$

because $g_k^\pm : G^\pm \to H^\pm$ and $g_k' : G' \to H'$ are homeomorphisms. The first equality follows from $G^- \cap G^+ = \varnothing$, $H^- \cap H^+ = \varnothing$. To show the remaining two equalities we note

$$G^\pm \cap G' = \{x \in E^k;\ (x_1, \cdots, x_{k-1}) \in F_{k-1,1},\ x_k = \pm 1\} ,$$
$$H^\pm \cap H' = \{y \in E^{k+1};\ (y_1, \cdots, y_k) \in F_{k,2},\ y_{k+1} = \pm 1\} .$$

If $x \in F_{k,1}$, $y \in F_{k+1,2}$, $\dot{g}_k(x) = y$, then:

$$\begin{aligned} & x \in G^\pm \cap G' \\ \Longleftrightarrow\ & (x_1, \cdots, x_{k-1}) \in F_{k-1,1},\ x_k = \pm 1 \\ \Longleftrightarrow\ & (y_1, \cdots, y_k) = g_{k-1}(x_1, \cdots, x_{k-1}) \in F_{k,2},\ y_{k+1} = x_k = \pm 1 \\ \Longleftrightarrow\ & y \in H^\pm \cap H' . \end{aligned}$$

(ii) For all $x \in F_{k,1}$ we have $\dot{g}_k(-x) = -\dot{g}_k(x)$. This follows directly from the definition of $\dot{g}_k$ if x is such that $x_k = \pm 1$. If $-1 < x_k < 1$ then $(x_1, \cdots, x_{k-1}) \in F_{k-1,1}$, and $(g_{k-1}(-x_1, \cdots, x_{k-1}), -x_k) = -(g_{k-1}(x_1, \cdots, x_{k-1}), x_k)$ follows from (a) of Theorem 1.1.

(iii) For all $n \in N$, $\dot{g}_k$ induces homeomorphisms

$$\dot{g}_k|_{F_{k,n}^{(\pm)}} : F_{k,n}^{(\pm)} \longrightarrow F_{k+1,n+1}^{(\pm)} .$$

In view of (i) and (ii), it is sufficient to show $\dot{g}_k(F_{k,n}^+) = F_{k+1,n+1}^+$. So let $x \in F_{k,1}$, $y \in F_{k+1,2}$, $y = \dot{g}_k(x)$: then:

$$\begin{aligned} & x \in F_{k,n}^+ \\ \Longleftrightarrow\ & \begin{cases} (1)\quad (x_1, \cdots, x_{k-1}) \in F_{k-1,n-1}^- \quad \text{and} \quad x_k = 1 , \\ \text{or } (2)\quad (x_1, \cdots, x_{k-1}) \in F_{k-1,n}^+ \quad \text{and} \quad x_k < 1 \end{cases} \\ \Longleftrightarrow\ & \begin{cases} (1)\quad (y_1, \cdots, y_k) = -g_{k-1}(-x_1, \cdots, -x_{k-1}) \in F_{k,n}^- \text{ and } y_{k+1} = 1 , \\ \text{or } (2)\quad (y_1, \cdots, y_k) = g_{k-1}(x_1, \cdots, x_{k-1}) \in F_{k,n+1}^+ \text{ and } y_{k+1} < 1 \end{cases} \\ \Longleftrightarrow\ & y \in F_{k+1,n+1}^+ . \end{aligned}$$

Finally we extend $\dot{g}_k$ to a homeomorphism $g_k : E^k \to F_{k+1,1}^+$. Denote by $\bar{y}$ the point $(1, 1, \cdots, 1) \in E^{k+1}$. For $x \in F_{k,1}$, $0 \leqq t \leqq 1$ we define

$$g_k(tx) := t\dot{g}_k(x) + (1 - t)\bar{y} .$$

From $\dot{g}_k(x) \in F_{k+1,2}$ we obtain immediately $g_k(tx) \in F_{k+1,1}^+$. For $z \in E^k$, the extension g_k can also be written as

$$g_k(z) = \begin{cases} g_k\left(\|z\|_\infty \dfrac{z}{\|z\|_\infty}\right) = \|z\|_\infty \dot{g}_k\left(\dfrac{z}{\|z\|_\infty}\right) + (1 - \|z\|_\infty)\bar{y} & \text{if } \quad z \neq 0 , \\ \bar{y} & \text{if } \quad z = 0 , \end{cases}$$

which shows that g_k is continuous. If $y \in F^+_{k+1,1}$, $y \neq \bar{y}$, then

$$\bar{y} + 2\frac{y - \bar{y}}{\|y - \bar{y}\|_\infty} \in F_{k+1,2} ,$$

$$z: = \frac{1}{2}\|y - \bar{y}\|_\infty \dot{g}_k^{-1}\left(\bar{y} + 2\frac{y - \bar{y}}{\|y - \bar{y}\|_\infty}\right) \in E^k ,$$

and $g_k(z) = \cdots = y$. This shows that g_k is surjective. In fact it is easily calculated that the expression obtained above yields the inverse

$$g_k^{-1}(y) = \begin{cases} \frac{1}{2}\|y - \bar{y}\|_\infty \dot{g}_k^{-1}\left(\bar{y} + 2\frac{y - \bar{y}}{\|y - \bar{y}\|_\infty}\right) & \text{if} \quad y \in F^+_{k+1,1} , \quad y \neq \bar{y} , \\ 0 & \text{if} \quad y = \bar{y} \end{cases}$$

of g_k. This shows that g_k is bijective as well as that g_k^{-1} is continuous.

So we have obtained a homeomorphism $g_k: E^k \to F^+_{k+1,1}$. The properties of $\dot{g}_k$ proved in (ii), (iii) show that g_k satisfies (a), (b) of Theorem 1.1.

COROLLARY 1.2. *For $n, k \in \boldsymbol{N}$, $1 \leqq n \leqq k$, there exists a homeomorphism $f_{k,n}: F_{k-n+1,1} \to F_{k,n}$ which also satisfies $f_{k,n}(-x) = -f_{k,n}(x)$ for all $x \in F_{k-n+1,1}$.*

Proof. From Theorem 1.1 we obtain that

$$f_{k,n}: = g_{k-1} \circ g_{k-2} \circ \cdots \circ g_{k-n+2} \circ g_{k-n+1} |_{F_{k-n+1,1}}$$

is the desired homeomorphism.

Let us note that, for our application in §2, it would have been sufficient to show that the mappings considered in Theorem 1.1 and Corollary 1.2 are continuous.

2. Alternating elements of subspaces of $\boldsymbol{R}^k$ and $C(J)$.

THEOREM 2.1. *Let $k \in \boldsymbol{N}$. Let $L \subset \boldsymbol{R}^k$ be a linear subspace, $n: = \dim L (\leqq k)$. Then there exists $x = (x_1, \cdots, x_k) \in L$ with the following properties: $\|x\|_\infty \leqq 1$, and there exist $1 \leqq j_1 < j_2 < \cdots < j_n \leqq k$ such that $x_{j_r} = (-1)^r$ for all $r = 1, \cdots, n$.*

Proof.[2] With $F_{k,n}$ from §1, one has to show that $L \cap F_{k,n} \neq \varnothing$. There exist linear functionals $l_j: \boldsymbol{R}^k \to \boldsymbol{R}$, $j = 1, \cdots, k-n$, such that $L = \bigcap_{j=1}^{k-n} l_j^{-1}(0)$. Then $l: = (l_1, \cdots, l_{k-n})|_{F_{k,n}}: F_{k,n} \to \boldsymbol{R}^{k-n}$ is continuous, and $l(-x) = -l(x) (x \in F_{k,n})$ by the linearity of l. From Corollary 1.2 we obtain that $l \circ f_{k,n}: F_{k-n+1,1} \to \boldsymbol{R}^{k-n}$ is a continuous mapping satisfying $l \circ f_{k,n}(-x) = -l \circ f_{k,n}(x)$ for all $x \in F_{k-n+1,1}$. Now $F_{k-n+1,1} =$

[2] A simplified proof is sketched in "Added in proof, 1."

$\partial \mathring{E}^{k-n+1}$, where $\mathring{E}^{k-n+1} = \{x \in \boldsymbol{R}^{k-n+1}; \|x\|_\infty < 1\}$ is open, bounded, symmetric, and $0 \in \mathring{E}^{k-n+1}$, and therefore Borsuk's theorem implies that there exists $x \in F_{k-n+1,1}$ such that $l \circ f_{k,n}(x) = 0$ ([9, Corollary 3.29], [2, §10, Satz 3]). This shows $f_{k,n}(x) \in L \cap F_{k,n} \neq \varnothing$.

REMARK 2.2. One might be tempted to think that Theorem 2.1 is a special case of a more general statement which would say that, under the assumptions of Theorem 2.1, for any n-tuple $y = (y_1, \cdots, y_n) \in \{-1, 1\}^n$ there exist $x \in L$, $\|x\|_\infty \leqq 1$, and $1 \leqq j_1 < \cdots < j_n \leqq k$ such that $x_{j_r} = y_r (r = 1, \cdots, n)$. Such a statement, however, is not true, as the following example shows: Let $L \subset \boldsymbol{R}^3$ be the 2-dimensional space spanned by $(1, 1, -2), (1, -2, 1)$. Then there is no $x \in L$, $\|x\|_\infty \leqq 1$, which has $+1$ in two coordinates.

In order to state the next theorem we need a definition. By a *compact totally ordered set* we understand a totally ordered set $J = (J, \leqq)$ which is compact in the order topology (cf. [4, Exercise (6.96)]). By $C(J)$ we denote the Banach space of continuous real-valued functions, endowed with the supremum norm.

THEOREM 2.3. *Let J be a compact totally ordered set. Let $L \subset C(J)$ be a linear subspace, $\dim L \geqq n (\in \boldsymbol{N}_0)$. Then there exists $f \in L$ with the following properties: $\|f\|_\infty \leqq 1$, and there exist $t_1, \cdots, t_n \in J$, $t_1 < t_2 < \cdots < t_n$, such that $f(t_r) = (-1)^r$ for all $r = 1, \cdots, n$.*

Proof. Without restriction we may assume $\dim L = n$. There exist $s_1, \cdots, s_n \in J$ such that $\dim \{(f(s_1), \cdots, f(s_n)); f \in L\} = n$. From $\dim L = n$ it follows that for each $x \in \boldsymbol{R}^n$ there exists a unique $f \in L$ with $f(s_j) = x_j (j = 1, \cdots, n)$, that the mapping $\boldsymbol{R}^n \ni x \mapsto f \in C(J)$ thus defined is linear and continuous, and therefore

$$C := \sup\{\|f\|_\infty; |f(s_j)| \leqq 1 \quad \text{for all} \quad j = 1, \cdots, n\} < \infty .$$

We define the system $\mathscr{F} := \{F \subset J; F \text{ finite}, \{s_1, \cdots, s_n\} \subset F\}$, which is directed by inclusion. For each $F \in \mathscr{F}$, $F = \{t_1, \cdots, t_k\}$, $t_1 < t_2 < \cdots < t_k$, the set $\{(f(t_1), \cdots, f(t_k)); f \in L\}$ is then a linear subspace of $\boldsymbol{R}^k$, of dimension n. By Theorem 2.1 there exists $f^F \in L$: $|f^F(t_j)| \leqq 1 (j = 1, \cdots, k)$, and there exist $1 \leqq j_1 < \cdots < j_n \leqq k$ such that $f^F(t_{j_r}) = (-1)^r (r = 1, \cdots, n)$. We define $t^F = (t_1^F, \cdots, t_n^F) := (t_{j_1}, \cdots, t_{j_n})$, $z_F := (f^F, t^F)$. Then $(z_F)_{F \in \mathscr{F}}$ is a net in $L \times \{(t_1, \cdots, t_n) \in J^n; t_1 \leqq t_2 \leqq \cdots \leqq t_n\}$. Moreover $\|f^F\|_\infty \leqq C$ since $\{s_1, \cdots, s_n\} \subset F$ and $|f^F(t)| \leqq 1$ for all $t \in F$. Now $B_C := \{f \in L; \|f\|_\infty \leqq C\}$ is compact in $C(J)$ (and therefore equicontiuous), and $\{(t_1, \cdots, t_n) \in J^n; t_1 \leqq t_2 \leqq \cdots \leqq t_n\}$ is compact. This implies that the net $(z_F)_{F \in \mathscr{F}}$ has a cluster point $(f, (t_1, \cdots, t_n))$, $f \in B_C$, $(t_1, \cdots, t_n) \in J^n$, $t_1 \leqq \cdots \leqq t_n$.

Let $\varepsilon > 0$. Since B_C is equicontinuous and J is compact we obtain an open covering $(U_1^\varepsilon, \cdots, U_{m(\varepsilon)}^\varepsilon)$ of J such that for all $i \in \{1, \cdots, m(\varepsilon)\}$, $t, t' \in U_i^\varepsilon$, $f \in B_C$ we have $|f(t) - f(t')| \leqq \varepsilon$. For $r = 1, \cdots, n$ we choose i_r such that $t_r \in U_{i_r}^\varepsilon$. Let $t \in J$. Since $(f, (t_1, \cdots, t_n))$ is a cluster point of $(z_F)_{F \in \mathscr{F}}$, there exists $F \in \mathscr{F}$, $t \in F$, such that $\|f - f^F\|_\infty \leqq \varepsilon$, $t_r^F \in U_{i_r}^\varepsilon (r = 1, \cdots, n)$. This implies

$$\begin{aligned}|f(t_r) - (-1)^r| &= |f(t_r) - f^F(t_r^F)| \\ &\leqq |f(t_r) - f^F(t_r)| + |f^F(t_r) - f^F(t_r^F)| \leqq 2\varepsilon \,, \\ |f(t)| &\leqq |f(t) - f^F(t)| + |f^F(t)| \leqq \varepsilon + 1 \,.\end{aligned}$$

Since $\varepsilon > 0$ and $t \in J$ were arbitrary, we obtain $f(t_r) = (-1)^r (r = 1, \cdots, n)$, and $\|f\|_\infty \leqq 1$. Finally, $f(t_r) \neq f(t_{r+1})$ together with $t_r \leqq t_{r+1}$ implies $t_r < t_{r+1} (r = 1, \cdots, n-1)$.

3. (P1) implies (P2). Let the function $h: [0, \infty) \to [0, \infty)$ be a *modulus of continuity*, i.e., h is nondecreasing, and $h(0) = \lim_{s \to 0} h(s) = 0$. We denote

$$\begin{aligned}\mathrm{Lip}_h[0, 1] := \{f: [0, 1] \longrightarrow \boldsymbol{R};\ f \text{ continuous, there exists } C \geqq 0 \\ \text{such that } |f(t) - f(t')| \leqq Ch(|t - t'|) \text{ for all } t, t' \in [0, 1]\} \,.\end{aligned}$$

$\mathrm{Lip}_h[0, 1]$, endowed with the norm $\|f\|_h := \|f\|_\infty + |f|_h$, where

$$|f|_h := \inf\{C \geqq 0;\ |f(t) - f(t')| \leqq Ch(|t - t'|) \quad \text{for all} \quad t, t' \in [0, 1]\} \,,$$

is a Banach space.

For $X := \mathrm{Lip}_h[0, 1]$, $Y := C[0, 1]$, the Arzelà-Ascoli theorem implies that (P1) is satisfied. We are going to show that (P2) is also satisfied, and that this implies (P1) $\Rightarrow$ (P2) in general.

THEOREM 3.1. *Let h be a modulus of continuity. Let $X := \mathrm{Lip}_h[0, 1]$, $Y := C[0, 1]$. Then* (P2) *is satisfied, more precisely*

$$\psi(n) \geqq 1 + 2h\left(\frac{1}{n-1}\right)^{-1} \quad (n \in \boldsymbol{N},\ n \geqq 2)$$

(where $h(1/(n-1))^{-1} = \infty$ if $h(1/(n-1)) = 0$).

Proof. Let $L \subset \mathrm{Lip}_h[0, 1]$ be a subspace of dimension $\geqq n \geqq 2$. By Theorem 2.3 there exist $f \in L$, $\|f\|_\infty = 1$, $0 \leqq t_1 < \cdots < t_n \leqq 1$ such that $f(t_r) = (-1)^r (r = 1, \cdots, n)$. It follows that there exists $r \in \{2, \cdots, n\}$ such that $t_r - t_{r-1} \leqq 1/(n-1)$. This implies

$$\|f\|_h \geqq \|f\|_\infty + \frac{|f(t_r) - f(t_{r-1})|}{h(|t_r - t_{r-1}|)} \geqq 1 + 2h\left(\frac{1}{n-1}\right)^{-1} ,$$

$$N_L = \sup\{\|f\|_h;\ f \in L, \|f\|_\infty \leqq 1\} \geqq 1 + 2h\left(\frac{1}{n-1}\right)^{-1},$$

therefore $\psi(n) \geqq 1 + 2h(1/n-1))^{-1}$. (If $h(1/(n-1)) = 0$, then $\mathrm{Lip}_h[0, 1]$ consists of the constant functions, $\dim \mathrm{Lip}_h[0, 1] = 1$, $\psi(n) = \infty$.)

THEOREM 3.2. (P1) *implies* (P2).

Proof.[3] Let (X, Y) satisfy (P1). Without restriction we may assume that X is dense in Y, and that X and Y are Banach spaces over $\boldsymbol{R}$. Since the unit ball B of X is relatively compact in Y we conclude that Y is separable. ($\bar{B}^Y$ is a compact metric space, therefore separable; $X = \bigcup_{n \in N} nB$ is dense in Y.) Now a theorem of Banach and Mazur states that the separable Banach space Y is norm isomorphic to a closed subspace of $C[0, 1]$ ([1, Ch. XI, §8, Théorème 9], cf. [5, §21.3, (6)]). So we may assume without restriction that Y is a closed subspace of $C[0, 1]$.

Now the unit ball B of X is relatively compact in $C[0, 1]$, therefore uniformly equicontinuous. This implies that $h: [0, \infty) \to [0, \infty)$,

$$h(s) := \sup\{|f(t) - f(t')|;\ t, t' \in [0, 1], |t - t'| \leqq s, f \in B\},$$

is a modulus of continuity. By the definition of h, we have $|f|_h \leqq 1$ for all $f \in B$. Let N be the norm of the injection $\mathrm{id}: (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$. Then $\|f\|_h = \|f\|_\infty + |f|_h \leqq N\|f\|_X + 1 \leqq N + 1 (f \in B)$ shows that the injection $\mathrm{id}: (X, \|\cdot\|_X) \to \mathrm{Lip}_h[0, 1]$ is continuous. Now the desired statement follows from Theorem 3.1 together with Lemma 3.3 proved subsequently.

LEMMA 3.3. *Let* (X, Y) *satisfy* (P2). *Let* $\hat{X} \subset X$ *be a Banach space, with continuous embedding* $\hat{X} \hookrightarrow X$. *Then* $(\hat{X}, Y)$ *satisfies* (P2). *If* N *is the norm of the embedding* $\hat{X} \hookrightarrow X$, *then* $\psi(n; \hat{X}, Y) \geqq N^{-1}\psi(n; X, Y)$, $\varphi(K; \hat{X}, Y) \leqq \varphi(N \cdot K; X, Y)$.

Proof. Let $L \subset \hat{X}$ be a subspace of dimension $\geqq n$. Then $L \subset X$, and

$$\begin{aligned}\psi(n; X, Y) &\leqq \sup\{\|x\|_X;\ x \in L, \|x\|_Y \leqq 1\}\\ &\leqq N \sup\{\|x\|_{\hat{X}};\ x \in L, \|x\|_Y \leqq 1\}.\end{aligned}$$

This implies $\psi(n; \hat{X}, Y) \geqq N^{-1}\psi(n; X, Y)$, and therefore (P2) for $(\hat{X}, Y)$.

If $K \geqq 0$, and $L \subset \hat{X}$ is a linear subspace with

$$\|x\|_{\hat{X}} \leqq K\|x\|_Y (x \in L),$$

[3] An alternative proof is sketched in "Added in proof, 2."

then $\|x\|_X \leqq N\|x\|_{\hat{X}} \leqq NK\|x\|_Y (x \in L)$ shows that $\dim L \leqq \varphi(N \cdot K; X, Y)$. This proves the inequality for φ.

EXAMPLE 3.4. Let $0 < \alpha \leqq 1$, $h_\alpha(s) := s^\alpha (s \geqq 0)$. Then $X := \mathrm{Lip}_{h_\alpha}[0, 1]$ is just the space of Hölder continuous functions, with Hölder exponent α. Let $Y := C[0, 1]$. (This is a special case of a situation considered in [10, VI. 3].) We are going to calculate

$$\psi(n; \alpha) = 1 + 2(n-1)^\alpha \quad (n \in \boldsymbol{N})$$

for this case. From Theorem 3.1 we obtain "$\geqq$" for $n \geqq 2$. To obtain "$\leqq$" we consider

$$L := \left\{ f \in C[0, 1];\ f \text{ affine linear on all intervals } \left[\frac{i-1}{n-1}, \frac{i}{n-1}\right], \right.$$
$$\left. i = 1, \cdots, n-1 \right\}.$$

Then $L \subset \mathrm{Lip}_{h_\alpha}[0, 1]$, $\dim L = n$. If $f \in L$, $\|f\|_\infty \leqq 1$, then it is easy to show $|f(t) - f(t')| \leqq 2(n-1)^\alpha |t - t'|^\alpha (t, t' \in [0, 1])$, and therefore $\|f\|_{h_\alpha} \leqq 1 + 2(n-1)^\alpha$. This implies "$\leqq$".

$\psi(1) = 1$ is true in the situation of Theorem 3.1: "$\geqq$" follows from $\|f\|_\infty \leqq \|f\|_h (f \in \mathrm{Lip}_h[0, 1])$, and "$\leqq$" is obtained by considering the constant functions.

4. Uniformly closed spaces of functions of bounded variation. For a compact totally ordered set J, we denote by $CBV(J)$ the space of continuous functions of bounded variation,

$$CBV(J) = \{f : J \longrightarrow \boldsymbol{R};\ f \text{ continuous},\ |f|_V < \infty\},$$

where

$$|f|_V := \sup \left\{ \sum_{j=1}^{l} |f(t_j) - f(t_{j-1})|;\ t_0, \cdots, t_l \in J,\ t_0 \leqq \cdots \leqq t_l \right\}.$$

$CBV(J)$, endowed with the norm $\|f\|_V := \|f\|_\infty + |f|_V$, is a Banach space.

THEOREM 4.1. *Let J be a compact totally ordered set, $X := CBV(J)$, $Y := C(J)$. Then (X, Y) satisfies* (P2); *more precisely $\psi(n) \geqq 2n - 1 (n \in \boldsymbol{N})$.*[4]

Proof. Let $L \subset CBV(J)$ be a linear subspace of dimension $\geqq n$. By Theorem 2.3 there exist $f \in L$, $\|f\|_\infty = 1$, $t_1, \cdots, t_n \in J$, $t_1 < \cdots < t_n$, such that $f(t_r) = (-1)^r (r = 1, \cdots, n)$. This implies $\|f\|_V = \|f\|_\infty + |f|_V \geqq 1 + 2(n-1)$, $N_L \geqq 1 + 2(n-1)$. This proves $\psi(n) \geqq 1 + 2(n-1)$.

[4] cf. "Added in proof, 3."

REMARK 4.2. (a) Theorem 4.1 implies that $(CBV(J), C(J))$ satisfies (P3). For $J = [0, 1]$, this statement is due to Mokobodzki and Rogalski [6, Théorème 9]; cf. also [7, Théorème 16]. A more general statement has been obtained by Pajor (cf. [7, Théorème 22]).

(b) For $1 \leqq p \leqq \infty$ we consider $Y: = C[0, 1]$, $X: = W_p^1(0, 1)$ (the Sobolev space of real, absolutely continuous functions whose derivative is in $L_p(0, 1)$). As norm on $W_p^1(0, 1)$ we take $\|f\|_{cp}: = \|f\|_\infty + \|f'\|_p$. For $p > 1$, property (P1) is satisfied (cf. [6, proof of Proposition 1]). In [6], [7] it is shown that, for $p > 1$, the function $\varphi(K; p): = \varphi(K; W_p^1(0, 1), C[0, 1])$ is finite for all $K < \infty$, but no explicit upper bound was obtained. Also the problem was posed if $\lim_{p \to 1, p > 1} \varphi(K; p)$ is finite. ([6, Problème 6], [7, Problème 7]; let us note that our function $\varphi(K; p)$ is slightly different from $\varphi(K, p)$ considered in [6], [7], but it is easy to see that the formulations of the problem are equivalent.)

From $|f|_V = \|f'\|_1 \leqq \|f'\|_p (f \in W_p^1(0, 1))$, which implies that the embeddings $W_p^1(0, 1) \hookrightarrow W_1^1(0, 1) \hookrightarrow CBV[0, 1]$ have norm $\leqq 1$, and Lemma 3.3 we obtain $\varphi(K; p) \leqq \varphi(K; 1) \leqq \varphi(K; CBV[0, 1], C[0, 1])$. To calculate the last quantity, let $L \subset CBV[0, 1]$ be a linear subspace. If K is a bound for the mapping $\mathrm{id}: (L, \|\cdot\|_\infty) \to (L, \|\cdot\|_V)$, and $n = \dim L$, then Theorem 4.1 implies $K \geqq 2n - 1$, $n \leqq [(K+1)/2] (= \max\{n \in N; n \leqq (K+1)/2\})$. This shows $\varphi(K; p) \leqq [(K+1)/2]$. The subspace $L \subset W_p^1(0, 1)$, defined in Example 3.4, is an n-dimensional subspace for which $N_L = 2n - 1$; thus we obtain

$$\varphi(K; p) = \left[\frac{K+1}{2}\right].$$

(Cf. [7, example at the end of §III].) This solves the problem mentioned above. At the same time we have calculated $\varphi(K; 1)$ ([6, Problème 11], [7, Problème 19]).

5. Examples.

EXAMPLE 5.1. Let I be an index set. Let $1 \leqq p < q \leqq \infty$, $X: = l_p(I)$, $Y: = l_q(I)$. (Here the elements of $l_p(I)$ are taken $\boldsymbol{K}$-valued, $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$.) If I is infinite, then (P1) is not satisfied. It follows from [3, Théorème 2, (2)] that (P3) is satisfied. We are going to show that (P2) is also satisfied. In order to do this and to calculate $\psi(n; p, q): = \psi(n; l_p(I), l_q(I))$ we proof an estimate.

LEMMA 5.2. *Let $L \subset c_0(I)$ be a subspace with* $\dim L \geqq n (\in N_0)$. *Then there exists $x \in L$, $\|x\|_\infty \leqq 1$,* $\operatorname{card}\{\iota \in I; |x(\iota)| = 1\} \geqq n$.

Proof. (i) If $J \subset I$, $\operatorname{card} J < n$, then there exists $0 \neq y \in L$ such that $y(\iota) = 0$ for all $\iota \in J$. Proof: If we define the linear forms $f_\iota: L \to \boldsymbol{K}, f_\iota(x) := x(\iota)$ (where $x = (x(\iota); \iota \in I)$), then $\bigcap_{\iota \in J} f_\iota^{-1}(0)$ is a subspace of L, of codimension $\leqq \operatorname{card} J < n$. This shows that $\bigcap_{\iota \in J} f_\iota^{-1}(0)$ cannot be all of L.

(ii) Let $x \in L$, $\|x\|_\infty \leqq 1$, $\operatorname{card}\{\iota \in I; |x(\iota)| = 1\} \leqq n - 1$. Then there exists $\hat{x} \in L$, $\|\hat{x}\|_\infty \leqq 1$, $\operatorname{card}\{\iota \in I; |\hat{x}(\iota)| = 1\} > \operatorname{card}\{\iota \in I; |x(\iota)| = 1\}$. Proof: From (i) we obtain $0 \neq y \in L$, $y(\iota) = 0$ for all $\iota \in I$ with $x(\iota) = 1$. It is easy to see that there exists $t > 0$ such that $\|x + ty\|_\infty = 1$, $\operatorname{card}\{\iota \in I; |x(\iota) + ty(\iota)| = 1\} \geqq \operatorname{card}\{\iota \in I; |x(\iota)| = 1\} + 1$. (Take $t := \inf\{s \geqq 0; \|x + sy\|_\infty > 1\}$, and use $x(\iota) \to 0$, $y(\iota) \to 0$ for "$\iota \to \infty$".)

(iii) From (ii) the desired statement follows by induction.

There is a slight similarity between Lemma 5.2 and Theorem 2.3. Indeed, for $I = \boldsymbol{N}$, we may consider the compact totally ordered set $J = \boldsymbol{N} \cup \{\infty\}$ and obtain the statement of Lemma 5.2 from Theorem 2.3.

ESTIMATE 5.3. *Let* $1 \leqq p < \infty$; *let* $L \subset l_p(I)$ *be a subspace,* $\dim L \geqq n$. *Then* $\sup\{\|x\|_p; x \in L, \|x\|_\infty \leqq 1\} \geqq n^{1/p}$.

Proof. For the element $x \in L$ obtained from Lemma 5.2 we have $\|x\|_\infty \leqq 1$, $\|x\|_p = (\sum_{\iota \in I} |x(\iota)|^p)^{1/p} \geqq n^{1/p}$.

Continuation of Example 5.1. We are going to show

$$\psi(n; p, q) = \begin{cases} n^{(1/p)-(1/q)} & \text{for } n \in \boldsymbol{N}_0 \text{ with } n \leqq \operatorname{card} I\,, \\ \infty & \text{for } n \in \boldsymbol{N}_0 \text{ with } n > \operatorname{card} I\,, \end{cases}$$

thereby establishing (P2).

Let $L \subset l_p(I)$ be a subspace, $\dim L \geqq n$ (then necessarily $n \leqq \operatorname{card} I$), and let N_L be the norm of the mapping $\mathrm{id}: (L, \|\cdot\|_q) \to (L, \|\cdot\|_p)$. For $x \in L$, we then have

$$\|x\|_q^q \leqq \|x\|_p^p \|x\|_\infty^{q-p} \leqq N_L^p \|x\|_q^p \|x\|_\infty^{q-p}\,,$$
$$\|x\|_p \leqq N_L \|x\|_q \leqq N_L^{q/(q-p)} \|x\|_\infty\,,$$

and therefore $\sup\{\|x\|_p; x \in L, \|x\|_\infty \leqq 1\} \leqq N_L^{q/(q-p)}$. Now Estimate 5.3 implies $n^{1/p} \leqq N_L^{q/(q-p)}$, $n^{(1/p)-(1/q)} \leqq N_L$, which shows $\psi(n) \geqq n^{(1/p)-(1/q)}$.

To show equality, we take $J \subset I$, $\operatorname{card} J = n$, and consider $L := \{x \in \boldsymbol{K}^I; x(\iota) = 0 \text{ for all } \iota \notin J\}$. If $x \in L$, then Hölder's inequality implies $\|x\|_p \leqq n^{(1/p)-(1/q)} \|x\|_q$, which shows $N_L \leqq n^{(1/p)-(1/q)}$.

EXAMPLE 5.4. Let μ be a probability measure on a measure space. Let $1 \leqq p < \infty$, $X := L_\infty(\mu)$, $Y := L_p(\mu)$. Except if μ is atomic, (P1) is not satisfied. By [3, Théorème 1], (X, Y) satisfies (P3). From a different proof for this fact, given in [8, Theorem 5.2], we obtain that also (P2) is satisfied: If $L \subset L_\infty(\mu)$, $\dim L \geqq n$, and $N_L = \sup\{\|f\|_\infty;\ f \in L,\ \|f\|_p \leqq 1\}$, then the proof in [8, loc. cit.] shows

$$N_L \geqq n^{1/p} \quad \text{for} \quad p \geqq 2\,, \qquad N_L \geqq n^{1/2} \quad \text{for} \quad 1 \leqq p \leqq 2\,.$$

This implies

$$\psi(n; p) \geqq n^{1/p} \quad \text{for} \quad p \geqq 2\,, \qquad \psi(n; p) \geqq n^{1/2} \quad \text{for} \quad 1 \leqq p \leqq 2\,.$$

It seems to the author that the best bound should be $\psi(n; p) \geqq n^{1/p}$ for all $p \in [1, \infty)$. This is correct for $p \geqq 2$: Taking μ = Lebesgue measure on [0, 1], and considering

$$L:=\left\{f\colon [0,1] \longrightarrow K;\ f \text{ constant on } \left(\frac{j-1}{n}, \frac{j}{n}\right) \text{ for all } j = 1, \cdots, n\right\},$$

one obtains $\psi(n; p) \leqq n^{1/p}$ for this special case. The above distinction for $p \geqq 2$, $p \leqq 2$ comes from the fact that the bound is first calculated for $p = 2$.

REMARK 5.5. We presented Examples 5.1 and 5.4 because for these Examples (P1) is not satisfied but (P3) is satisfied, and moreover even (P2) is satisfied. This raises the question for an example satisfying (P3) but not (P2), or else whether it can be proved that (P2) and (P3) are equivalent.[5] Also, for noncountable index set I, Example 5.1 provides an example of nonseparable spaces satisfying (P2).

Added in proof. 1. In this remark we sketch a simplified version of the proof of Theorem 2.1.

From the proof of Theorem 2.1 it is clear that for this proof it would be sufficient to know that for the set $F_{k,n}$ the following Borsuk's tpye theorem is valid.

THEOREM A.1. *Let* $k, n \in N$, $n \leqq k$. *Let* $l\colon F_{k,n} \to R^{k-n}$ *be continuous and odd* (i.e., $l(-x) = -l(x)$). *Then there exists* $x \in F_{k,n}$ *such that* $l(x) = O$.

The whole object of § 1 in our context is to prove the existence of the homeomorphism $f_{k,n}\colon F_{k-n+1,1} \to F_{k,n}$ of Corollary 1.2; this homeomorphism is used in the proof of Theorem 2.1 to obtain implicitly

[5] cf. "Added in proof, 4."

the statement of Theorem A.1. We are going to indicate a simpler proof of Theorem A.1.

LEMMA A.2. *For $k, n\in \boldsymbol{N}$, $1\leqq n\leqq k-1$, there exists $h_{k,n}\colon F_{k,1}\to \boldsymbol{R}$, continuous and odd, such that $h_{k,n}^{-1}(0)\cap F_{k,n}=F_{k,n+1}$.*

Proof. Define $\dot{h}_{k,n}\colon F_{k,n}\to \boldsymbol{R}$ by

$$\dot{h}_{k,n}(x):=\begin{cases}\operatorname{dist}(x, F_{k,n+1}) & \text{if } x\in F_{k,n}^{+},\\ -\operatorname{dist}(x, F_{k,n+1}) & \text{if } x\in F_{k,n}^{-}.\end{cases}$$

Standard arguments show that $\dot{h}_{k,n}$ can be extended to a continuous and odd mapping $h_{k,n}\colon F_{k,1}\to \boldsymbol{R}$, which then has the desired properties.

Proof of Theorem A.1. Extend l to a continuous and odd mapping $\hat{l}\colon F_{k,n}\to \boldsymbol{R}^{k-n}$. Apply Borsuk's theorem ([9, Corollary 3.29], [2, § 10, Satz 3]) to the mapping $(\hat{l}, h_{k,n-1}, h_{k,n-2}, \cdots, h_{k,1})\colon F_{k,1}\to \boldsymbol{R}^{k-1}$, to obtain $x\in F_{k,1}$ with $(\hat{l}, h_{k,n-1}, \cdots, h_{k,1})(x)=O$. Now $O=h_{k,1}(x)=h_{k,2}(x)=\cdots=h_{k,n-1}(x)$ imply $x\in F_{k,n}$, and so $l(x)=\hat{l}(x)=O$.

The author is indebted to N. Rogler for a discussion as a consequence of which he found this proof of Theorem A.1.

2. An alternative (and very natural) proof of Theorem 3.2 (P1) $\Rightarrow$ (P2)) was communicated by R. Tandler. His proof exploits directly the fact that the $4K$-ball B_{4K} in X is precompact in Y. The number of translates of the unit ball in Y which is needed to cover B_{4K} is shown to be an estimate for $\varphi(K; X, Y)$.

3. In § 4 we were restricted to the case of real valued functions because Theorem 2.3 is valid only for real valued functions. On the other hand, if E is a finite dimensional (real or complex) Banach space, then (P3) is known for ($X:=CBV(J; E)$, $Y:=C(J; E)$) [10, Satz IV.4]. In this remark we want to indicate how to carry over Theorem 4.1 to the case of $\boldsymbol{R}^m$-valued functions. (This also covers the case of $\boldsymbol{C}^m$-valued functions.)

We consider $\boldsymbol{R}^m$ endowed with the norm $\|\cdot\|_\infty$. Let $I:=\{1, \cdots, m\}$, J a totally ordered compact set; then $I\times J$, with lexicographical order, is a totally ordered compact set. Define $j\colon C(J; \boldsymbol{R}^m)\ni f\mapsto \hat{f}\in C(I\times J)$ by $\hat{f}(i, t):=f_i(t)$ $(i\in I, t\in J)$. If $L\subset CBV(J\colon \boldsymbol{R}^m)$ is a subspace of dimension $\geqq n$, then $\hat{L}:=j(L)\subset CBV(I\times J)$ has dimension $\geqq n$, and by Theorem 2.3 there exists $\hat{f}\in \hat{L}$ such that $\|\hat{f}\|_\infty=1$, $|\hat{f}|_V\geqq 2(n-1)$. Then $\|f\|_\infty=\|\hat{f}\|_\infty=1$, and $\|f\|_V\geqq 1+m^{-1}(|\hat{f}|_V-2(m-1))\geqq 2(n/m)-1$. For $X=CBV(J; \boldsymbol{R}^m)$, $Y=C(J\colon \boldsymbol{R}^m)$, this implies $\psi(n\colon X, Y)\geqq 2(n/m)-1$, and therefore (P2).

4. In this remark we indicate an example satisfying (P3) but not (P2); cf. Remark 5.5.

For $m \in \boldsymbol{N}$, let $l_1(m)$ be $\boldsymbol{R}^m$, with norm $\|\cdot\|_1$. For $p = 1, 2$, we denote the l_p-sum of $(l_1(m); m \in \boldsymbol{N})$ by

$$\sum^p l_1(m) := \left\{x = (x_m); x_m \in l_1(m), \|x\| := \left(\sum_{m \in N} \|x_m\|^p\right)^{1/p} < \infty\right\} .$$

Then $(X := \sum^1 l_1(m),\ Y := \sum^2 l_1(m))$ satisfies (P3) but not (P2):

Obviously $\psi(n; X, Y) = 1$ for all $n \in \boldsymbol{N}$, therefore (P2) is not satisfied.

Let $L \subset X$ be a subspace which is closed in Y. It is easy to see that Y is reflexive, and therefore so is L. The norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent on L, by the closed graph theorem. Assume that L is not finite dimensional. Since L is also closed in X, and X is $l_1(\boldsymbol{N})$, it would follow that L contains $l_1(\boldsymbol{N})$ [1, XII, § 2, Théorème 1], so that L could not be reflexive. This yields a contradiction.

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