

PERTURBING EMBEDDINGS IN CODIMENSION TWO

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Let $f: X^n \rightarrow W^{n+2}$ be a fixed embedding of manifolds, assume X compact, and let $g: X \rightarrow W$ be an embedding close to f in the C^0 topology. In general, g and f will not be concordant. What small perturbation of g will yield an embedding concordant to f ?

In this paper, our goal is to replace g by a new embedding while altering the image manifold $g(X)$ as little as possible. In case X is simply connected, the problem was solved by Cappell and Shaneson as follows: if n is odd, $g(X)$ is already concordant to f [5]. If n is even and f has trivial normal bundle, $g(X)$ may be replaced by its connected sum with a knot in W [4]. The current paper applies previous work of the author [9, 10] to study the nonsimply connected case.

Recall that embeddings $f, g: X \rightarrow W$ are *concordant* provided there exists an embedding $H: X \times I \rightarrow W \times I$ with $H(x, 0) = (f(x), 0)$ and $H(x, 1) = (g(x), 1)$ for $x \in X$. Results will be stated for smooth manifolds and embeddings, but are true as well for piecewise linear or topological manifolds and locally flat embeddings. We assume throughout that the manifold topology of W is induced by a metric which in turn induces the supremum metric d on the space of embeddings of X in W .

The first two results describe codimension two embeddings of the product of a sphere and a compact manifold.

THEOREM A. *Let $f: S^n \times M^k \rightarrow W^{n+k+2}$ ($n \geq 2, n+k \geq 4$) be an embedding with trivial normal bundle. Assume that M is compact, and either $Wh(\pi_1(M)) = 0$, or $\pi_1(M)$ is cyclic and $n+k$ is odd. Let $Y \subset S^n \times M$ be a tubular neighborhood of $s \times M$, where $s \in S^n$ is a basepoint.*

Then for $g: S^n \times M \rightarrow W$ sufficiently close to f , there exists a diffeomorphism $\psi: S^n \times M \rightarrow S^n \times M$, and an embedding $h: S^n \times M \rightarrow W$, such that $h\psi$ is concordant to f , and h coincides with g outside Y .

In the above result, the phrase “ g sufficiently close to f ” means that there exists $\varepsilon > 0$ such that $d(f, g) < \varepsilon$ implies the existence of h with the desired properties.

In the special case $M = T^k$, the k -torus, Theorem A remains true if $n \geq 4$ and S^n is replaced by any simply connected manifold [§1, Example (ii)].

As an important case of Theorem A, let $W = S^{n+2} \times M$. Then $f: S^n \times M \rightarrow S^{n+2} \times M$ is called a “parametrized knot” in M provided it is homotopic to the standard inclusion; see [5, 9] for studies of parametrized knot theory.

THEOREM B. *Let $f: S^n \times M^k \rightarrow S^{n+2} \times M^k$ be a parametrized M -knot, i.e., an embedding homotopic to the standard inclusion i_0 . Assume $n \geq 2$, $n + k \geq 4$, and either $\pi = \pi_1(M)$ is cyclic, or $Wh(\pi) = 0$, or $n + k$ is even and π is finite abelian of odd order. Let $Y \subset S^n \times M$ be a tubular neighborhood of $s \times M$, where $s \in S^n$ is a basepoint. Then there exist diffeomorphisms $\phi: S^{n+2} \times M \rightarrow S^{n+2} \times M$ and $\psi: S^n \times M \rightarrow S^n \times M$, and a parametrized M -knot h , such that $h\psi$ is concordant to ϕf , and h coincides with i_0 outside Y .*

Thus the standard inclusion may be perturbed only near a single copy of M in order to produce a placement concordant to an ambient diffeomorphism of f .

The above two results rely strongly on the product structure of $S^n \times M$ in order to obtain a good hold on the normal invariants of the “complementary maps” [5, 9] of the various embeddings. By applying results of [10], we obtain results for more general manifolds. Recall that two compact manifolds Z and Y are *normally cobordant rel boundary* provided there exists a simple homotopy equivalence $h: Z \rightarrow Y$, a diffeomorphism of boundaries, whose normal invariant in $[Y, G/0]$ vanishes.

THEOREM C. *Let $f: X^n \rightarrow W^{n+2}$ be an embedding with trivial normal bundle; assume $n \geq 4$, X compact, and $Wh(\pi_1(X)) = 0$. Let $Y \subset X$ be a compact codimension zero submanifold, and denote by \bar{C} the closure in X of $C = X - Y$. Assume that the inclusion $Y \subset X$ induces an isomorphism of fundamental groups.*

Then for $g: X \rightarrow W$ an embedding sufficiently close to f , there exists $h: X \rightarrow W$, concordant to f , such that $h(X)$ contains $g(\bar{C})$, and $Z = h(X) - g(C)$ is normally cobordant rel boundary to Y .

It follows easily that Z and Y have diffeomorphic tubular neighborhoods in W [§2, Lemma 3].

To apply Theorem C to the perturbation problem, let Y be a “small” submanifold of X , e.g., a tubular neighborhood of the 2-skeleton. Then the image manifolds $h(X)$ and $g(X)$ intersect at least in their large common submanifold $g(C)$, and the remaining part $Z = h(X) - g(C)$ resembles closely the manifold $g(Y)$ which would have remained in the unperturbed case $h = g$.

1. Let ξ be a 2-plane bundle over a compact manifold X . We first recall from [5, 10] the theory of embeddings of X in $E(\xi)$, the total space of the disc bundle of ξ . If X has nonempty boundary, all embeddings, homotopies, concordances, etc. are relative to the boundary. Let $\dot{E}(\xi)$ be the total space of the circle bundle of ξ .

A *semilocal knot* in ξ is an embedding $f: X \rightarrow E(\xi)$ homotopic to the zero section i_0 . Call f and g *cobordant* provided f is concordant to ϕg for some diffeomorphism $\phi: (E(\xi), \dot{E}(\xi)) \rightarrow (E(\xi), \dot{E}(\xi))$, homotopic to the identity as map of pairs. The set of cobordism equivalence classes, denoted $C(X, \xi)$ or just $C(X)$, admits an abelian group structure which is defined geometrically by composition of embeddings. See [5, 10] for details.

To compute $C(X)$, let Φ be the commutative square

$$\begin{CD} \pi_1(\dot{E}(\xi)) @>id>> \pi_1(\dot{E}(\xi)) \\ @VidVV @VV\mathcal{F}V \\ \pi_1(\dot{E}(\xi)) @>\mathcal{F}>> \pi_1(E(\xi)) \end{CD}$$

where \mathcal{F} is induced by inclusion. Let $S_x(\xi) = S: [\Sigma E, G/0] \rightarrow L_{n+3}(\mathcal{F})$ be the Wall surgery obstruction for the pair $(E(\xi), \dot{E}(\xi))$, relative to $E(\xi|\partial X)$; here n is the dimension of X . Then

PROPOSITION [5, 5.2]. *For $n \geq 4$, there is an exact sequence*

$$0 \longrightarrow C(X) \xrightarrow{\Sigma} \Gamma_{n+3}(\Phi) \xrightarrow{\rho} \text{cok } S .$$

The middle term is an algebraic K theoretic functor introduced in [5] for the study of homology equivalent manifolds. The homomorphism Σ is the surgery obstruction of a normal map whose domain is the closed complement of a tubular neighborhood of an embedding, and ρ is a natural homomorphism.

For ∂X nonempty, the above result holds provided a semilocal knot is required to coincide with the zero section on ∂X [10]. This permits the study of naturality properties for codimension zero embeddings $j: Y \rightarrow X$. Let C be the closure of the complement of the image of j ; then $\partial C = \partial Y \cup \partial X$. Given a semilocal knot $f: Y \rightarrow E(\xi|j(Y))$, define $j(f) = f \cup i_0|_C: X = Y \cup C \rightarrow E(\xi|j(Y)) \cup E(\xi|C) = E(\xi)$. This defines a natural homomorphism $j_*: C(Y) \rightarrow C(X)$. Let

$$\mathcal{F}_X: \pi_1(\dot{E}(\xi)) \longrightarrow \pi_1(E(\xi)) \quad \text{and} \quad \mathcal{F}_Y: \pi_1(\dot{E}(\xi|j(Y))) \longrightarrow \pi_1(E(\xi|j(Y)))$$

be induced by inclusions. Then [5, 5.2] easily implies

LEMMA 1. *If $j_*: \pi_1(Y) \rightarrow \pi_1(X)$ is an isomorphism, then j_* :*

$C(Y) \rightarrow C(X)$ is injective. If in addition the induced map on Wall groups satisfies $j_*(\text{Image } S_Y) = \text{Image } (S_X) \subset L_{n+3}(\mathcal{F}_X)$, then j_* is an isomorphism.

Here $S_X: [\Sigma E(\xi), G/0] \rightarrow L_{n+3}(\mathcal{F}_X)$ and $S_Y: [\Sigma E(\xi|j(Y)), G/0] \rightarrow L_{n+3}(\mathcal{F}_Y)$ are the appropriate surgery obstruction maps.

In order to study perturbations of embeddings, we shall require $j_*: C(Y) \rightarrow C(X)$ to be an epimorphism. Call $j: Y \rightarrow X$ a *perfect embedding* if this is the case.

PROPOSITION 1. *Assume dimension $X = n \geq 4$ and $f: X \rightarrow E(\xi)$ is a semilocal knot. If $j: Y \rightarrow X$ is a perfect embedding, there exist*

(i) *a diffeomorphism $\phi: (E(\xi), \dot{E}(\xi)) \rightarrow (E(\xi), \dot{E}(\xi))$, homotopic as a map of pairs to the identity, and*

(ii) *a semilocal knot $g: X \rightarrow E(\xi)$, concordant to ϕi_0 , such that g and f coincide on $X - Y$.*

In other words, f may be made cobordant to the zero section if it is allowed to be changed on Y .

Proof. Let $[h] \in C(Y)$ be a cobordism class for which $j_*([h]) = -[f] \in C(X)$. Let $\bar{f}: E(\xi) \rightarrow E(\xi)$ be the canonical extension of f [5, Lemma 4.2], and set $g = \bar{f} \circ j(h)$. By definition of the group operation in $C(X)$, $[g] = [f] + [j(h)] = [i_0]$. Since $j(h)|_{X - Y}$ coincides with i_0 , the result follows.

In order to remove the ambient diffeomorphism from the above result, it is necessary to restrict the bundle ξ . Let $p'_n: L_n(\pi_1(X)) \rightarrow L_{n+2}(\mathcal{F}_X)$ be defined by using the projection $p: E(\xi) \rightarrow X$ to induce a normal map to $(E(\xi), \dot{E}(\xi))$ from a normal map to X . Call the bundle ξ *n-perfect* provided p'_{n+2} is surjective and p'_{n+1} is injective. The next result follows from the proof of [4, Lemma 2].

LEMMA 2. *Let ξ be an n-perfect 2-plane bundle over a compact manifold X^n , $n \geq 4$. If a diffeomorphism $\phi: (E(\xi), \dot{E}(\xi)) \rightarrow (E(\xi), \dot{E}(\xi))$ is homotopic as a map of pairs to the identity, then ϕ is concordant to a bundle map covering a diffeomorphism $\psi: X \rightarrow X$.*

THEOREM 1. *Let $f: X^n \rightarrow W$ ($n \geq 4$) be a codimension two embedding with n-perfect normal bundle ξ . Suppose the inclusion $Y \subset X$ of a codimension zero submanifold is a perfect embedding.*

Then for any $g: X \rightarrow W$ sufficiently close to f , there exists a diffeomorphism $\psi: X \rightarrow X$, homotopic to the identity, and an em-

bedding $h: X \rightarrow W$ which coincides with g outside Y , such that h is concordant to $f\psi$.

Proof. For g sufficiently close to f , $g(X)$ will lie in $E(\xi)$, and g and f will be homotopic as maps to $E(\xi)$. Thus g is a semilocal knot in $E(\xi)$, and f plays the role of the zero section. Now apply the last two results.

We now give examples of bundles ξ and embeddings $j: Y \rightarrow X$ which satisfy the hypotheses. The results of [5, 4] show that in many cases with $\pi_1(X)$ trivial, the conclusion of Theorem 1 holds when Y is either empty or a disc.

If ξ is a trivial bundle and $\pi = \pi_1(X)$, then p'_n factors as $L_n(\pi) \rightarrow L_n^h(\pi) \rightarrow L_{n+2}(\pi \times Z \rightarrow \pi)$. The second map is an isomorphism obtained by crossing with a disc [11]; the first is the forgetful homomorphism, which fits into an exact sequence involving $H^*(Z_2, Wh(\pi))$. If particular, ξ is n -perfect if $H^{n+2}(Z_2, Wh(\pi))$ vanishes, e.g., if $Wh(\pi)$ vanishes or π is finite cyclic and n is odd [8, 1]. If ξ is nontrivial but n is even and $\pi_1(X) = 0$, then ξ is n -perfect; see [4].

We now list examples of perfect embeddings. In all cases, assume ξ is trivial.

EXAMPLE (i). Let $j: D^n \times M \rightarrow S^n \times M$ be the Cartesian product of id_M with an embedding of a disc in S^n . The homomorphisms S for $D^n \times M$ and $S^n \times M$ in the exact sequence [5, 5.2] may be identified with the surgery obstruction maps

$$\begin{aligned} \sigma: [\Sigma(S^n \times M), G/0] &\longrightarrow L_{n+1}^h(\pi_1(M)) \quad \text{and} \\ \sigma: [\Sigma(D^n \times M), G/0] &\longrightarrow L_{n+1}^h(\pi_1(M)), \end{aligned}$$

which have the same image by [9, Lemma 15.1]. Lemma 1 and the Five Lemma complete the result. This example, together with Theorem 1, implies Theorem A of the Introduction.

EXAMPLE (ii). Suppose $j: T^n \times D^k \rightarrow Y$ induces a fundamental group isomorphism, T^n the n -torus. If $k \geq 4$, then $S: [\Sigma(T^n \times D^k)] \rightarrow L_{n+k+1}(nZ)$ is surjective [3]. Now apply Lemma 1.

EXAMPLE (iii). Assume $j: Y \subset X$ induces an isomorphism of finite cyclic odd order fundamental groups. Then the surgery obstruction $\sigma: [\Sigma X, G/0] \rightarrow L_{n+1}^h(\pi_1(X))$ is zero [3] and may be identified with $S_X: [\Sigma E, G/0] \rightarrow L_{n+3}(\mathcal{F})$ provided ξ is trivial. Now apply Lemma 1 to see that j is a perfect embedding.

As an application of Example (i) above, we settle a question

raised by the author's study of $G_n(M)$, the set of cobordism classes of "parametrized M -knots," i.e., embeddings $f: S^n \times M \rightarrow S^{n+2} \times M$ homotopic to the standard embedding i_0 . Two such embeddings f and g are *cobordant* provided there exist diffeomorphisms $\phi: S^{n+2} \times M \rightarrow S^{n+2} \times M$ and $\psi: S^n \times M \rightarrow S^n \times M$, each commuting up to homotopy with projection to M , such that $\phi f \psi$ is concordant to g . See [5, 9] for details.

Proof of Theorem B. We may assume that Y contains the image of an embedding $j: D^n \times M \rightarrow S^n \times M$ as in Example (i) above. Consider the composite

$$C(D^n \times M) \xrightarrow{i_\#} C(S^n \times M) \xrightarrow{i} G_n(M).$$

The semilocal knot groups are those for the trivial bundle; the map i is induced by the inclusion $S^n \times M \times D^2 \subset S^{n+2} \times M$, and is a well defined isomorphism by [9, 15.3]. By Example (i) above and Lemma 1, $j_\#$ is also an isomorphism. The stated result follows from the surjectivity of the composite. The injectivity of the composite implies that the cobordism class of a parametrized M -knot is determined by its behavior near a single copy of M in $S^n \times M$.

2. In order to prove Theorem C, we review the results of [10] on embeddings in $E(\xi)$ of manifolds simple homotopy equivalent to X . As before, ξ is a 2-plane bundle over the compact manifold X^n .

DEFINITION. A *fake semilocal knot in ξ* is an embedding $f: Z \rightarrow E(\xi)$, where Z is a manifold of dimension n , such that for some simple homotopy equivalence of pairs $h: (X, \partial X) \rightarrow (Z, \partial Z)$, a diffeomorphism on the boundary, the composite $f \circ h$ is homotopic, relative to ∂X , to the zero section.

Two fake knots $f_i: Z_i \rightarrow E(\xi)$ ($i = 1, 2$) are *cobordant* provided there exist diffeomorphisms $\psi: Z_1 \rightarrow Z_2$ and $\phi: E(\xi) \rightarrow E(\xi)$, the latter homotopic as a map of pairs to the identity, such that $\phi f_2 \psi$ is concordant to f_1 . Let $\hat{C}(X, \xi)$ be the set of cobordism equivalence classes.

To compute $\hat{C}(X, \xi)$, recall from §1 the natural homomorphism $j_*: \Gamma_i(\Phi) \rightarrow L_i(\mathcal{F})$. As before, $p^i: L_{i-2}(\pi) \rightarrow L_i(\mathcal{F})$ is obtained by inducing normal maps. Define

$$\Gamma_i(\Phi, \xi) = j_i^{-1}(\text{Image } p^i) \subset \Gamma_i(\Phi).$$

THEOREM 3. *Let ξ be a 2 plane bundle over a compact manifold X^n ($n \geq 4$). There is a natural surgery obstruction $\Sigma: \hat{C}(X, \xi) \rightarrow \Gamma_{n+3}(\Phi)$ which is injective with image $\Gamma_{n+3}(\Phi, \xi)$.*

Proof. This follows from minor modifications of the proof of [10, Theorem 1]. Note that $\hat{C}(X, \xi)$ acquires an abelian group structure; this is interpreted geometrically in [10].

The following result shows that $\hat{C}(X, \xi)$ often consists of local knots which are only mildly fake. Assume the hypotheses of Theorem 3.

LEMMA 3. *If ∂X is nonempty, and $f: Z \rightarrow E(\xi)$ is a fake semi-local knot with tubular neighborhood T , then Z is normally cobordant, rel boundary, to X , and T is diffeomorphic to $E(\xi)$.*

Proof. Let $h: X \rightarrow Z$ be a simple homotopy equivalence of pairs with $f \circ h$ homotopic to the zero section, and let $\bar{h}: E(\xi) \rightarrow T$ be a covering bundle map. As in [10], there is a “characteristic map” $\hat{F}: E(\xi) \rightarrow E(\xi) \cup_{\dot{E}(\xi) \times 0} \dot{E}(\xi) \times I \approx E(\xi)$, such that $\hat{F}|T = (\bar{h})^{-1}$, where \hat{F} is a simple homotopy equivalence with vanishing normal invariant in $[E(\xi), G/0]$. By naturality of normal invariants [2, 13], $(\bar{h})^{-1}$, \bar{h} , and h all have vanishing normal invariants. In particular, Z is normally cobordant to X in all cases. If ∂X is nonempty, Van Kampen’s theorem implies easily that the inclusion $\partial E(\xi) \subset E(\xi)$ induces a fundamental group isomorphism. By the $\pi - \pi$ theorem [13], the simple homotopy equivalence \bar{h} is homotopic as a map of pairs (although *not* relative to $E(\xi|\partial X)$) to a diffeomorphism.

Now suppose $j: Y \rightarrow X$ is a codimension zero embedding. As in §1, j induces a natural homomorphism $j_*: \hat{C}(Y, \xi|j(Y)) \rightarrow \hat{C}(X, \xi)$, defined by gluing a fake semilocal knot in $\xi|j(Y)$ to the zero section of $\xi|X - Y$.

For most applications, it is necessary to assume that $p^!: L_{n+1}(\pi) \rightarrow L_{n+3}(\mathcal{F})$ is surjective, where $\pi = \pi_1(X)$ and \mathcal{F} is induced by including $\dot{E}(\xi)$ in $E(\xi)$.

PROPOSITION 2. *Assume ξ is a 2-plane bundle over the compact manifold X^n , and let $j: Y^n \rightarrow X^n$ induce a fundamental group isomorphism. Then $p^!_X$ is surjective if and only if $p^!_Y$ is surjective. In this case $j_*: \hat{C}(Y, \xi|j(Y)) \rightarrow \hat{C}(X, \xi)$ is an isomorphism.*

Proof. The first assertion follows from the naturality of $p^!$. Theorem 3 now yields an isomorphism $\Sigma_X: \hat{C}(X) \rightarrow \Gamma_{n+3}(\Phi_X)$, where Φ_X is determined by $\pi_1(\dot{E}(\xi)) \rightarrow \pi_1(E(\xi))$; similarly $\Sigma_Y: \hat{C}(Y) \rightarrow \Gamma_{n+3}(\Phi_Y)$ is an isomorphism. Since j induces an isomorphism of the squares Φ_X and Φ_Y , the result follows by the naturality of Σ .

In order to prove Theorem C we combine restrictions on p^i that occur in the above result and in §1.

PROPOSITION 3. *Let $f: X^n \rightarrow W$ ($n \geq 4$) be a codimension two embedding with normal bundle ξ , and let $j: Y^n \rightarrow X^n$ induce a fundamental group isomorphism. Assume p_{n+2}^1 is surjective and p_{n+1}^1 is an isomorphism. Then for any embedding $g: X \rightarrow W$ which is sufficiently close to f , there exists an embedding $\theta: X \rightarrow E(\xi)$ such that*

- (i) $\bar{g}\theta$ is concordant to f , where $\bar{g}: E(\xi) \rightarrow W$ is an embedding extending g
- (ii) $\theta(X) \cap E(\xi|(X - Y)) = i_0(X - Y)$, and
- (iii) $\theta(X) \cap E(\xi|j(Y))$ is normally cobordant, rel boundary, to Y .

Proof. As in §1, assume $g(X) \subset E(\xi)$ and view g as a semilocal knot in ξ . By Proposition 2, $j_*: \hat{C}(Y) \rightarrow \hat{C}(X)$ is an isomorphism; hence $[i_0] = [g] + j_*[k]$ for some fake knot $k: Z \rightarrow E(\xi|j(Y))$. Since g is a nonfake semilocal knot, addition of a fake knot to g may be defined by composition; see [10] for definitions. Thus $[i_0] = [\bar{g} \circ j(k)]$ and there is a commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{g'} & E(\xi) \\
 \downarrow \psi & & \downarrow \phi \\
 Z \bigcup_{\partial Y} \bar{C} & \xrightarrow{j(k)} & E(\xi) \xrightarrow{\bar{g}} E(\xi)
 \end{array}$$

in which $\bar{C} = X - \text{interior}(Y)$, $j(k) = k \cup i_0|_{\bar{C}}$, g' is concordant to i_0 , and ϕ and ψ are diffeomorphisms. By Lemma 2, ϕ is concordant to a bundle map covering a diffeomorphism $\psi_1: X \rightarrow X$. Set $\theta = j(k) \circ \psi \circ \psi_1^{-1}$; then $\bar{g}\theta$ is concordant to i_0 . Properties (ii) and (iii) follow from the definition of $j(k)$ and from Lemma 3.

In order to satisfy the hypotheses, it seems necessary to restrict to $Wh(\pi_1(X)) = 0$ and ξ the trivial bundle. Theorem A now follows by taking $h = \bar{g}\theta$.

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