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GENERALIZATIONS OF THE ROBERTSON FUNCTIONS

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We study a class of analytic functions which unifies a number of classes previously studied, including functions with boundary rotation at most $k\pi$, functions convex of order ρ and the Robertson functions, i.e., functions f for which zf' is α-spirallike. We obtain representation theorems for this general class, and using a simple variational formula, also obtain sharp bounds on the modulus of the second coefficient of the series expansion of these functions. Using a univalence criterion due to Ahlfors, we determine a condition on the parameters k, α, and ρ which will ensure that a function in this class is univalent. This result improves previously published results for various subclasses and is sharp for the class of functions f for which zf' is α-spirallike of order ρ.

1. Let $P_\alpha^k(\rho)$ denote the class of regular functions $p(z)$ in $E = \{z: |z| < 1\}$ such that $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\text{Re}\,\{e^{i\alpha}p(z)\text{-}\rho\cos\alpha\}}{1-\rho} \right| d\theta \leqq k\pi\cos\alpha\ ,$$

$k \geqq 2$, $0 \leqq \rho < 1$, α real, $|\alpha| < \pi/2$, $z = \text{r}e^{i\theta}$, $0 \leqq r < 1$.

Let $V_\alpha^k(\rho)$ denote the class of functions regular in E with $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \in P_\alpha^k(\rho)\ ,$$

k, α, and ρ as above. $V_0^k(0)$ is the class of functions with bounded boundary rotation. $V_\alpha^k(0)$ is a generalization of this class which has been studied recently ([7] and [13]). Padmanabhan and Parvatham [9] have studied properties of $V_0^k(\rho)$. In this paper we study properties of $V_\alpha^k(\rho)$ which unlike $V_0^k(\rho)$ contains functions whose boundary rotation is not necessarily bounded. A function f belongs to $V_\alpha^2(\rho)$ if and only if

$$\text{Re}\left\{e^{i\alpha}\left[\frac{1 + zf''(z)}{f'(z)}\right]\right\} > \rho\cos\alpha\ ,$$

ρ and α as above. When $\rho = 0$, we obtain the class of functions $f(z)$ for which $zf'(z)$ is α-spirallike, which has been studied by M.S. Robertson [10], Libera and Ziegler [6], Bajpai and Mehrok [2], and Kulshrestha [5]. The case when $k = 2$ but ρ and α are not zero has been studied by Chichra [4] who denoted the class F_α^ρ. This class also has been studied by Sizuk [12], who has called $zf'(z)$ α-spiral-shaped of order ρ. The class $V_0^2(\rho)$ is the class of functions which are convex of order ρ, introduced by M. S. Robertson in 1936.

LEMMA 1. *If $p(z) \in P_\alpha^k(\rho)$, then*

$$e^{i\alpha}p(z) = \frac{\cos\alpha}{2\pi}\int_0^{2\pi}\frac{1+(1-2\rho)ze^{i\theta}}{2-ze^{i\theta}}d\psi(\theta) + i\sin\alpha\ , \tag{1.1}$$

where $\psi(\theta)$ is a function with bounded variation in $[0, 2\pi]$ *satisfying*

$$\int_0^{2\pi} d\psi(\theta) = 2\pi \quad and \quad \int_0^{2\pi}|d\psi(\theta)| \leqq k\pi\ . \tag{1.2}$$

Proof. Let

$$g(z) = \frac{e^{i\alpha}p(z) - \rho\cos\alpha - i\sin\alpha}{(1-\rho)\cos\alpha}\ ,$$

and let

$$u(z) = \operatorname{Re}\{g(z)\} = \operatorname{Re}\left\{\frac{e^{i\alpha}p(z) - \rho\cos\alpha}{(1-\rho)\cos\alpha}\right\}\ .$$

Then since $p(z) \in P_\alpha^k(\rho)$, $\int_0^{2\pi}|u(re^{i\theta})|d\theta \leqq k\pi$, and according to a representation theorem due to Paatero [8],

$$\frac{e^{i\alpha}p(z) - \rho\cos\alpha - i\sin\alpha}{(1-\rho)\cos\alpha} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{i\theta}}{1-ze^{i\theta}}d\psi(\theta)\ ,$$

where $\psi(\theta)$ has bounded variation and satisfies condition (1.2) above. The conclusion of the lemma follows.

Now let $f(z) \in V_\alpha^k(\rho)$. By a theorem due to Padmanabhan and Parvatham [9], the integral in (1.1)

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1+(1-2\rho)ze^{i\theta}}{1-ze^{i\theta}}d\psi(\theta) = 1 + zf_0''(z)/f_0'(z)\ ,$$

for some f_0 in $V_0^k(\rho)$. So

$$e^{i\alpha}\left[1+\frac{zf''(z)}{f'(z)}\right] = \cos\alpha\left[1+\frac{zf_0''(z)}{f_0'(z)}\right] + i\sin\alpha\ .$$

$$\frac{f''(z)}{f'(z)} = e^{i\alpha}\cos\alpha\left[\frac{1}{z}+\frac{f_0''(z)}{f_0'(z)}\right] + i\frac{e^{-i\alpha}\sin\alpha - 1}{z}\ .$$

Integrating, we obtain

LEMMA 2. *$f(z)$ is in $V_\alpha^k(\rho)$ if and only if there is a function $f_0(z)$ in $V_0^k(\rho)$ such that*

$$f'(z) = [f_0'(z)]^{e^{-i\alpha}\cos\alpha} .$$

The function $f_0(z)$ in $V_0^k(\rho)$ has associated with it a function $g_0(z)$ in $V_0^k(0)$. ([9], *Lemma* 2.)

LEMMA 3. *$f(z)$ is in $V_\alpha^k(\rho)$ if and only if there is a function $g_0(z)$ in $V_0^k(0)$ such that*

$$f'(z) = [g_0'(z)]^{(1-\rho)e^{-i\alpha}\cos\alpha} .$$

LEMMA 4. *$f(z)$ is in $V_\alpha^k(\rho)$ if and only if there exists a function $g(z)$ in $V_\alpha^k(0)$ such that*

$$f'(z) = [g'(z)]^{(1-\rho)} .$$

Proof. The function $[g_0'(z)]^{e^{-i\alpha}\cos\alpha}$ determines a function $g_\alpha'(z)$, where $g_\alpha(z)$ is in $V_\alpha^k(0)$ [7].

From Paatero's representation theorem for functions with bounded variation [8], we obtain the following representation.

THEOREM 1. *$f(z)$ is in $V_\alpha^k(\rho)$ if and only if there exists a function $\psi(\theta)$ with bounded variation on $[0, 2\pi]$ satisfying condition* (1.2) *and*

$$f'(z) = \exp\left\{\frac{-(1-\rho)e^{-i\alpha}\cos\alpha}{\pi}\int_0^{2\pi}\log(1 - ze^{i\theta})d\psi(\theta)\right\} .$$

THEOREM 2. *$f(z)$ is in $V_\alpha^k(\rho)$ if and only if*

(A) *there exist starlike functions S_1, S_2 such that*

$$f'(z) = \left\{\frac{\left[\dfrac{S_1(z)}{z}\right]^{(k+2)/4}}{\left[\dfrac{S_2(z)}{z}\right]^{(k-2)/4}}\right\}^{(1-\rho)e^{-i\alpha}\cos\alpha}$$

(B) *there exist α-spiral functions T_1, T_2 such that*

$$f'(z) = \left\{\frac{\left[\dfrac{T_1(z)}{z}\right]^{(k+2)/4}}{\left[\dfrac{T_2(z)}{z}\right]^{(k-2)/4}}\right\}^{1-\rho} .$$

(C) *there exist functions L_1, L_2 in $V_0^2(0)$ such that*

$$f'(z) = \left\{\frac{[L_1'(z)]^{(k+2)/4}}{[L_2'(z)]^{(k-2)/4}}\right\}^{(1-\rho)e^{-i\alpha}\cos\alpha} .$$

(D) *there exist functions* H_1, H_2 *in* $V_0^2(\rho)$ *such that*

$$f'(z) = \left\{\frac{[H_1'(z)]^{(k+2)/4}}{[H_2'(z)]^{(k-2)/4}}\right\}^{e^{-i\alpha}\cos\alpha} .$$

Proof. (A) follows from Lemma 3 and Brannan's representation for functions with bounded boundary rotation [3]. (B) follows from (A) since $s(z)$ is starlike if and only if $T(z) = z[s(z)/z]^{e^{-i\alpha}\cos\alpha}$ is α-spirallike. (C) follows from (A) because of the fact that $H(z)$ is convex if and only if $zH'(z) = S(z)$ is starlike. (D) follows trivially from (C).

2. Properties of functions in $V_\alpha^k(\rho)$.

COROLLARY 1. *Suppose* $f(z) = z + a_2z^2 + \cdots$ *is in* $V_\alpha^k(\rho)$. *Then* $|a_2| \leqq k(1-\rho)\cos\alpha/2$, *and this bound is sharp.*

Proof. It is well known that if g_0 is in $V_0^k(0)$, then $|g_0''(0)| \leqq k$, so the result follows directly from Lemma 3. This bound is sharp for the function $f(z)$ in $V_\alpha^k(\rho)$ defined by

$$f'(z) = \left\{\left[\frac{(1-z)^{(k-2)/2}}{(1+z)^{(k+2)/2}}\right]\right\}^{(1-\rho)e^{-i\alpha}\cos\alpha}$$

LEMMA 5. *If* $f(z)$ *is in* $V_\alpha^k(\rho)$, *then* $F(z)$ *defined by*

$$F'(z) = \frac{f'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f'(a)(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha}} , \ F(0) = 0, \ |a| < 1, \ |z| < 1 ,$$

is also in $V_\alpha^k(\rho)$.

Proof. By Lemma 2, for $f(z)$ in $V_\alpha^k(\rho)$, there exists $f_0(z)$ in $V_0^k(\rho)$ such that $f'(z) = [f_0'(z)]^{e^{-i\alpha}\cos\alpha}$. By Lemma 3 in [9],

$$\frac{f_0'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f_0'(a)(1+\bar{a}z)^{2(1-\rho)}} \text{ is the derivative of}$$

a function in $V_0^k(\rho)$. Hence

$$\left[\frac{f_0'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f_0'(a)(1+\bar{a}z)^{2(1-\rho)}}\right]^{e^{-i\alpha}\cos\alpha} = \frac{f'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f'(a)(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha}}$$

is the derivative of a function in $V_\alpha^k(\rho)$.

THEOREM 3. *If* $f(z)$ *is in* $V_\alpha^k(\rho)$ *and* $0 < k(1-\rho)\cos\alpha \leqq 1$, *then* $f(z)$ *is univalent in* $|z| < 1$.

Proof. By the previous lemma, if $f(z)$ is in $V_\alpha^k(\rho)$, then $F(z)$ defined by

$$F'(z) = \frac{f'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f'(a)(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha}}\,, \quad F(0) = 0\,,$$

is in $V_\alpha^k(\rho)$ also, with $|a| < 1$ and $|z| < 1$. Then

$$\begin{aligned} F''(z) = \Big[&(1+az)^{2(1-\rho)e^{-i\alpha}\cos\alpha} f''\left(\frac{z+a}{1+\bar{a}z}\right)\cdot\frac{1-|a|^2}{(1+\bar{a}z)^2} \\ &- 2(1-\rho)e^{-i\alpha}\cos\alpha(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha-1}\bar{a}f'\left(\frac{z+a}{1+\bar{a}z}\right)\Big] \\ &\times [f'(a)(1+\bar{a}z)^{4(1-\rho)e^{-i\alpha}\cos\alpha}]^{-1}\,, \end{aligned}$$

$$F''(0) = \frac{f''(a)}{f'(a)}(1-|a|^2) - 2(1-\rho)e^{-i\alpha}\cos\alpha\ \bar{a}\,.$$

Replacing a by z, using Corollary 1 of Theorem 2, and multiplying through by $|z|$, we have

$$\begin{aligned} &\left|\frac{zf''(z)}{f'(z)}(1-|z|^2) - 2(1-\rho)e^{-i\alpha}\cos\alpha|z|^2\right| \\ &\qquad \leqq k(1-\rho)\cos\alpha|z| < k(1-\rho)\cos\alpha\,. \end{aligned}$$

Ahlfors' univalence criterion [1], with $c = 2(1-\rho)e^{-i\alpha}\cos\alpha$, shows that f is univalent in E when $0 < k(1-\rho)\cos\alpha \leqq 1$.

COROLLARY 1. *If* $f(z)$ *is in* $V_\alpha^k(0)$, f *is univalent in* E *whenever*

(2.1) $$0 < \cos\alpha \leqq 1/k\,.$$

This simplifies and improves bounds previously published for this class [7].

COROLLARY 2. *If* $f(z)$ *is in* $V_0^k(\rho)$, *then* f *is univalent in* E *for*

(2.2) $$\rho \geqq \frac{k-1}{k}\,.$$

Previously, it was shown in [9] *that f is univalent for* $\rho \geqq (k+1)/(k+2)$.

COROLLARY 3. *If $f(z)$ is in $V_\alpha^2(\rho)$, then $f(z)$ is univalent in E when $0 < \cos\alpha \leqq 1/2(1-\rho)$. f need not be univalent if* $\cos\alpha > 1/[2(1-\rho)]$.

Chichra [4] has shown that for each α, $1/[2(1-\rho)] < \cos\alpha < 1$, there exists a function $f(z)$ in $F_\alpha^\rho = V_\alpha^2(\rho)$ such that $f(z)$ is not univalent in E. Hence the problem of univalence in $V_\alpha^2(\rho)$ is solved.

3. We may use the same function f as in [4] to study conditions on k, α, and ρ which will allow functions in $V_\alpha^k(\rho)$ to be nonunivalent. Let

$$g(z) = \frac{1}{\mu}[(1-z)^{-\mu} - 1] , \tag{3.1}$$

and note

$$g'(z) = \frac{1}{(1-z)}^{\mu+1} .$$

$g'(z)$ has the form given in Theorem 2C, with $L_1'(z) = (1-z)^{-1}$ and $L_2'(z) = 1$ and

$$\mu + 1 = e^{-i\alpha}\cos\alpha(1-\rho)(k+2)/4 . \tag{3.2}$$

Hence $g(z)$ is in $V_\alpha^k(\rho)$ and, from an earlier result due to Royster [11], will not be univalent in $|z| < 1$ when $|\mu+1| > 1$ and $|\mu - 1| > 1$. The first condition requires that

$$\cos\alpha(1-\rho)(k+2)/4 > 1 , \tag{3.3}$$

while the second condition simplifies to

$$\cos^2\alpha(1-\rho)(k+2)\left[\frac{(1-\rho)(k+2)}{16} - 1\right] > -3 . \tag{3.4}$$

We may use these conditions to analyze the nonunivalence of functions in subclasses of $V_\alpha^k(\rho)$ which have been previously studied. When $\rho = 0$, the conditions defined by (2.1), (3.3) and (3.4) appear in Fig. 1. All functions in $V_\alpha^k(0)$ with k and α corresponding to points in region 1 are univalent, by (2.1). In region 3, $(k+2)\cos\alpha/4 > 1$ and condition (3.4) is satisfied for all $k > 6$ when $0 < \cos\alpha < \sqrt{3}/2$; for $\sqrt{3}/2 \leqq \cos\alpha < 1$, (3.4) is equivalent to $k > 6 - 4[4\cos^2\alpha - 3]^{1/2}/\cos\alpha$. When $g(z)$ defined by (3.1) is chosen so as to correspond with points in region 3, it will not be univalent. When

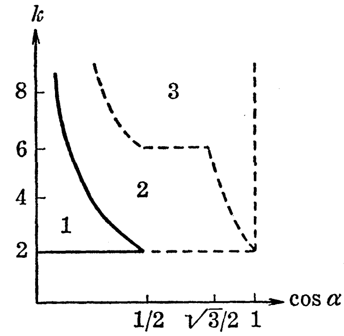

FIGURE 1

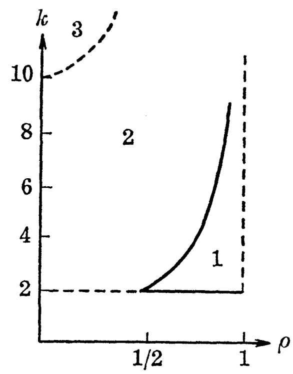

FIGURE 2

k and α correspond to points in region 2, it is an open question whether such f in $V_\alpha^k(0)$ will be univalent.

Fig. 2 is the corresponding diagram for univalence in the class $V_0^k(\rho)$. Region 1 depicts inequality (2.2), and all functions g defined by (3.1) with k, ρ satisfying (3.2) for $\alpha = 0$ are univalent in $|z|<1$. Conditions (3.3) and (3.4) require that $\rho < (k-10)/(k+2)$, and for these values of ρ and k (in region 3), $g(z)$ will not be univalent. Region 2 shows those values of k and ρ for which the univalency of functions in $V_0^k(\rho)$ is an open question. We note that when $k = 2$, the equation (3.1) defines the function used by Chichra in showing that there exist functions f in $F_\alpha^\rho = V_\alpha^2(\rho)$ where f is not univalent in $|z| < 1$, for $1/2(1-\rho) < \cos\alpha < 1$.

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