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BETWEENNESS RELATIONS IN PROBABILISTIC METRIC SPACES

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Four distinct versions of the betweenness concept for probabilistic metric spaces are defined and studied. Conditions under which some or all of the properties of metric betweenness are satisfied are determined and the relationships among the different concepts are investigated.

1. Introduction. In his original paper on probabilistic metric spaces [8] K. Menger, in addition to introducing the basic concepts and axioms, introduced a definition of betweenness, developed some of its properties and showed that this relation was generally weaker than ordinary metric betweenness. Shortly thereafter, A. Wald [25] introduced a different definition of betweenness, based on a different triangle inequality, and showed that his relation did have all the properties of metric betweenness. Subsequently, J. F. C. Kingman [6] and F. Rhodes [16] studied betweenness in "Wald spaces" and H. Sherwood [23] considered a probabilistic version of the concept. Otherwise the subject has lain dormant—primarily because adequate tools for its analysis were not available. Our recent work on the structure of semigroups on the space of probability distribution functions [9, 11, 12, 17] and the development of "characteristic functions" for certain classes of these semigroups [10, 13, 14] has changed this state of affairs. Thus we return to the study of betweenness in probabilistic metric spaces. We focus our attention on four different versions of this concept. The first of these is the straightforward generalization of Wald's betweenness from Wald spaces to arbitrary probabilistic metric spaces. We show that this relation satisfies some, but generally not all, of the usual properties of metric betweenness, determine sufficient conditions for the validity of those properties which are not always satisfied and show that in some instances these conditions are also necessary. The second betweenness relation applies to a restricted but nevertheless very large class of probabilistic metric spaces. It always satisfies the metric betweenness properties and, whenever it is comparable to the first relation, it is either identical or weaker. The third relation, which applies to the same class of spaces as the second, is obtained by an extension of Kingman's idea from Wald spaces to this class. It is a metric betweenness for certain naturally defined metrics and is always weaker than the second relation. The last relation is Menger's betweenness. We reformulate Menger's definition in terms of triangle functions and show that in

simple spaces and Wald spaces, Wald's and Menger's concepts coincide. A more detailed study of Menger's betweenness still remains to be carried out.

In order to present our results we need to recapitulate some of the basic definitions and known results from the theory of probabilistic metric (PM) spaces. Recall that such a space is an ordered triple $(S, \mathscr{F}, \tau)$, where S is a set, τ is a triangle function, and $\mathscr{F}$ is a mapping from $S \times S$ into the set of distribution functions

$$\Delta^+ = \{F: \boldsymbol{R} \to [0, 1] \mid F \text{ is nondecreasing, left-continuous and } F(0) = 0\} \tag{1.1}$$

such that, for all p, q, r in S,

(I) $F_{pq} = \varepsilon_0$ if and only if $p = q$,

(II) $F_{pq} = F_{qp}$,

(III) $F_{pr} \geqq \tau(F_{pq}, F_{qr})$.

Here $F_{pq} = \mathscr{F}(p, q)$; ε_0 is the distribution function defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leqq 0, \\ 1, & 0 < x; \end{cases} \tag{1.2}$$

and a *triangle function* τ is a binary operation on Δ^+ satisfying

(α) $\tau(F, \varepsilon_0) = F$,

(β) $\tau(F_1, G_1) \geqq \tau(F_2, G_2)$, whenever $F_1 \geqq F_2$, $G_1 \geqq G_2$,

(γ) $\tau(F, G) = \tau(G, F)$,

(δ) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Thus τ is a commutative, order-preserving semigroup operation, with unit ε_0, on Δ^+.

A sequence $\{F_n\}$ in Δ^+ converges weakly to $F \in \Delta^+$, and we write $F_n \xrightarrow{w} F$, if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at every continuity point x of the limit function F. This mode of convergence is metrizable (an explicit metric is exhibited in [24]) and the space Δ^+ is compact in the induced metric topology. If the triangle function τ is (uniformly) continuous then the collection of sets $\{N_p(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda > 0,\ p \in S\}$, where

$$N_p(\varepsilon, \lambda) = \{q \in S \mid F_{qp}(\varepsilon) > 1 - \lambda\} \tag{1.3}$$

is a neighborhood basis for a metrizable topology on S [18, 21], called the *ε, λ-topology*. Moreover, a sequence $\{q_n\}$ in S converges to $q \in S$ in this topology if and only if $F_{qq_n} \xrightarrow{w} \varepsilon_0$.

When τ is convolution (a continuous triangle function), then $(S, \mathscr{F})$ is a *Wald space*; and $(S, \mathscr{F})$ is a *Menger space* when τ is of the form τ_T, where, for any $F, G \in \Delta^+$ and any real x,

$$\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v)) \tag{1.4}$$

and T is a (left-continuous) *t-norm*, i.e., a (left-continuous) binary operation on the unit interval $[0, 1]$ such that

(a) $T(a, 1) = a$,

(b) $T(c, d) \geqq T(a, b)$, whenever $c \geqq a$, $d \geqq b$,

(c) $T(a, b) = T(b, a)$,

(d) $T(T(a, b), c) = T(a, T(b, c))$.

In this paper we generally assume that T is continuous. This implies that τ_T is continuous on Δ^+ [17]. The most important continuous t-norms are $\mathrm{Min}(a, b)$, $\mathrm{Prod}(a, b) = ab$, and $T_m(a, b) = \mathrm{Max}\{a + b - 1, 0\}$.

DEFINITION 1.1. A t-norm T is *Archimedean* if it is continuous on $[0, 1] \times [0, 1]$ and such that $T(a, a) < a$ for all $a \in (0, 1)$; and T is *strict* if it is continuous on $[0, 1] \times [0, 1]$ and strictly increasing in each place on $(0, 1] \times (0, 1]$.

It is immediate that every strict t-norm is Archimedean. The t-norms of Definition 1.1 are completely characterized by the following representation theorem [7]:

THEOREM 1.1. *The t-norm T is Archimedean if and only if there exists a continuous and increasing function $h: [0, 1] \to [0, 1]$, with $h(1) = 1$, such that*

$$T(a, b) = h^{[-1]}(h(a) \cdot h(b)) , \tag{1.5}$$

where

$$h^{[-1]}(x) = \begin{cases} 0 , & 0 \leqq x \leqq h(0) , \\ h^{-1}(x) , & h(0) \leqq x \leqq 1 , \end{cases} \tag{1.6}$$

and h^{-1} is the usual inverse of h on $[h(0), 1]$. Furthermore, T is strict if and only if $h(0) = 0$, in which case $h^{[-1]} = h^{-1}$.

The function h in (1.5) is called a *multiplicative generator* of the t-norm T and $h^{[-1]}$ is the *pseudo-inverse* of h.

2. **Wald-betweenness.** If (S, d) is a metric space and p, q, r are three distinct points of S then q is said to lie between p and r—and one writes pqr—if and only if $d(p, r) = d(p, q) + d(q, r)$. This relation has the following properties [3]:

(B1) If pqr then rqp.

(B2) If pqr then neither qrp nor rpq.

(B3) (a) If pqr and prs then pqs.

(b) If pqr and prs then qrs.

(B4) The set $B(p, r) \cup \{p, r\}$ is closed in the metric topology, where $B(p, r) = \{q \mid pqr\}$.

In his paper [25] Wald considered the straightforward generalization of metric betweenness given by: If p, q, r are three distinct points of a Wald space, then q lies between p and r if and only if

$$F_{pr} = F_{pq} * F_{qr} , \tag{2.1}$$

where $*$ denotes convolution. In this section we consider Wald's betweenness for an arbitrary triangle function τ.

DEFINITION 2.1. Let $(S, \mathscr{F}, \tau)$ be a PM space and let p, q, r be three distinct points of S. Then q is *Wald-between* p and r—and we write $W(pqr)$—if $F_{pr} \neq \varepsilon_\infty$ and

$$F_{pr} = \tau(F_{pq}, F_{qr}) , \tag{2.2}$$

where $\varepsilon_\infty \in \Delta^+$ is defined by $\varepsilon_\infty(x) = 0$ for all x.

Since ε_∞ and ε_0 are, respectively, the mimimal and maximal elements of Δ^+, for any $F \in \Delta^+$ we have

$$\varepsilon_\infty \leqq \tau(\varepsilon_\infty, F) \leqq \tau(\varepsilon_\infty, \varepsilon_0) = \varepsilon_\infty , \tag{2.3}$$

whence $\tau(\varepsilon_\infty, F) = \varepsilon_\infty$. Thus if $W(pqr)$ then none of F_{pr}, F_{pq}, F_{qr} is equal to ε_∞. Note also that, since p, q, r are distinct, $W(pqr)$ implies that none of F_{pr}, F_{pq}, F_{qr} is equal to ε_0.

LEMMA 2.1. *Let the triangle function τ be continuous on Δ^+. Then the following are equivalent:*

(i) *There exist $F, G \in \Delta^+$, both different from ε_∞ and ε_0, such that $\tau(F, G) = F$.*

(ii) *There exists an $H \in \Delta^+$, different from ε_∞ and ε_0, such that $\tau(H, H) = H$,* i.e., *there exists a nontrivial idempotent in the semigroup (Δ^+, τ).*

Proof. Clearly (ii) implies (i) on letting $F = G = H$. Therefore suppose (i) holds and let $G^2 = \tau(G, G)$ and $G^{n+1} = \tau(G, G^n)$, for $n = 2, 3, \cdots$. Then $\varepsilon_0 > G \geqq G^2 \geqq \cdots \geqq G^n \geqq \cdots$, whence the weak limit of the sequence $\{G^n\}$ exists and is distinct from ε_0. Denote this limit by H. Since τ is continuous, we have

$$H = \lim_{n\to\infty} G^{2n} = \lim_{n\to\infty} \tau(G^n, G^n) = \tau(H, H) ,$$

whence H is idempotent. Next, using (i) and the continuity of τ yields

$$\begin{aligned} F &= \tau(F, G) = \tau(\tau(F, G), G) = \tau(F, G^2) \\ &= \cdots = \tau(F, G^n) = \tau(F, \lim_{n\to\infty} G^n) = \tau(F, H) . \end{aligned}$$

Since $F \neq \varepsilon_\infty$, it follows from (2.3) that $H \neq \varepsilon_\infty$. Thus H satisfies (ii) and the proof is complete.

COROLLARY 2.1. *If τ is continuous then the semigroup (Δ^+, τ) has no nontrivial idempotents if and only if*

$$\tau(F, G) < F \tag{2.4}$$

whenever F and G are both different from ε_∞ and ε_0.

On Δ^+, the conditions (i) and (ii) of Lemma 2.1 are not satisfied by convolution nor by any τ_T when T is Archimedean [11]. Thus these semigroups have no nontrivial idempotents.

DEFINITION 2.2. A triangle function τ is *strictly increasing* on a subset $\mathscr{S}$ of Δ^+ if, for any $F, G, H \in \mathscr{S}$, $\tau(F, G) > \tau(F, H)$ whenever $F \neq \varepsilon_\infty$ and $G > H$.

It is easily seen that if the cancellation law holds in (Δ^+, τ) then τ is strictly increasing on Δ^+; and that if τ is strictly increasing on Δ^+ then (Δ^+, τ) has no nontrival idempotents.

THEOREM 2.1. *Let $(S, \mathscr{F}, \tau)$ be a PM space. Then the Wald-betweenness relation*:

(i) *Always satisfies the betweenness properties* (B1) *and* (B3a).

(ii) *Satisfies* (B2) *whenever τ is continuous and has no nontrivial idempotents.*

(iii) *Satisfies* (B2) *and* (B3b) *whenever τ is strictly increasing on* Ran $\mathscr{F}$, *the range of $\mathscr{F}$.*

(iv) *Satisfies* (B4), *with respect to the ε, λ-topology on S, whenever τ is continuous.*

Proof. (i) The property (B1) is trivial. Now suppose $W(pqr)$ and $W(prs)$. Then $F_{pr} = \tau(F_{pq}, F_{qr})$, $F_{ps} = \tau(F_{pr}, F_{rs})$ and, in view of the triangle inequality, $F_{qs} \geqq \tau(F_{qr}, F_{rs})$ and $F_{ps} \geqq \tau(F_{pq}, F_{qs})$. Thus

$$\begin{aligned} F_{ps} = \tau(\tau(F_{pq}, F_{qr}), F_{rs}) &= \tau(F_{pq}, \tau(F_{qr}, F_{rs})) \\ &\leqq \tau(F_{pq}, F_{qs}) \leqq F_{ps} , \end{aligned} \tag{2.5}$$

whence $W(pqs)$.

(ii) Suppose $W(pqr)$. Then, by the remarks after Definition 2.1, none of F_{pq}, F_{qr}, F_{pr} is equal to either ε_0 or ε_∞. Thus, using Corollary 2.1,

$$\begin{aligned} \tau(F_{qr}, F_{rp}) &= \tau(F_{qr}, \tau(F_{pq}, F_{qr})) \\ &= \tau(F_{pq}, \tau(F_{qr}, F_{qr})) \leqq \tau(F_{pq}, \tau(F_{qr}, \varepsilon_0)) \\ &= \tau(F_{pq}, F_{qr}) < F_{pq} , \end{aligned} \tag{2.6}$$

whence $W(qrp)$ does not hold. Similarly $W(rpq)$ also does not hold.

(iii) To prove (B2), we have only to note that the display (2.6) remains valid.

Now suppose $W(pqr)$ and $W(prs)$. If $F_{qs} > \tau(F_{qr}, F_{rs})$ then the first inequality in (2.5) would be strict, which cannot be. Thus also $W(qrs)$, whence (B3b) holds.

(iv) Suppose that $\{q_n\}$ is a sequence in $B(p, r) \cup \{p, r\}$ such that $q_n \to q_0 \in S$ in the ε, λ-topology. If $q_0 = p$ or $q_0 = r$, we are done. Otherwise the points p, q_0, r are distinct and we may also assume without loss of generality that all the triples (p, q_n, r) consist of distinct points, so that $W(pq_nr)$ for all n. Then, using the fact that $F_{q_0q_n} \xrightarrow{w} \varepsilon_0$, we have

$$\begin{aligned} F_{pr} &\geqq \tau(F_{pq_0}, F_{q_0r}) \geqq \tau(\tau(F_{pq_n}, F_{q_nq_0}), \tau(F_{q_0q_n}, F_{q_nr})) \\ &= \tau(\tau(F_{pq_n}, F_{q_nr}), \tau(F_{q_0q_n}, F_{q_0q_n})) \\ &= \tau(F_{pr}, \tau(F_{q_0q_n}, F_{q_0q_n})) \xrightarrow{w} F_{pr} , \end{aligned}$$

since τ is continuous. Thus $W(pq_0r)$, whence $q_0 \in B(p, r)$, and the proof is complete.

Note. The proofs of (i) and (iii) are generalizations of Wald's arguments [25], and the proof of (iv) is a generalization of the argument used by F. Rhodes in [16].

Theorem 2.1 applies in the following special cases:

(i) $\tau =$ convolution. In this case Wald [25] has shown that τ is strictly increasing on the subset $\mathscr{D}^+$ of Δ^+ given by

$$\mathscr{D}^+ = \{F \in \Delta^+ \mid \lim_{x \to \infty} F(x) = 1\} , \tag{2.7}$$

and Wald's argument extends to Δ^+. Equivalently, this follows from the validity of the cancellation law in the semigroup $(\Delta^+, *)$.

(ii) $\tau = \tau_{\text{Min}}$ and $\operatorname{Ran} \mathscr{F} \subset \mathscr{D}^+$. In this case, as was shown in [9], the cancellation law holds in the semigroup $(\mathscr{D}^+, \tau_{\text{Min}})$.

For example, if $(\mathscr{S}, \mathscr{F})$ is the *simple space* generated by the metric space (S, d) and the distribution function $G \in \mathscr{D}^+$, so that for any distinct $p, q \in S$,

$$F_{pq}(x) = G(x/d(p, q)) , \tag{2.8}$$

then $(S, \mathscr{F})$ is a Menger space under τ_{Min} [18]. Moreover, it is easy to show that in this case Wald-betweenness and d-metric betweenness are equivalent.

(iii) $\tau = \tau_T$ where T is a strict t-norm and $\operatorname{Ran} \mathscr{F} \subseteqq \Delta_T^+$ (see Definition 3.2). In this case the cancellation law holds in (Δ_T^+, τ_T) [10].

The following examples show that the relation $W(pqr)$ may fail to satisfy either (B2) or (B3b) when the corresponding hypotheses of Theorem 2.1 are not satisfied.

EXAMPLE 2.1. Let τ be a triangle function for which there exists a nontrivial $H \in \Delta^+$ such that $\tau(H, H) = H$. Let $(S, \mathscr{F})$ be the equilateral PM space in which $F_{pq} = H$ for any pair of distinct points p and q. Then $W(pqr)$ holds for all triples of distinct points in S and thus (B2) fails.

EXAMPLE 2.2. In [9] it was shown that the cancellation law fails in the semigroup $(\Delta^+, \tau_{\text{Prod}})$. The counterexample which established this fact will serve us here as well. Let $S = \{p, q, r, s\}$ and first define $\mathscr{F}$ via:

$$F_{pq}(x) = \begin{cases} 0 , & x \leqq 0 , \\ x , & 0 \leqq x \leqq 1 , \\ 1 , & 1 \leqq x ; \end{cases}$$

$$F_{qr}(x) = \varepsilon_0(x - 1) ;$$

$$F_{rs}(x) = \begin{cases} 0 , & x \leqq 0 , \\ 1/2 , & 0 < x \leqq 1.45 , \\ 1 , & 1.45 < x ; \end{cases}$$

and $F_{pr} = \tau_{\text{Prod}}(F_{pq}, F_{qr})$, $F_{ps} = \tau_{\text{Prod}}(F_{pr}, F_{rs})$, $F_{qs} = \tau_{\text{Prod}}(F_{qr}, F_{rs})$. Now, by (i) of Theorem 2.1, it follows that

$$F_{ps} = \tau_{\text{Prod}}(F_{pq}, F_{qs}) , \tag{2.9}$$

whence we have $W(pqr)$, $W(prs)$, $W(qrs)$, and $W(pqs)$. Using the above, it is easily verified that $(S, \mathscr{F}, \tau_{\text{Prod}})$ is a PM space in which Wald-betweenness satisfies (B1)-(B4). However, as shown in [9], we can alter F_{qs} slightly and still maintain equality in (2.9). In fact, if $G \in \Delta^+$ is given by

$$G(x) = \begin{cases} 0 , & x \leqq 1 , \\ 1/2 , & 1 < x \leqq 2.4 , \\ .55 , & 2.4 < x \leqq 2.45 , \\ 1 , & 2.45 < x ; \end{cases}$$

then $G > F_{qs}$, but $\tau_{\text{Prod}}(F_{pq}, G) = \tau_{\text{Prod}}(F_{pq}, F_{qs})$. Thus, if we let $F'_{qs} = G$ and let the remaining distance distribution functions be defined as before then we obtain a space $(S, \mathscr{F}')$ which is still a PM space under τ_{Prod}. Furthermore in this new space $W(pqr)$, $W(prs)$, and $W(pqs)$ hold but, by construction, $W(qrs)$, and hence (B3b), fails.

Using some of the results of [9], similar counterexamples can be constructed for any triangle function τ_T, where T is a continuous t-norm.

We conclude this section with an interesting particular instance of Wald betweenness.

Let $(S, \mathscr{F})$ be a *pseudo-metrically generated* PM *space*, that is to say, let S be a given set, let $(D, \mathscr{B}, \mu)$ be a probability space whose elements are pseudo-metrics on S, and let $\mathscr{F}$ be defined via

$$F_{pq}(x) = \mu\{d \in D \mid d(p, q) < x\} . \tag{2.10}$$

Then $(S, \mathscr{F})$ is a Menger space under τ_{T_m} [22]. Furthermore, T_m is the strongest t-norm for the class of pseudo-metrically generated spaces [4]. (Note: This does not mean that T_m is the strongest t-norm for all such spaces. For example, simple spaces, which are Menger spaces under Min, are pseudo-metrically generated. In general, for Wald-betweenness to be a meaningful relation in a specific Menger space, it must be with respect to the strongest t-norm for that space.)

As regards Wald-betweenness with respect to τ_{T_m}, we have the following:

THEOREM 2.2. *If $(S, \mathscr{F})$ is a pseudo-metrically generated PM space and p, q, r are distinct points of S, then q is Wald-between p and r,* i.e., *$W(pqr)$ holds with respect to τ_{T_m}, if and only if q is between p and r for almost all pseudo-metrics d in the generating collection D and either $d(p, q)$ or $d(q, r)$ is constant for almost all d in D.*

Proof. For any $p, q \in S$, the mapping $(pq): D \to \boldsymbol{R}^+$ defined by $(pq)(d) = d(p, q)$ is a nonnegative random variable whose distribution function is the function F_{pq} given by (2.10). Note that $F_{pq} \in \mathscr{D}^+$, where $\mathscr{D}^+$ is given by (2.7). Furthermore, for any $p, q, r \in S$ the joint distribution function of (pq) and (qr) exists and is given by

$$F_{pq,qr}(u, v) = \mu\{d \mid d(p, q) < u,\ d(q, r) < v\} .$$

Let C be the connecting copula of (pq) and (qr) [20], so that

$$F_{pq,qr}(u, v) = C(F_{pq}(u), F_{qr}(v)) .$$

Then the distribution function of the random variable $(pq) + (qr)$, i.e., $d(p, q) + d(q, r)$, is given by

$$F_{pq+qr} = \sigma_C(F_{pq}, F_{qr}) , \tag{2.11}$$

where σ_C is the binary operation on Δ^+ defined via

$$\sigma_C(F_{pq}, F_{qr})(x) = \iint_{u+v<x} dC(F_{pq}(u), F_{qr}(v)) .$$

Since each of the pseudo-metrics $d \in D$ satisfies the ordinary triangle inequality, we have

$$F_{pq+qr}(t) = \mu\{d \mid d(p, q) + d(q, r) < t\} \leqq \mu\{d \mid d(p, r) < t\} = F_{pr}(t) . \tag{2.12}$$

So much holds in general. To proceed with the proof, suppose first that $W(pqr)$, i.e., that $F_{pr} = \tau_{T_m}(F_{pq}, F_{qr})$. Then (2.11) and (2.12) yield

$$\sigma_C(F_{pq}, F_{qr}) \leqq \tau_{T_m}(F_{pq}, F_{qr}) .$$

Now it is known that $\tau_{T_m}(F, G) \leqq \sigma_C(F, G)$ for any F, $G \in \Delta^+$ [17]. Hence

$$\sigma_C(F_{pq}, F_{qr}) = \tau_{T_m}(F_{pq}, F_{qr}) . \tag{2.13}$$

But F_{pq}, F_{qr} belong to the subspace $\mathscr{D}^+$ of Δ^+; and in this case it can be shown[1] that (2.13) holds if and only if, for some a, $b > 0$, either $F_{pq} = \varepsilon_a$ or $F_{qr} = \varepsilon_b$, where $\varepsilon_a(x) = \varepsilon_0(x - a)$, $\varepsilon_b(x) = \varepsilon_0(x - b)$. Thus for almost all $d \in D$, either $d(p, q) = a$ or $d(q, r) = b$. Suppose $d(p, q) = a$. Then

$$F_{pr}(x) = \tau_{T_m}(\varepsilon_a, F_{qr})(x) = F_{qr}(x - a) ,$$

which, combined with the fact that $d(p, r) \leqq d(p, q) + d(q, r)$, yields $d(p, r) = a + d(q, r)$ for almost all $d \in D$. Similarly, if $d(q, r) = b$ then $d(p, r) = d(p, q) + b$ for almost all $d \in D$. This proves the first half of the theorem.

In the other direction, suppose that $d(p, r) = d(p, q) + d(q, r)$ and $d(p, q) = a > 0$, for almost all $d \in D$. Then $F_{pq} = \varepsilon_a$ and

$$F_{pr}(x) = F_{qr}(x - a) = \tau_{T_m}(F_{pq}, F_{qr})(x) ,$$

whence $W(pqr)$; and similarly if $d(q, r) = b > 0$.

COROLLARY 2.2. *In a pseudo-metrically generated PM space Wald-betweenness, with respect to τ_{T_m}, satisfies all the properties of metric betweenness.*

Proof. Since τ_{T_m} is continuous and has no nontrivial idempotents, only (B3b) needs verification. But this is immediate.

To illustrate Theorem 2.2, let L be the set of all Lebesgue measurable functions on [0, 1]. For any $t \in [0, 1]$ and any f, $g \in L$, let d_t be the pseudo-metric on L defined by

$$d_t(f, g) = |f(t) - g(t)| . \tag{2.14}$$

[1] The proof, which is rather lengthy, is given in [**12**; Theorem 7 and Corollary].

Let $D = \{d_t \mid t \in [0, 1]\}$, let μ be the measure on D induced by Lebesgue measure λ on $[0, 1]$ and let $\mathscr{F}$ be defined via

$$(2.15) \qquad F_{fg}(x) = \mu\{d_t \mid d_t(f, g) < x\} = \lambda\{t \mid |f(t) - g(t)| < x\} .$$

Then $(L, \mathscr{F})$ is a pseudo-metrically generated space—and it is easy to show that T_m is the strongest t-norm under which it is a Menger space. Furthermore, for any distinct f, g, h in L, g is Wald-between f and h if and only if the graph of g lies between the graph of f and the graph of h almost everywhere and either $|f(t) - g(t)| = a > 0$ almost everywhere or $|g(t) - h(t)| = b > 0$ almost everywhere. In particular, if $f(t) = h(t)$ on a set of positive measure then there is no g such that $W(fgh)$. It follows that in L Wald-betweenness is considerably stronger than betweenness with respect to either the L_∞ (ess sup) metric or the L_1 metric (L_1 betweenness is simply pointwise almost-everywhere betweenness). However, for $1 < p < \infty$, Wald-betweenness is not comparable to betweenness with respect to the L_p metric.

The space in the above example is an E-space [22]; and since any E-space is a pseudo-metrically generated space, the above discussion generalizes at once to yield:

COROLLARY 2.3. *Let $(S, \mathscr{F})$ be an E-space, of mappings from the probability space $(\Omega, \mathscr{A}, P)$ into the metric space (M, d). Let p, q, r be distinct elements of S. Then $W(pqr)$ if and only if $q(t)$ is between $p(t)$ and $r(t)$ in (M, d) for almost all $t \in \Omega$ and either $d(p(t), q(t)) = a > 0$ for almost all $t \in \Omega$ or $d(q(t), r(t)) = b > 0$ for almost all $t \in \Omega$.*

3. **Envelope-betweenness.** It is desirable to have a betweenness relation which satisfies (B1)-(B4) even when the triangle function τ is not strictly increasing. When τ is of the form τ_T, for some Archimedean t-norm T, such a relation exists. In order to define and study it, we need some of the elements of the theory of the conjugate transform for τ_T-semigroups.[2] This is the analog of the Laplace transform for the convolution semigroup $(\Delta^+, *)$.

Throughout the rest of this paper, unless explicitly state otherwise, T will denote an Archimedean t-norm, h a fixed multiplicative generator of T, and $h^{[-1]}$ the pseudo-inverse of h.

[2] The conjugate transform was first defined by W. Fenchel [**5**] and later, independently, by R. Bellman and W. Karush [**1, 2**] who also developed many of its properties. Their results apply directly to the semigroup $(\Delta^+, \tau_{\text{Prod}})$. The development of the theory of this transform, its inverse transform, limit theorems, etc., for the semigroups (Δ^+, τ_T), when T is an arbitrary Archimedean t-norm, is the central topic of [**10**]. The details are given in [**13**], [**14**], and [**15**].

DEFINITION 3.1. The *T-conjugate transform for the semigroup* (Δ^+, τ_T) is the mapping C_T defined for any $F \in \Delta^+$ via:

$$C_T F(z) = \sup_{x \geqq 0} e^{-xz} hF(x) , \qquad \text{for all } z \geqq 0 , \tag{3.1}$$

where $hF \in \Delta^+$ is given by

$$hF(x) = \begin{cases} 0 , & x \leqq 0 , \\ h(F(x)) , & 0 < x . \end{cases} \tag{3.2}$$

T-conjugate transforms are completely characterized by the following:

THEOREM 3.1. *Let* $A_T = \{C_T F \mid F \in \Delta^+\}$. *Then*

$$\mathscr{A}_T = \{\phi\colon [0, \infty) \longrightarrow [h(0), 1] \mid \phi \textit{ is nonincreasing, positive, continuous and log-convex}\} \cup \{\theta_T\} , \tag{3.3}$$

where $\theta_T(z) = h(0)$ *for all* $z \geqq 0$.

DEFINITION 3.2. (i) C_T^* is the mapping defined for any $\phi \in \mathscr{A}_T$ via

$$C_T^* \phi(x) = h^{[-1]}(\inf_{z \geqq 0} e^{xz} \phi(z)) , \qquad \text{for all } x , \tag{3.4}$$

and where, in addition, $C_T^* \phi$ is normalized so as to be left-continuous.

(ii) $F \in \Delta^+$ is *T-log-concave* if $\log (hF)$ is concave on (b_F, ∞), where

$$b_F = \sup \{x \mid hF(x) = 0\} . \tag{3.5}$$

Furthermore,

$$\Delta_*^+ = \{F \in \Delta^+ \mid F \text{ is } T\text{-log-concave}\} . \tag{3.6}$$

(iii) For any $F \in \Delta^+$, $\overline{hF}$ is the function with the following properties: $\overline{hF}(x) = 0$ for $x \leqq b_F$; on (b_T, ∞) the graph of $\log (\overline{hF})$ is the concave hull of the graph of $\log (hF)$.

(iv) For any $F \in \Delta^+$, the *T-log-concave envelope of F* is the function F_T in Δ_T^+ given by

$$F_T = h^{[-1]}(\overline{hF}) . \tag{3.7}$$

REMARK. The conjugate transform C_T defined by (3.1) clearly depends on the choice of multiplicative generator h. However, any other multiplicative generator of T is of the form h^λ, for some $\lambda > 0$. From this it follows that T-log-concavity is independent of the particular choice of multiplicative generator, whence Δ_T^+ is completely determined by T alone. Similarly, T-log-concave envelopes depend only on T. Furthermore, if C'_T is the conjugate transform determined by h^λ, for some $\lambda > 0$, then for any $F \in \Delta^+$,

$$C'_T F(z) = [C_T F(z/\lambda)]^\lambda ,$$

whence $\log C'_T F(z) = \lambda \log C_T F(z/\lambda)$ so that the transforms determined by distinct multiplicative generators of the same t-norm are essentially equivalent.

The elements of the theory of the T-conjugate transform which will be needed in the sequel are listed in:

THEOREM 3.2. *For any $F, G, H \in \Delta^+$ and any $\phi, \theta \in A_T$, we have:*

(C 1) $C_T\tau_T(F, G)(z) = \max [h(0), C_T F(z)\cdot C_T G(z)]$, *for all* $z \geqq 0$. *Thus, if T is strict,* $C_T\tau_T(F, G) = C_T F\cdot C_T G$.

(C 2) $C_T\colon \Delta_T^+ \to \mathscr{A}_T$ *is one-one, onto, with inverse* C_T^*.

(C 3) *If* $F \geqq G$ *then* $C_T F \geqq C_T G$.

(C 4) *If* $\phi \geqq \theta$ *then* $C_T^*\phi \geqq C_T^*\theta$.

(C 5) $F_T \geqq F$.

(C 6) *If* $F \geqq G$ *then* $F_T \geqq G_T$.

(C 7) $C_T F_T = C_T F$.

(C 8) $C_T^* C_T F = F_T$.

(C 9) *If* $F \in \Delta_T^+$ *then* $F_T = F$.

(C10) $\tau_T(F_T, G_T)$ *is T-log-concave if and only if* $C_T(\tau_T(F, G)) = C_T F\cdot C_T G$.

(C11) $C_T^*(C_T F\cdot C_T G) = \tau_T(F_T, G_T)$.

(C12) $(\tau_T(F, G))_T \geqq \tau_T(F_T, G_T)$, *with equality if T is strict.*

(C13) *If* $\tau_T(F_T, G_T) = \tau_T(F_T, H_T) \neq \varepsilon_\infty$ *and is T-log-concave then* $G_T = H_T$.

(C14) *If T is strict then* (Δ_T^+, τ_T) *is a subsemigroup in which the cancellation law holds.*

(C15) *If, for some* $a > 0$, $G(x) = F(ax)$, *for all* x, *then* $C_T G(z) = C_T F(z/a)$, *for all* $z \geqq 0$.

(C16) *If* $F_n \xrightarrow{w} F$ *then* $C_T F_n(z) \to C_T F(z)$, *for all* $z > 0$.

(C17) *For any* $z > 0$, $C_T F_n(z) \to 1$ *if and only if* $F_n \xrightarrow{w} \varepsilon_0$.

(C18) *For any* $z > 0$, $C_T F(z) = 1$ *if and only if* $F = \varepsilon_0$.

(C19) $F_T = \varepsilon_0$ *if and only if* $F = \varepsilon_0$.

(C20) $F_T = \varepsilon_\infty$ *if and only if* $F = \varepsilon_\infty$.

Perusal of the above shows that the essential properties of T-conjugate transforms, as well as their usefullness as analytical tools, are independent of the particular choice of multiplicative generator in (3.1).

DEFINITION 3.3. Let $(S, \mathscr{F}, \tau_T)$ be a PM space where T is an Archimedean t-norm. Let p, q, r be distinct points of S. Then q is *envelope-between* p and r—and we write $E(pqr)$—if $F_{pr} \neq \varepsilon_\infty$ and

$$(F_{pr})_T = \tau_T((F_{pq})_T, (F_{qr})_T) , \tag{3.8}$$

where, for any $F \in \Delta^+$, F_T is the T-log-concave-envelope of F.

Again, $E(pqr)$ implies that F_{pr}, F_{pq}, and F_{qr} are all different from both ε_∞ and ε_0.

THEOREM 3.3. *Let $(S, \mathscr{F}, \tau_T)$ be a PM space, where T is an Archimedean t-norm. Then the betweenness relation $E(pqr)$ satisfies* (B1)-(B4).

Proof. To simplify the notation, we will denote the T-log-concave envelope F_T of any $F \in \Delta^+$ by $\bar{F}$ and τ_T, C_T by τ and C, respectively.

Again (B1) is trivial. To prove (B2) we merely replace the distribution functions in (2.6) by their T-log-concave envelopes. As noted previously, Corollary 2.1 applies to (Δ^+, τ_T) when T is Archimedean and (C19), (C20) imply that none of $\bar{F}_{pq}$, $\bar{F}_{qr}$, $\bar{F}_{pr}$ is equal to either ε_0 or ε_∞.

To establish (B3) suppose $E(pqr)$ and $E(prs)$ hold, so that

$$\bar{F}_{pr} = \tau(\bar{F}_{pq}, \bar{F}_{pr}) \quad \text{and} \quad \bar{F}_{ps} = \tau(\bar{F}_{pr}, \bar{F}_{rs}) .$$

Note that, by (III), (C6) and (C12), we have

$$\bar{F}_{qs} \geqq \overline{\tau(F_{qr}, F_{rs})} \geqq \tau(\bar{F}_{qr}, \bar{F}_{rs}) \tag{3.9}$$

and, similarly, $\bar{F}_{ps} \geqq \tau(\bar{F}_{pq}, \bar{F}_{qs})$. Hence, replacing the distribution functions in (2.5) by their T-log-concave envelopes yields $E(pqs)$. Next, using this fact, (C7) and (C10), we have

$$\begin{aligned} CF_{pq} \cdot CF_{qs} &= C\tau(\bar{F}_{pq}, \bar{F}_{qs}) = C\bar{F}_{ps} \\ &= C\tau(\bar{F}_{pr}, \bar{F}_{rs}) = CF_{pr} \cdot CF_{rs} \\ &= C\tau(\bar{F}_{pq}, \bar{F}_{qr}) \cdot CF_{rs} = CF_{pq} \cdot CF_{qr} \cdot CF_{rs} . \end{aligned} \tag{3.10}$$

Since $F_{pq} \neq \varepsilon_\infty$, it follows from (3.1) that $CF_{pq}(z) > 0$ for all $z \geqq 0$. Thus, cancelling CF_{pq} in (3.10) yields:

$$CF_{qs} = CF_{qr} \cdot CF_{rs} ,$$

whence, by (C8) and (C11), we have

$$\bar{F}_{qs} = C^* CF_{qs} = C^*(CF_{qr} \cdot CF_{rs}) = \tau(\bar{F}_{qr}, \bar{F}_{rs})$$

and $E(qrs)$.

The proof of (B4) follows as in Theorem 2.1 (iv), using the facts that:

(i) τ_T is continuous with respect to weak convergence on Δ^+;

(ii) taking T-log-concave envelopes is a continuous operation, i.e., if $F_{q_0 q_n} \xrightarrow{w} \varepsilon_0$ then $\bar{F}_{q_0 q_n} \xrightarrow{w} \bar{\varepsilon}_0 = \varepsilon_0$; and

(iii) $\bar{F}_{pr} \geqq \tau(\bar{F}_{pq}, \bar{F}_{qr})$ for any $p, q, r \in S$. This completes the proof.

COROLLARY 3.1. *Let $\mathscr{F}_T$ be the mapping defined on $S \times S$ by $\mathscr{F}_T(p, q) = (\mathscr{F}(p, q))_T$. Then $(S, \mathscr{F}_T, \tau_T)$ is also a PM space. Furthermore, in $(S, \mathscr{F}_T, \tau_T)$ Wald-betweenness and envelope-betweenness coincide.*

Note that Wald-betweenness and envelope-betweenness coincide in any PM space $(S, \mathscr{F}, \tau_T)$ for which $\mathrm{Ran}\,\mathscr{F} \subseteqq \Delta_T^+$.

For strict t-norms, Wald-betweenness is stronger than envelope-betweenness, for we have:

THEOREM 3.4. *Let $(S, \mathscr{F}, \tau_T)$ be a PM space, where T is a strict t-norm. Then, for any distinct p, q, r in S, $W(pqr)$ implies $E(pqr)$.*

Proof. Suppose $W(pqr)$ so that $F_{pr} = \tau_T(F_{pq}, F_{qr})$. Then, using (C12), we have

$$(F_{pr})_T = (\tau_T(F_{pq}, F_{qr}))_T = \tau_T((F_{pq})_T, (F_{qr})_T) .$$

In PM spaces $(S, \mathscr{F}, \tau_T)$, where T is Archimedean but not strict, the relations $W(pqr)$ and $E(pqr)$ are generally not comparable.

4. Conjugate-metric-betweenness.

In [6] J. F. C. Kingman showed that in a Wald space the function d defined on $S \times S$ by

$$d(p, q) = -\log\left[\int_0^\infty e^{-x} dF_{pq}(x)\right] \tag{4.1}$$

is a metric on S; that d-metric beteenness and Wald-betweenness are equivalent; and that the d-metric topology and the ε, λ-topology are also equivalent.

The right-hand side of (4.1) is just the negative of the logarithm of the Laplace transform of F_{pq} evaluated at 1; and since the T-conjugate transform is related to the semigroup (Δ^+, τ_T) just as the Laplace transform is related to $(\Delta^+, *)$, we are led to the following:

THEOREM 4.1. *Let $(S, \mathscr{F}, \tau_T)$ be a PM space, with T Archimedean, and such that $F_{pq} \neq \varepsilon_\infty$ for any $p, q \in S$. Let C_T be the T-conjugate transform on (Δ^+, τ_T), as given by (3.1); and for any $z > 0$ let d_z be* *the mapping defined on* $S \times S$ *by*

$$d_z(p, q) = -\log C_T F_{pq}(z) . \tag{4.2}$$

Then d_z *is a metric on* S. *Furthermore, the* d_z*-metric topology is equivalent to the* ε, λ*-topology.*

Proof.[3] By (3.3), clearly $0 \leqq d_z(p, q) < \infty$ for any $p, q \in S$. If $d_z(p, q) = 0$ then $CF_{pq}(z) = 1$, whence by (C18) $F_{pq} = \varepsilon_0$ and $p = q$. Conversely, if $p = q$ then $d_z(p, q) = 0$.

Since $F_{pq} = F_{qp}$ we have $d_z(p, q) = d_z(q, p)$.

For any $p, q, r \in S$, using (III), (C3), and (C1), we have

$$CF_{pr}(z) \geqq C(\tau(F_{pq}, F_{qr}))(z) \geqq CF_{pq}(z) \cdot CF_{qr}(z) ,$$

whence, using (4.2), $d_z(p, r) \leqq d_z(p, q) + d_z(q, r)$. Thus d_z is a metric on S.

Next let $\{p_n\}$ be a sequence in S and let $p \in S$. Then, by (4.2) and (C17), the following are equivalent:

(i) $d_z(p_n, p) \to 0$;
(ii) $CF_{p_n p}(z) \to 1$;
(iii) $F_{p_n p} \xrightarrow{w} \varepsilon_0$;
(iv) $p_n \to p$ in the ε, λ-topology.

This completes the proof.

For any $z > 0$ the function d_z defined by (4.2) will be called a *conjugate-metric* on S. Clearly, any two conjugate metrics induce the same topology on S. But more is true:

THEOREM 4.2. *Under the hypotheses of Theorem* 4.1, *if* $z \geqq w > 0$ *then, for any* $p, q \in S$,

$$d_w(p, q) \leqq d_z(p, q) \leqq \frac{z}{w} d_w(p, q) . \tag{4.3}$$

Proof. The first inequality in (4.3) follows immediately from the fact that CF_{pq} is nonincreasing. Next, since the function $f(y) = -\log CF_{pq}(y)$ is concave and nonnegative on $[0, \infty)$, we have

$$(f(w) - f(0))/w \geqq (f(z) - f(0))/z .$$

Thus $zf(w) \geqq wf(z) + (z - w)f(0) \geqq wf(z)$, which yields the second inequality in (4.3) and completes the proof.

It follows from (4.3) that for any $z, w > 0$, the conjugate metrics d_z, d_w are equivalent.

[3] As in the proof of Theorem 3.3, in the proofs given in this section we suppress reference to the subscript T, and denote F_T by $\bar{F}$.

DEFINITION 4.1. Suppose the hypotheses of Theorem 4.1 are satisfied and let p, q, r be three distinct points of S. Then *q is between p and r with respect to the conjugate metric d_z*—and we write $M_z(pqr)$—if $d_z(p, r) = d_z(p, q) + d_z(q, r)$.

Since $M_z(pqr)$ is a metric betweenness relation, it is immediate that (B1)-(B4) are satisfied. There is also a connection between the relations $\{M_z(pqr) \mid z > 0\}$ and envelope-betweenness $E(pqr)$.

THEOREM 4.3. *Let $(S, \mathscr{F}, \tau_T)$ be a PM space, with T Archimedean. Then, for any distinct $p, q, r \in S$, $E(pqr)$ if and only if $M_z(pqr)$ for all $z > 0$.*

Proof. Suppose $E(pqr)$, so that $\bar{F}_{pr} = \tau(\bar{F}_{pq}, \bar{F}_{qr})$. Then, by (C7) and (C10), we have

$$\begin{aligned} CF_{pr} = C\bar{F}_{pr} &= C\tau(\bar{F}_{pq}, \bar{F}_{qr}) \\ &= C\bar{F}_{pq} \cdot C\bar{F}_{qr} = CF_{pq} \cdot CF_{qr} , \end{aligned}$$

whence it follows from (4.2) and Definition 4.1 that $M_z(pqr)$ for all $z > 0$.

Conversely, if $M_z(pqr)$ holds for all $z > 0$, then for all $z > 0$ we have

$$CF_{pr}(z) = CF_{pq}(z) \cdot CF_{qr}(z) , \tag{4.4}$$

and the continuity of the conjugate transform on $[0, \infty)$ yields (4.4) at $z = 0$, and hence for all $z \geqq 0$. Thus, using (C8) and (C11), we have

$$\bar{F}_{pr} = C^*CF_{pr} = C^*(CF_{pq} \cdot CF_{qr}) = \tau(\bar{F}_{pq}, \bar{F}_{qr}) ,$$

whence $E(pqr)$.

COROLLARY 4.1. *If T is strict then, for any distinct $p, q, r \in S$, $W(pqr)$ implies $M_z(pqr)$ for all $z > 0$.*

In the case of convolution the betweenness relations obtained via (4.1) are independent of the particular point at which the Laplace transform is evaluated. In contrast, there are virtually no connections among the conjugate metric betweenness relations $\{M_z(pqr)\}$ for different values of z. This is brought out by the following example:

EXAMPLE 4.1. Let $S = \{p, q, r\}$ and let $\tau = \tau_{\text{Prod}}$. Let $\phi(z) = \exp(-\sqrt{z})$, for $z \geqq 0$. Then ϕ is a Prod-conjugate transform. Let

$F_{pq} = F_{qr} = C^*\phi$. To define F_{pr} we proceed as follows: Choose a fixed $w > 0$ and let

$$f(z) = \begin{cases} (-2/\sqrt{w})z\ , & 0 \leqq z \leqq w\ , \\ -2\sqrt{w}\ , & w \leqq z\ . \end{cases}$$

Thus f is linear between the points (0, 0) and $(w, -2\sqrt{w})$ and constant thereafter. Note also that the points (0, 0) and $(w, -2\sqrt{w})$ lie on the graph of

$$-2\sqrt{z} = \log(\phi^2(z)) = \log(CF_{pq}(z)\cdot CF_{qr}(z))\ .$$

In particular, $f(z) \geqq -2\sqrt{z}$ and, since f is convex on $[0, \infty)$, the function $\theta(z) = \exp(f(z))$ is also a Prod-conjugate transform. Let $F_{pr} = C^*\theta$. Then we have

$$CF_{pr}(z) = \theta(z) \geqq CF_{pq}(z)\cdot CF_{qr}(z)\ , \qquad \text{for all } z \geqq 0. \tag{4.5}$$

Using (C11), it follows that

$$F_{pr} = C^*\theta \geqq C^*(CF_{pq}\cdot CF_{qr}) = \tau(F_{pq}, F_{qr})\ ,$$

which, together with the obvious inequalities $F_{pq} = \tau(\varepsilon_0, F_{qr}) \geqq \tau(F_{pr}, F_{rq})$ and $F_{qr} \geqq \tau(F_{qp}, F_{pr})$, yields that S is a PM space under τ_{Prod}. However, by construction, equality in (4.5) holds only when $z = 0$ or $z = w$. Thus $M_w(pqr)$ holds while $M_z(pqr)$ fails for any other $z > 0$.

The above example can obviously be modified so that $M_z(pqr)$ holds for any z in the finite set of positive numbers $\{w_1, \cdots, w_n\}$ and fails otherwise. Similarly, it can be carried over from Prod to any Archimedean t-norm.

We conclude this discussion with a simple illustrative example. Let $(S, \mathscr{F})$ be the *α-simple space* generated by the metric space (S, d) and the distribution function G, where $G \in \mathscr{D}^+$ is continuous and strictly increasing on $[0, \infty)$. Thus, for distinct p, $q \in S$,

$$F_{pq}(x) = G(x/d^\alpha(p, q))\ . \tag{4.6}$$

If $\alpha \leqq 1$ then d^α is also a metric on S and $(S, \mathscr{F})$ is the simple space generated by the metric space (S, d^α) and G; in this case remark (ii) after Theorem 2.1 applies directly. Suppose therefore that $\alpha > 1$. Then, as shown in [19], $(S, \mathscr{F})$ is a Menger space under τ_T, where T is strict and multiplicatively generated by

$$h(x) = \exp[-(G^{-1}(x))^{1/(1-\alpha)}]\ , \tag{4.7}$$

and G^{-1} is the inverse of the restriction of G to $[0, \infty)$. Thus, for distinct p, $q \in S$,

$$(4.8)\qquad hF_{pq}(x) = \begin{cases} 0\,, & x \leqq 0\,,\\ \exp\left[-(x/(d(p,q))^{\alpha})^{1/(1-\alpha)}\right]\,, & x > 0\,. \end{cases}$$

Evaluating the right-hand side of (3.1) yields that, for $z \geqq 0$,

$$C_T F_{pq}(z) = \exp\left[-\alpha(\alpha-1)^{(1-\alpha)/\alpha} z^{1/\alpha} d(p,q)\right]\,,$$

whence, substituting in (4.2), we find that

$$(4.9)\qquad d_z(p,q) = \alpha(\alpha-1)^{(1-\alpha)/\alpha} z^{1/\alpha} d(p,q)\,.$$

Thus, for each $z > 0$, d_z is a constant multiple of the metric d and consequently, for any distinct points $p, q, r \in S$, we have $M_z(pqr)$ if and only if q is between p and r with respect to the metric d. It follows that in this case the betweenness relation $M_z(pqr)$ is independent of z. Furthermore, since it is clear from (4.8) that each F_{pq} is T-log-concave, we have, finally, that in an α-simple space with $\alpha > 1$, the following are equivalent:

(i) $W(pqr)$;

(ii) $E(pqr)$; and

(iii) q is between p and r with respect to the metric d.

5. Menger-betweenness. Let T be a t-norm and let $(S, \mathscr{F}, \tau_T)$ be a PM space. In [8] Menger postulated that q lies between p and r if these three points are distinct and if, for all x, y,

$$(5.1)\qquad 1 - F_{pr}(x+y) \geqq T(1 - F_{pq}(x), 1 - F_{qr}(y))\,.$$

The probabilistic interpretation of (5.1) is that, for all x, y,

$$(5.2)\qquad \begin{aligned} &\text{Prob}\,[\text{dist}\,(p,r) \geqq x+y] \\ &\quad \geqq T(\text{Prob}\,[\text{dist}\,(p,q) \geqq x], \text{Prob}\,[\text{dist}\,(q,r) \geqq y])\,. \end{aligned}$$

The condition (5.1) may be restated in a more perspicacious manner. First of all, let T^* be the *t-conorm* of T, i.e., the function defined for all a, b in $[0, 1]$ by

$$(5.3)\qquad T^*(a,b) = 1 - T(1-a, 1-b)\,.$$

Then (5.1) reads:

$$(5.4)\qquad F_{pr}(x+y) \leqq T^*(F_{pq}(x)\cdot F_{qr}(x))\,.$$

Next, if T^* is continuous then the binary operation τ_{T^*} defined on Δ^+ via

$$(5.5)\qquad \tau_{T^*}(F,G)(x) = \inf_{u+v=x} T^*(F(u), G(v))\,,$$

is a continuous triangle function [17]. Since (5.4) holds for all x, y, it follows that q is *between* p and r *in Menger's sense*—and we write $M(pqr)$—if these three points are distinct and if

$$F_{pr} \leqq \tau_{T^*}(F_{pq}, F_{qr}) . \tag{5.6}$$

This is the desired reformulation of (5.1). It states that $M(pqr)$ if and only if $\tau_{T^*}(F_{pq}, F_{qr})$ is an upper bound for F_{pr}. Since

$$\tau_T(F, G) \leqq \tau_{T^*}(F, G) , \tag{5.7}$$

for any $F, G \in \Delta^+$ and any t-norm T, the condition (5.6) is consistent with the triangle inequality, which states that $\tau_T(F_{pq}, F_{qr})$ is a lower bound for F_{pr}.

It follows at once from (5.7) that $W(pqr)$ implies $M(pqr)$, i.e., that Wald-betweenness implies Menger-betweenness. If $T = \text{Min}$ then $T^* = \text{Max}$ and a simple calculation shows that $\tau_{\text{Min}} = \tau_{\text{Max}}$. Thus in PM spaces under τ_{Min}, e.g., in simple spaces the relations $W(pqr)$ and $M(pqr)$ coincide.

In Wald spaces the probabilistic distances between points are generally assumed to be given by independent random variables. Thus, for any three points p, q, r, the triangle inequality states that, for all x,

$$\begin{aligned} F_{pr}(x) &= \text{Prob}\,[\text{dist}\,(p, r) < x] \\ &\geqq \text{Prob}\,[\text{dist}\,(p, q) + \text{dist}\,(q, r) < x] = (F_{pq} * F_{qr})(x) . \end{aligned}$$

Similarly, in this context the analog of (5.2) is

$$\begin{aligned} 1 - F_{pr}(x) &= \text{Prob}\,[\text{dist}\,(p, r) \geqq x] \\ &\geqq \text{Prob}\,[\text{dist}\,(p, q) + \text{dist}\,(q, r) \geqq x] = 1 - (F_{pq} * F_{qr})(x) , \end{aligned}$$

for all x, i.e.,

$$F_{pr} \leqq F_{pq} * F_{qr} , \tag{5.8}$$

whence, in view of the triangle inequality, $F_{pr} = F_{pq} * F_{qr}$. Thus, in a Wald space, the relations $W(pqr)$ and $M(pqr)$ also coincide.

Generally, however, equality in (5.7) holds only under very restrictive circumstances. It fails, for example, for any t-norm T for which $T(a, b) < \text{Min}\,(a, b)$ for all $a, b \in (0, 1)$, and thus for any Archimedean t-norm. When this is the case, Menger's betweenness restricts F_{pr} to a certain interval in Δ^+. To gain some insight into this situation, we consider several examples.

Let $(S, \mathscr{F})$ be a pseudo-metrically generated PM space (see § 2). Since T_m is the strongest t-norm for this class of spaces and since $T_m^*(a, b) = \text{Min}\,(a + b, 1)$, (5.6) becomes

$$F_{pr}(x) \leqq \inf_{u+v=x} \text{Min}\,(F_{pq}(u) + F_{qr}(v), 1) \tag{5.9}$$

and we have the following:

THEOREM 5.1. *If $(S, \mathscr{F})$ is pseudo-metrically generated and if $p, q, r \in S$ are such that q is between p and r for almost all metrics $d \in D$, then q is Menger-between p and r,* i.e., *$M(pqr)$ holds.*

Proof. Let $x > 0$ be given and choose $u, v \geqq 0$ such that $u + v = x$. If $d(p, r) = d(p, q) + d(q, r)$ and $d(p, r) < x$ then either $d(p, q) < u$ or $d(q, r) < v$. Consequently,

$$\begin{aligned} F_{pr}(x) &= \mu\{d \mid d(p, r) < x\} \leqq \mu\{d \mid d(p, q) < u\} + \mu\{d \mid d(q, r) < v\} \\ &= F_{pq}(u) + F_{qr}(v) , \end{aligned}$$

and the theorem follows.

The converse is false. To see this, consider again the space $(L, \mathscr{F})$, where L is the set of Lebesgue measurable functions on $[0, 1]$ and $\mathscr{F}$ is given by (2.15). Let $f(x) = x$ and $h(x) = 0$, for $x \in [0, 1]$; and let $g(x)$ be given by

$$g(x) = \begin{cases} 1/8 , & 0 \leqq x \leqq 1/8 , \\ x , & 1/8 \leqq x \leqq 1/2 , \\ 0 , & 1/2 < x \leqq 1 . \end{cases}$$

Since $g(x) > f(x) + h(x)$ for $0 \leqq x < 1/8$, $g(x)$ is not between $f(x)$ and $h(x)$ for almost all $x \in [0, 1]$. Nevertheless, a straightforward computation shows that $M(fgh)$ holds.

Comparing the above with the known properties of betweenness with respect to the usual L_p metrics on L, we find that here Menger-betweenness is strictly weaker than betweenness in any L_p-metric for $1 \leqq p < \infty$ and not comparable to betweenness in the L_∞ metric; and comparing with the results of § 2 shows that Menger-betweenness is a much weaker relation than Wald-betweenness.

As a final example, consider the α-simple space generated by (S, d) and the strict distribution function G. In this case, using (4.6) and (4.7), some calculation yields that $M(pqr)$ is equivalent to the inequality

$$(5.10) \quad d^\alpha(p, q)H(u) + d^\alpha(q, r)H(v) \leqq d^\alpha(p, r)H(u + v), \text{ for all } u, v \geqq 0 ,$$

where H is the strictly increasing function from $\boldsymbol{R}^+$ to $\boldsymbol{R}^+$ given by

$$(5.11) \qquad H(x) = G^{-1}[1 - G(1/x^{\alpha-1})] .$$

If we choose $\alpha = 2$ and let G be a strict distribution satisfying

$1 - G(x) = G(1/x)$ (for example, $G(x) = x/2$ for $0 \leqq x \leqq 1$ and $G(x) = 1 - 1/2x$ for $x \geqq 1$) then $H(x) = x$ and (5.10) reduces to

$$(5.12) \quad d^2(p, q)u + d^2(q, r)v \leqq d^2(p, r)(u + v) , \qquad \text{for all } u,\ v \geqq 0 .$$

The inequality (5.12) holds if and only if $d(p, r) \geqq \operatorname{Max}(d(p, q), d(q, r))$. In particular, when (S, d) is the Euclidean plane, the set of all points q between two given points p and r is the closed convex region bounded by two circular arcs of radius $d(p, r)$, one with center at p, the other with center at r. Thus, if p, q, r are vertices of an equilateral triangle then $M(pqr)$, $M(rpq)$ and $M(qrp)$ all hold.

Note that since $H(0) = 0$, setting, respectively, $u = 0$ and $v = 0$ in (5.10) yields that $d(p, r) \geqq \operatorname{Max}(d(p, q), d(q, r))$ is a necessary condition for $M(pqr)$. In our particular example—and indeed, whenever $H(u) + H(v) \leqq H(u + v)$—it is also sufficient.

One might conjecture that, in general, $\{q \mid M(pqr)\}$ is a "convex" set having p and r on its boundary. In any event, Menger-betweenness is a relation which merits further study.

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