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ON THE AVERAGE NUMBER OF REAL ZEROS OF A CLASS OF RANDOM ALGEBRAIC CURVES

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Let $a_1, a_2, \cdots,$ be a sequence of dependent normal random variables with mean zero, variance one and the correlation between any two random variables is $\rho, 0<\rho<1$. In this paper the average number of real zeros of $\sum_{k=1}^{n} a_k k^p x^k$, $0 \leqq p < \infty$ is estimated for large n and this average is asymptotic to $(2\pi)^{-1}[1+(2p+1)^{1/2} \log n$.

1. Let $a_1, a_2, \cdots$ be a sequence of dependent normal random variables with mean zero, variance one and joint density function.

$$|M|^{1/2}(2\pi)^{-n/2} \exp[-(1/2)\bar{a}'M\bar{a}] \tag{1.1}$$

where M^{-1} is the moment matrix with $\rho_{ij} = \rho$, $i \neq j$, $0 < \rho < 1$, $i, j = 1, 2, \cdots, n$. We estimate in this paper the average number of real zeros of

$$f(x) = \sum_{k=1}^{n} a_k k^p x^k \, , \qquad 0 \leqq p < \infty \tag{1.2}$$

and we state our result in the following theorem.

THEOREM. *The average number of real zeros of* (1.2) *in* $-\infty \leqq x \leqq \infty$, *when the random variables are dependent normal with joint density function* (1.1) *is* $(2\pi)^{-1}[1+(2p+1)^{1/2}] \log n$, *for larger* n.

When $p = 0$, that is, for the polynomial $\sum a_k x^k$, the average number of real zeros is estimated in Sambandham [5] and this average is $\pi^{-1} \log n$. Since the maxima or minima of $\sum a_k x^k$ is only half of the average number of real zeros of $\sum k a_k x^{k-1}$, by giving $p = 1$ in the theorem we get the average number of maxima of $\sum a_k x^k$. This average has been already estimated in Sambandham and Bhatt [6] and its value is $(4\pi)^{-1}[1 + 3^{1/2}] \log n$.

When the random variables are independent and normally distributed Das [2] estimated the average number of real zeros of [1.2] and this average is $\pi^{-1}[1 + (2p + 1)^{1/2}] \log n$. Under the same condition the average number of maxima of $\sum a_k x^k$ is $(2\pi)^{-1}[1 + 3^{1/2}] \log n$ and the average number of real zeros of $\sum a_k x^k$ is $(2/\pi) \log n$. These two results are respectively in Das [1] and Kac [3].

We note that the average number of zeros and the average number of maxima in the case when the random variables are in dependent are twice that of the case when the random variables are dependent normal with a constant correlation.

This is because when the random variables are dependent with a constant correlation ρ, most of the random variables have a tendency to be of the same sign as they are interdependent. As the most of the random variables preserve the same sign $\sum a_k k^p x^k$ has a tendency of behaving like $\pm \sum |a_k| k^p x^k$. Under this condition when $x > 0$, the consecutive terms have a tendency to cancel each other and when $x < 0$ the cancellation does not become possible. This fact reduces the average number of real zeros for $x > 0$ to $o(\log n)$.

In view of the relation

$$\begin{aligned} f(x) &= n^p x^{n+1} \sum_{k=0}^{n-1} a_{n-k}(1 - kn^{-1})^p y^{k+1} \\ &\equiv n^p x^{n+1} P_n(y) , \qquad y = \frac{1}{x} \end{aligned}$$

the number of roots of in $(-\infty, -1) \cup (1, \infty)$ equals with probability one, the number of roots of the polynomial $P_n(y)$ in $(-1, 1)$. Proceeding the method here we can easily show that the number of zeros of the polynomial $\sum_{k=0}^{n} a_k x^k$ in $(-1, 1)$ remain true for $P_n(y)$ in $(-1, 1)$. Hence we get from Sambandham [5]

$$M_n(1, \infty) = o(\log n) \tag{1.3}$$

and

$$M_n(-\infty, 1) \sim (2\pi)^{-1} \log n . \tag{1.4}$$

Therefore our further discussion will be on the average number of real zeros of (1.2) in $(-1, 1)$.

If we show that

$$M_n(-1, 0) \sim (2\pi)^{-1}(2p + 1)^{1/2} \log n \tag{1.5}$$

and

$$M_n(0, 1) = o(\log n) \tag{1.6}$$

in view of the relations (1.3) and (1.4) we get the proof of the theorem. To prove (1.5) and (1.6) we proceed as follows:

2. Let $M_n(a, b)$ denote the average number of real zeros of (1.2) in (a, b). Then following the method in Sambandham [5] we get

$$M_n(a, b) = \int_a^b [(A_p C_p - B_p^2)^{1/2} / A_p] dx \tag{2.1}$$

where

$$A_p \equiv A_p(x) = (1-\rho)\sum_{k=1}^{n} k^{2p}x^{2k} + \rho\Big(\sum_{k=1}^{n} k^p x^k\Big)^2$$
$$B_p \equiv B_p(x) = (1-\rho)\sum_{k=1}^{n} k^{2p+1}x^{2k-1} + \rho\Big(\sum_{k=1}^{n} k^p x^k\Big)\Big(\sum_{k=1}^{n} k^{p+1}x^{k-1}\Big)$$
$$C_p \equiv C_p(x) = (1-\rho)\sum_{k=1}^{n} k^{2p+2}x^{2k-2} + \rho\Big(\sum_{k=1}^{n} k^{p+1}x^{k-1}\Big)^2$$

if $A_pC_p - B_p^2 > 0$ in (a, b) which is easily seen to hold as in Sambandham [5].

Since

$$\sum_{k=1}^{n} k^p x^k = \Big\{x\frac{d}{dx}\cdots\Big[x\frac{d}{dx}\Big(x\frac{d}{dx}\sum_{k=1}^{n} x^k\Big)\Big]\Big\}$$
$$= \Big\{x\frac{d}{dx}\cdots\Big[x\frac{d}{dx}\Big(\frac{1-x^{n+1}}{1-x}\Big)\Big] \tag{2.2}$$

we can sum the values of A_p, B_p and C_p. This calculations show that for large n and $0 \leqq x \leqq 1 - (\log\log n/n)$

$$\frac{A_1C_1 - B_1^2}{A_0C_0 - B_0^2}\cdot\frac{A_2C_2 - B_2^2}{A_1C_1 - B_1^2}\cdots\frac{A_pC_p - B_p^2}{A_{p-1}C_{p-1} - B_{p-1}^2} < \frac{L_1(x, p)}{(1-x)^{4p}}$$

and

$$\frac{A_1}{A_0}\cdot\frac{A_2}{A_1}\cdots\frac{A_p}{A_{p-1}} > \frac{L_2(x, p)}{(1-x)^{2p}}$$

since each

$$\frac{A_iC_i - B_i^2}{A_{i-1}C_{i-1} - B_{i-1}^2} < \frac{L_3(x, p)}{(1-x)^4}$$

and

$$\frac{A_i}{A_{i-1}} > \frac{L_4(x, p)}{(1-x)^2}\ .$$

Here and in the following $L(x, p)$ with subscripts are bounded positive values of x and all of them are greater than zero. Therefore we find

$$\frac{(A_pC_p - B_p^2)^{1/2}}{A_p} < L_5(x, p)\frac{(A_0C_0 - B_0^2)^{1/2}}{A_0} < \frac{L_6(x, p)}{(1-x)^{1/2}}\ .$$

Therefore (2.1) reduces to

$$M_n\Big(0, 1 - \frac{\log\log n}{n}\Big) = 0(1)\ . \tag{2.3}$$

Since always

$$\frac{(A_pC_p - B_p^2)^{1/2}}{A_p} < n$$

$$M_n\left(1 - \frac{\log\log n}{n},\ 1\right) = 0(\log\log n) \tag{2.4}$$

(2.3) and (2.4) proves (1.6). Now we proceed to prove (1.5).

When $-1 \leqq x \leqq 0$ we find that in A_p, B_p and C_p the first terms in the right hand side are dominant and in this case we get

$$\frac{(A_pC_p - B_p^2)^{1/2}}{A_p} < \frac{(A_0C_0 - B_0^2)^{1/2}}{A_0} L_7(2, p) < \frac{L_8(x, p)}{1 - x^2} .$$

Therefore for $-1 + \eta \leqq x \leqq 0$, where $\eta = \exp[-(\log n)^{1/3}]$ we get

$$M_n(-1 + \eta, 0) = 0(\log n)^{1/3} \tag{2.5}$$

For $-1 \leqq x \leqq -1 + \delta/n$, where $\delta = (\log n)^{1/2}$, we have

$$\frac{(A_pC_p - B_p^2)^{1/2}}{A_p} < n$$

and therefore

$$M_n\left(-1,\ -1 + \frac{\delta}{n}\right) = 0(\log n)^{1/2} . \tag{2.6}$$

For x in the interval $(-1 + \delta/n, 1 - \eta)$ we follow the method suggested by Logan and Shepp [4], which was used by Das [2] also.

3. We put

$$\begin{aligned} \mu_\varepsilon(x) &= 1 \quad \text{if} \quad -\varepsilon < x < \varepsilon \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

From Kac [3] we get

$$M_n(a, b) = \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_a^b E\{\mu_\varepsilon(f(x))\,|f'(x)|\}dx . \tag{3.1}$$

The combined variable $(f(x),\ f'(x))$ has characteristic function,

$$\varphi(z, w) = E\{\exp[i\,f(x)z + if'(x)w]\} .$$

The probability density $p(\xi, \eta)$ for $f(x) = \xi$ and $f'(x) = \eta$ is given by

$$p(\xi, \eta) = (2\pi)^{-2} \iint_{-\infty}^{\infty} \exp[-i\xi z - i\eta w]\varphi(z, w)dz\,dw .$$

Therefore the chance that $u \leqq f(x) < u + du$ and $u \leqq f'(x) < v + dv$ hold together in $p(u, v)du\,du$. As the x's very both f and f' assume values from $-\infty$ to ∞ independently to one another so that

$$E[\mu_\varepsilon(f)|f'|] = \iint_{-\infty}^{\infty} \mu_\varepsilon(u)|v|\,du\,dv\ .$$

Let us write

$$F(u) = \int_{-\infty}^{\infty} |v| p(u, v) dv\ .$$

Then $F(u)$ is continuous and therefore we get

$$\begin{aligned}\lim_{\varepsilon\to 0}(2\varepsilon)^{-1}E[\mu_\varepsilon(f)|f'|] &= \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_{-\varepsilon}^{\varepsilon} F(u)du \\ &= F(0)\ .\end{aligned}$$

Since a and b are finite and $(2\varepsilon)^{-1}\int_a^b \mu_\varepsilon(f)|f'|dx$ is bounded from (3.1) we get the Kac – Rice formula.

$$\begin{aligned}M_n(a, b) &= \lim_{\varepsilon\to 0}(2\varepsilon)\int_a^b E[\mu_\varepsilon(f)|f'|]dx \\ &= \int_a^b F(0)dx = \int_a^b dx \int_{-\infty}^{\infty} |\eta|\,p(0, \eta)d\eta\ . \qquad (3.2)\end{aligned}$$

We put $f(x) = \sum_{k=1}^n a_k b_k$ and $f'(x) = \sum_{k=1}^n a_k c_k$ so that

$$\begin{aligned}\varphi(z, w) = \exp\Big[\Big(-\frac{1}{2}\Big)\Big\{(1-\rho)\sum_{k=1}^n (b_k z + c_k w)^2 \\ + \rho\Big(\sum_{k=1}^n (b_k z + c_k w)\Big)^2\Big\}\Big]\end{aligned}$$

and

$$p(0, y) = (2\pi)^{-2}\int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} \exp(-iyw)\varphi(z, w)dz\ . \qquad (3.3)$$

Then for $\varepsilon > 0$ we have

$$\begin{aligned}&\int_{-\infty}^{\infty} |\eta| \exp(-\varepsilon|\eta|)p(0, \eta)d\eta \\ &= \mathrm{Re}\,\pi^{-2}\int_0^{\infty} \frac{dw}{(\varepsilon + iw_0)^2}\int_{-\infty}^{\infty} \varphi(z, w)dz \qquad (3.4)\end{aligned}$$

where Re stands for the real part. We need the following identity, valid for non zero P and Q,

$$\text{(3.5)} \qquad \operatorname{Re} \pi^{-2}\int_0^\infty \frac{dw}{(\varepsilon + iw)^2}\int_{-\infty}^\infty \exp\left[-\left(\frac{1}{2}\right)(Pz + Qw)^2\right]dz = 0 .$$

One way to see this is to allow b_k and c_k to be arbitrary in (3.4). If we take them each to be constant in k then the probability density $p(\xi, \eta)$ corresponding to $\xi = \sum a_k b_k = P\bar{x}$ and $\eta = \sum a_k c_k = Q\bar{y}$ degenerates and (3.5) follows. Further given P and Q, the constants $\bar{x}$ and $\bar{y}$ can be chosen such that $\bar{x}$ and $\bar{y}$ are normally distributed.

We choose P and Q such that

$$\text{(3.6)} \qquad \begin{aligned} p^2 &= (1-\rho)\sum_{k=1}^n b_k^2 + \rho\left(\sum_{k=1}^n b_k\right)^2 \\ PQ &= (1-\rho)\sum_{k=1}^n b_k c_k + \rho\left(\sum_{k=1}^n b_k\right)\left(\sum_{k=1}^n c_k\right) . \end{aligned}$$

From (3.4) and (3.5) we get

$$\text{(3.7)} \qquad \begin{aligned} \int_{-\infty}^\infty |\eta|\, p(0, \eta)d\eta = \pi^{-2}\int_0^\infty \frac{dw}{w^2}\int_{-\infty}^\infty \exp -\left(\frac{1}{2}\right)(Pz + Qw)^2 \\ -\exp\left\{-\left(\frac{1}{2}\right)\left[(1-\rho)\sum b_k z + c_k w)^2 \right.\right. \\ \left.\left. +\rho\left(\sum (b_k z + c_k w)\right)^2\right]\right\}\Bigg]dz . \end{aligned}$$

We put $z = w'u$, $w = -xw'$ and use Frullani's theorem to integrate on w'. The right hand side of (3.7) reduces to

$$\text{(3.8)} \qquad g_n(x) = \frac{1}{2\pi^2 x}\int_{-\infty}^\infty \log h_n(x, u)dx$$

where

$$h_n(x, u) = \frac{\{[(1-\rho) + \rho\lambda_1]u^2 - 2[(1-\rho)\lambda_2 + \rho\lambda_3]u + [(1-\rho)\lambda_4 + \rho\lambda_5]\}}{[(1-\rho) + \rho\lambda_1]\left[\mu - \dfrac{(1-\rho)\lambda_2 - \rho\lambda_3}{(1-\rho) + \rho\lambda_1}\right]}$$

where

$$\begin{aligned} \lambda_1 &\equiv \lambda_1(x) = \left(\sum_{k=1}^n k^p x^k\right)^2\left(\sum_{k=1}^n k^{2p}x^{2k}\right)^{-1} , \\ \lambda_2 &\equiv \lambda_2(x) = \left(\sum_{k=1}^n k^{2p+1}x^{2k}\right)\left(\sum_{k=1}^n k^{2p}x^{2k}\right)^{-1} \\ \lambda_3 &\equiv \lambda_3(x) = \left(\sum_{k=1}^n k^p x^k\right)\left(\sum_{k=1}^n k^{p+1}x^k\right)\left(\sum_{k=1}^n k^{2p}x^{2k}\right)^{-1} , \\ \lambda_4 &\equiv \lambda_4(x) = \left(\sum_{k=1}^n k^{2p+2}x^{2k}\right)\left(\sum_{k=1}^n k^{2p}x^{2k}\right)^{-1} \\ \lambda_5 &\equiv \lambda_5(x) = \left(\sum_{k=1}^n k^{p+1}x^k\right)^2\left(\sum_{k=1}^n k^{2p}x^{2k}\right)^{-1} . \end{aligned}$$

We put $a = -1 + \delta/n$, $b = -1 + \eta$, $x = -\exp(-t/2n)$, $u = nv/t$. Therefore

$$M_n\left(-1 + \frac{\delta}{n}, -1 + \eta\right) = \int_{-1+\delta/n}^{-1+\eta} g_n(x)dx$$

$$= (4\pi^2)^{-1}\int_{T_0}^{n\delta_0}\frac{dt}{t}\int_{-\infty}^{\infty} \log W_n(t, v)dv \tag{3.9}$$

where

$$T_0 = -2n \log\left(1 - \frac{\delta}{n}\right),$$

$$\delta_0 = -2 \log (1 - \eta),$$

$$W_n(t, x) = \frac{U_n(t, v)}{V_n(t, v)},$$

$$U_n(t, v) = [(1 - \rho) + \rho\lambda_{11}]v^2 - 2[(1 - \rho)\lambda_{21} + \rho\lambda_{31}]v + [(1 - \rho)\lambda_{41} + \rho\lambda_{51}],$$

$$V_n(t, v) = [(1 - \rho) + \rho\lambda_{11}]\left[u - \frac{(1 - \rho)\lambda_{21} + \rho\lambda_{31}}{(1 - \rho) + \rho\lambda_{11}}\right]^2,$$

$$\lambda_{11} \equiv \lambda_{11}(t) = \frac{\left[\sum_1^n (-1)^k k^p \exp\left(-\frac{kt}{2n}\right)\right]^2}{\left[\sum_1^n k^{2p} \exp\left(-\frac{kt}{n}\right)\right]},$$

$$\lambda_{21} \equiv \lambda_{21}(t) = \frac{t}{n}\frac{\left[\sum_1^n k^{2p+1} \exp\left(-\frac{kt}{n}\right)\right]}{\left[\sum_1^n k^{2p} \exp\left(-\frac{kt}{n}\right)\right]},$$

$$\lambda_{31} \equiv \lambda_{31}(t) = \frac{t}{n}\frac{\left[\sum_1^n (-1)^k k^p \exp\left(-\frac{kt}{2n}\right)\right]\left[\sum_1^n (-1)^k k^{p+1} \exp\left(-\frac{kt}{2n}\right)\right]}{\left[\sum_1^n k^{2p} \exp\left(-\frac{kt}{n}\right)\right]},$$

$$\lambda_{41} \equiv \lambda_{41}(t) = \frac{t^2}{n^2}\frac{\left[\sum_1^n k^{2p+2} \exp\left(-\frac{kt}{n}\right)\right]}{\left[\sum_1^n k^{2p} \exp\left(-\frac{kt}{n}\right)\right]},$$

and

$$\lambda_{51} \equiv \lambda_{51}(t) = \frac{t^2}{n^2} \frac{\left[\sum_1^n (-1)^k k^{p+1} \exp\left(-\frac{kt}{2n}\right)\right]^2}{\left[\sum_1^n k^{2p} \exp\left(-\frac{kt}{n}\right)\right]} .$$

From Das [2] we get

$$\lambda_{21} = (2p+1) + 0(te^{-t/2}) ,$$

$$\lambda_{41} = (2p+1)(2p+2) + 0(te^{-t/2}) ,$$

and using the idea in (2.2) we get

$$\lambda_{11} = 0\left(\frac{t}{n}\right)^{2p+1} ,$$

$$\lambda_{31} = 0\left(\frac{t}{n}\right)^{2p+2} ,$$

$$\lambda_{51} = 0\left(\frac{t}{n}\right)^{2p+3} .$$

Now

$$\begin{aligned}\int_{|v|>L} \log W_n(v, t)dv &= L \log \frac{(L^2 - 2\lambda L + \mu)(L^2 + 2\lambda L + \mu)}{(L^2 - 2\lambda L + \lambda^2)(L^2 + 2\lambda L + \lambda^2)} \\ &+ \lambda \log \frac{(L^2 - 2\lambda L + \mu)(L^2 + 2\lambda L + \mu)}{(L^2 - 2\lambda L + \lambda^2)(L^2 + 2\lambda L + \lambda^2)} \\ &+ 4(\mu - \lambda^2)\int_L^\infty \frac{(v^2 - \mu)dv}{(v^2 + \mu)^2 - 4\lambda^2 v^2}\end{aligned}$$

where

$$\lambda \equiv \lambda(t) = \frac{(1-\rho)\lambda_{21} + \rho\lambda_{31}}{(1-\rho) + \rho\lambda_{11}}$$

and

$$\mu \equiv \mu(t) = \frac{(1-\rho)\lambda_{41} + \rho\lambda_{51}}{(1-\rho) + \rho\lambda_{11}} .$$

For $(1/2)(\log n)^{1/2} \leqq t \leqq n\delta$ and when n is large, we find that λ_{31}, λ_{11} and λ_{51} are tending to zero, λ_{21} and λ_{41} are respectively asymptotic to $(2p+1)$ and $(2p+1)(2p+2)$ and

$$\pi^{-2}\int_{T_0}^{n\delta_0} \frac{dt}{t} \int_{-L}^{L} \log \frac{v^2 - 2\lambda v + \mu}{v^2 - 2\lambda v + \lambda^2} dv = o\left(\frac{1}{2}\log n\right) . \tag{3.10}$$

Further we note that

$$\int_0^L \frac{v^2 - 2rv + s}{v^2 - 2r'v + s'} dv = 0(s - s') + 0(r^2 - r'^2)$$

for large L. This makes

$$(4\pi^2)^{-1}\int_{T_0}^{n\delta_0} \frac{dt}{t}\int_{-L}^{L} \log \frac{v^2 - 2\lambda v + \mu}{v^2 - 2\lambda v + \mu^2} dv$$

$$= (4\pi^2)^{-1}\int_{T}^{n\delta_0} \frac{dt}{t}\int_{-L}^{L} \log \frac{v^2 - 2(2p+1)v + (2p+1)(2p+2)}{v^2 - 2(2p+1)v + (2p+1)^2} dv$$

$$+ \eta .$$

Where $|\eta| < \varepsilon \log n$ and ε is infinitely small. Taking L large we obtain from (3.9), (3.10) and (3.11)

$$M_n\left(-1 + \frac{\delta}{n}, -1 + n\right) = (2\pi)^{-1}(2p+1)^{1/2} \log n + o(\log n) .$$

Hence we have proved (1.5) combining this with the discussion in §§ 1 and 2 we get the proof of the theorem.

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