TAUBERIAN THEOREMS VIA BLOCK DOMINATED MATRICES

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The principle result is a general Tauberian theorem that can be applied to any regular real matrix summability method. The Tauberian condition is determined by the lengths of the blocks of consecutive terms that dominate the rows of the matrix. This theorem and its variants are then used to give a unified method of proving some of the classical Tauberian theorems for the methods of Abel, Borel, and Euler-Knopp. The general technique is also used to prove new Tauberian theorems for Nörlund and Taylor matrices.

1. **Introduction.** Let A be the matrix summability method that maps the sequence x into the sequence Ax, whose nth term is given by $(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$. Let Δx denote the sequence of differences: $(\Delta x)_n = x_n - x_{n+1}$. In this paper we shall study order-type Tauberian theorems in which the convergence of Ax together with a hypothesis such as $(\Delta x)_n = o(d_n)$ implies the convergence of x.

From each row of A that is in l^1, we can choose a block of consecutive terms whose total dominates the row sum. In order to state this precisely, let us introduce some notation and terminology. For each n, let $B_n = \{k \colon \mu(n) < k \leq \nu(n)\}$, where μ and ν are nonnegative integer sequences satisfying $\mu(n + 1) < \nu(n)$. Note that every integer greater than $\mu(0)$ belongs to at least one block B_n. Let L_n denote the length of B_n, i.e., $L_n = \nu(n) - \mu(n)$. We shall say that the matrix A is $\{B_n\}$-dominated if

$$(1) \qquad \liminf_n \left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \notin B_n} |a_{nk}| \right\} > 0 .$$

The main result of this paper asserts that if A is $\{B_n\}$-dominated, then $\max_{k \in B_n} |(\Delta x)_k| = o(L_n^{-1})$ is a Tauberian condition for A over the bounded sequences; if A is rowfinite, then a similar assertion holds without the assumption that x is bounded. The strength of this result lies not in its precision, but rather in its generality. It is easy to see that if $\liminf_n |\sum_{k=0}^{\infty} a_{nk}| > 0$ and each row is in l^1, then A admits a block sequence that satisfies (1). In case A is regular, the sequence $\{B_n\}$ can be chosen satisfying (1) and $\lim_n \mu(n) = \infty$. This helps us to define $\{B_n\}$ so that L_n does not increase too rapidly, thus obtaining a larger order estimate in the Tauberian condition. In § 3 the dominant blocks concept is used as a unified approach to proving some classical Tauberian theorems for the methods of Abel, Borel, and Euler-Knopp. In the final section, new Tauberian theorems are proved for Nörlund

and Taylor matrices.

2. **General Tauberian theorems.** The method of proof to be used here is suggested by an argument employed by Agnew [1] and Rado [8]. These authors considered matrices A in which each row was dominated by a single term, and their conclusions were that A is Mercerian, i.e., A is equivalent to ordinary convergence. In case A is $\{B_n\}$-dominated, we can use the slow oscillation of x to make the terms of these dominant blocks coincide with x_k's whose values are all nearly equal to some limit point α. Then $\sum_{k=0}^{\infty} a_{nk}x_k$ will approximate a multiple of α, and in this way the convergence of Ax prevents x from oscillating between two or more limit points.

THEOREM 1. *Suppose that A is a regular real matrix that is $\{B_n\}$-dominated; if x is a bounded sequence such that Ax is convergent and*

$$(2) \qquad \max_{k \in B_n} |(\Delta x)_k| = o(L_n^{-1}),$$

then x is convergent.

Proof. We may assume that x is a real number sequence. We suppose that x is nonconvergent, and we shall show that no real number r can be the limit of Ax. Let R denote the positive number $\limsup_k |x_k - r|$. Using the well-known Silverman-Töplitz conditions, we can write

$$(Ax)_n - r = o(1) + \sum_{k=0}^{\infty} a_{nk}(x_k - r).$$

Suppose $0 < \varepsilon < R$, and choose K such that $k \geq K$ implies $|x_k - r| < R + \varepsilon$. Then

$$|(Ax)_n - r| \geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - \sum_{k \notin B_n} |a_{nk}| \cdot |x_k - r|$$

$$\geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - \sum_{\substack{k \notin B_n \\ k \geq K}} |a_{nk}| \, |x_k - r|$$

$$> o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - (R + \varepsilon) \sum_{k \notin B_n} |a_{nk}|$$

$$\geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - R \sum_{k \notin B_n} |a_{nk}| - \varepsilon \|A\|,$$

where $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}|$.

Next we assert that for infinitely many n,

$$(3) \qquad k \in B_n \quad \text{implies} \quad |x_k - r| > R - \varepsilon.$$

This is established by first choosing N so that $L_n \max_{i \in B_n} |(\Delta x)_i| < \varepsilon/2$ whenever $n > N$; then select some n greater than N for which B_n contains an integer j satisfying $|x_j - r| > R - \varepsilon/2$. For any k in B_n, we have $|x_k - x_j| \leq L_n \max_{i \in B_n} |(\Delta x)_i| < \varepsilon/2$, so by the triangle inequality it follows that $|x_k - r| > R - \varepsilon$. Also, we note that all the numbers $\{x_k - r\}_{k \in B_n}$ have the same sign.

For any of the infinitely many n as established in the preceding paragraph, we have

$$|(Ax)_n - r| \geqq o(1) + (R - \varepsilon)\left| \sum_{k \in B_n} a_{nk} \right| - R \sum_{k \notin B_n} |a_{nk}| - \varepsilon \, \|A\|$$

$$\geqq o(1) - \varepsilon\left\{ \|A\| + \left| \sum_{k \in B_n} a_{nk} \right| \right\} + R\left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \notin B_n} |a_{nk}| \right\}$$

$$\geqq o(1) - 2\varepsilon \, \|A\| + R\left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \notin B_n} |a_{nk}| \right\}.$$

Since ε is an arbitrarily small positive number, it follows from (1) that $\liminf_n |(Ax)_n - r| > 0$. Therefore, Ax cannot have limit r, whence Ax is nonconvergent.

The proof of Theorem 1 can be applied to sequences that satisfy a gap condition, say $(\Delta x)_k = 0$, if $k \neq \kappa(m)$ where $m = 0, 1, 2, \cdots$, instead of an order type Tauberian condition. In order to ensure that (3) holds we must assume that the block B_n is contained in a single gap, i.e., $\kappa(m) \leqq \mu(n) < \nu(n) \leqq \kappa(m+1)$, for some m. The remainder of the proof can be used verbatim to yield the following general gap theorem. (Cf. [6, Theorem 1].)

COROLLARY 1. *Let* A *be a regular real matrix that is* $\{B_n\}$-*dominated and let* x *be a bounded gap sequence such that* $(\Delta x)_n = 0$ *if* $k \neq \kappa(m)$, $m = 0, 1, \cdots$. *If* $\{B_n\}$ *and* κ *satisfy* $\kappa(m) \leqq \mu(n) < \nu(n) \leqq \kappa(m+1)$ *for infinitely many* n, *then* Ax *and* x *either both converge or both diverge.*

Although the assumption that x is bounded places a considerable restriction on Theorem 1, we see by the following example that it is a necessary hypothesis. Let A be the matrix given by

$$a_{nk} = \begin{cases} 3/2, & \text{if } k = n, \\ -1/2, & \text{if } k = n^3, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that a dominant block sequence is $\{B_n\} = \{n\}$; thus $\mu(n) = n - 1$, $\nu(n) = n$, and $L_n \equiv 1$. Therefore, the Tauberian condition (2) reduces to $(\Delta x)_n = o(1)$. Consider the unbounded sequence $x: x_n = \log n$, if $n > 0$. Then $\lim_n (\Delta x)_n = 0$, and $(Ax)_n = 0$, if $n > 0$, but obviously

x is not convergent.

In order to prove a result similar to Theorem 1 in which the hypothesis that x is bounded is not necessary, we shall assume that A is row finite and terms to the right of the dominant blocks are zero. The argument is an adaptation of the idea used by Agnew in [1, § 2].

THEOREM 2. *Suppose A is a regular real matrix that is $\{B_n\}$-dominated and $a_{nk} = 0$ whenever $k > \nu(n)$; if x is a sequence such that Ax is convergent and $\max_{k \in B_n} |(\varDelta x)_k| = o(L_n^{-1})$, then x is convergent.*

Proof. Again, we assume that x is real, and we suppose that x is nonconvergent; by virtue of Theorem 1, we need to consider only the case where x is unbounded. Let H and ε be arbitrary positive numbers, and choose N such that $n > N$ implies that $L_n \max_{kk \in B_n} |(\varDelta x)_k| < \varepsilon$. Since x is unbounded, we can select n greater than N for which B_n contains an integer p such that $|x_p| > \max_{k < p} \{|x_k|, H\}$. We also have $\max_{k \in B_n} |x_k - x_p| < \varepsilon$, which implies that for all k in B_n, $|x_k| > |x_p| - \varepsilon$ and all $\{x_k\}_{k \in B_n}$ have the same sign. For such an n we have

$$
\begin{aligned}
|(Ax)_n| &\geqq \left| \sum_{k \in B_n} a_{nk} x_k \right| - \sum_{k \leqq \mu(n)} |a_{nk} x_k| \\
&> (|x_p| - \varepsilon) \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \leqq \mu(n)} |a_{nk} x_p| \\
&\geqq \left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \leqq \mu(n)} |a_{nk}| \right\} H - \varepsilon \|A\|.
\end{aligned}
$$

Since H and ε are arbitrary, it follows from (1) that $\lim \sup_n |(Ax)_n| = \infty$. Hence, Ax is nonconvergent.

As above, this theorem gives rise to a gap theorem, which we state next.

COROLLARY 2. *Let A be a regular real matrix that is $\{B_n\}$-dominated so that $a_{nk} = 0$ whenever $k > \nu(n)$, and let x be a gap sequence such that $(\varDelta x)_n = 0$ if $k \neq \kappa(m)$, $m = 0, 1, \cdots$. If $\{B_n\}$ and κ satisfy $\kappa(m) \leqq \mu(n) < \nu(n) \leqq \kappa(m + 1)$ for infinitely many n, then Ax and x either both converge or both diverge.*

In order to apply Theorems 1 and 2 to particular classes of matrices, it will be helpful if we first make some observations and remarks establishing some relationships between the Tauberian condition (2) and the more customary order estimates $(\varDelta x)_k = o(d_k)$ or $O(d_k)$. Suppose that $(\varDelta x)_k = o(d_k)$, where d is a nonincreasing sequence. Define $r_n = \max_{k > n} |(\varDelta x)_k / d_k|$, so $\lim_n r_n = 0$ and

$$(4) \qquad\qquad L_n \max_{k \in B_n} | (\Delta x)_k | \leqq L_n d_{\mu(n)} r_{\mu(n)} .$$

For a particular matrix, if we can construct the blocks $\{B_n\}$ so that $L_n d_{\mu(n)} = O(1)$, then we can infer from (4) that (2) holds. These remarks and similar observations based on $(\Delta x)_k = O(d_k)$ can be summarized formally as follows:

LEMMA 1.   *If $(\Delta x)_k = o(d_k)$, where d is nonincreasing, and A is $\{B_n\}$-dominated so that $\lim_n \mu(n) = \infty$ and $L_n d_{\mu(n)} = O(1)$, then (2) holds.*

LEMMA 2.   *If $(\Delta x)_k = O(d_k)$, where d is nonincreasing, and A is $\{B_n\}$-dominated so that $\lim_n \mu(n) = \infty$ and $L_n d_{\mu(n)} = o(1)$, then (2) holds.*

These lemmas provide some insight into the relationship between o- and O-Tauberian conditions as well as the relative difficulty of working with the weaker O-condition. For example, if $d_k = k^{-1}$, as in Tauber's original theorem [9], then $L_n d_{\mu(n)} = \{\nu(n) - \mu(n)\}/\mu(n)$, which yields the implications

$$(5) \qquad (\Delta x)_k = o(k^{-1}) \quad \text{and} \quad \nu(n) = O(\mu(n)) \quad \text{imply (2)}$$

and

$$(6) \qquad (\Delta x)_k = O(k^{-1}) \quad \text{and} \quad \nu(n) \sim \mu(n) \quad \text{imply (2)} .$$

For a given matrix, it is obviously easier to construct $\{B_n\}$ so that $\nu(n) = O(\mu(n))$ rather than $\nu(n) \sim \mu(n)$. Indeed, for the Cesàro matrix of order 1, it is impossible for $\{B_n\}$ to satisfy both (1) and $\nu(n) \sim \mu(n)$; but if we choose $\mu(n) = [n/3]$ and $\nu(n) = n$, then $\{B_n\}$ satisfies (1) and $\nu(n) = O(\mu(n))$. The resulting o-Tauberian theorem will be proved in the next section for Cesàro matrices C_j of all positive orders; our purpose in citing the C_1 case at this time is to demonstrate the fact that the general dominant block theory does not yield O-theorems. We must conclude, as did Lorentz [5, p. 228], that "these O-Tauberian theorems are connected with much more delicate properties of the matrix" than those made use of in Theorems 1 and 2.

3. **Applications to known Tauberian theorems.**   In this section we demonstrate how the block Tauberian results can be applied to some classical summability methods to derive well-known o-Tauberian theorems. (See e.g., [2] and [7].)   First we consider the Cesàro matrix of order j, where j is a positive integer, which is the lower triangular matrix given by

$$C_j(n, k) = \binom{n - k + j - 1}{j - 1} \binom{n + j}{j}^{-1} \qquad\qquad (k \leqq n) .$$

THEOREM 3. *If the sequence x is C_j-summable for some $j > 0$ and $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

Proof. In order to apply Theorem 2, we must determine μ and ν so that (1) holds. First define $\nu(n) = n$. If $1/2 < \beta < 1$, we wish to define $\mu = \mu(n)$ so that $\sum_{k=\mu+1}^{n} C_j(n, k) \geqq \beta$. By manipulating binomial coefficients, we find that

$$\sum_{k=\mu+1}^{n} C_j(n, k) = \left(1 - \frac{\mu + 1}{n + j}\right)\left(1 - \frac{\mu + 1}{n + j - 1}\right) \cdots \left(1 - \frac{\mu + 1}{n + 1}\right)$$
$$\geqq \left(1 - \frac{\mu + 1}{n + 1}\right)^j .$$

Therefore, we want to have $\mu \leqq -1 + (n + 1)(1 - \beta^{1/j})$, which is achieved by defining $\mu(n) = [n(1 - \beta^{1/j})]$, where $[t]$ denotes the greatest integer not exceeding t. Since $\nu(n) = O(\mu(n))$, property (5) guarantees that (2) holds. Hence, by Theorem 2, $(\Delta x)_k = o(k^{-1})$ is a Tauberian condition for C_j.

We remark that the preceding theorem applies to nonintegral j as well, because C_j includes C_r whenever $0 < r < j$.

Next we turn our attention to Abel summability, which assigns to a sequence x the limit value $\lim_{t \to 1^-} \{(1 - t) \sum_{k=0}^{\infty} x_k t^k\}$. Since the dominant blocks technique is designed for matrix methods, we replace the continuous parameter t with a sequence $\{t_n\}$ such that $0 < t_n < 1$ and $\lim_n t_n = 1$. Thus the Abel limit of x is equal to $\lim_n (Ax)_n$, where A is the matrix given by $a_{nk} = (1 - t_n)t_n^k$. It is clear that this Abel matrix method includes the continuous Abel method, so it also includes C_j for every $j > -1$. The constraint is that A is not row finite, and therefore, the Tauberian theorem that we can prove by using Theorem 1 will not apply to unbounded sequences.

THEOREM 4. *If x is a bounded sequence that is summable by a regular Abel matrix and $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

Proof. Suppose $\varepsilon > 0$. Since $\sum_{k=\mu+1}^{\nu} a_{nk} = t_n^{1+\mu} - t_n^{1+\nu}$, we first choose $\mu(n)$ as large as possible while still satisfying $t_n^{1+\mu(n)} \geqq 1/2 + 2\varepsilon$, say $\mu(n) = [\log (1/2 + 2\varepsilon)/\log t_n]$. Then choose $\nu(n)$ to be the least integer that is not less than $\log \varepsilon / \log t_n$. This yields

$$\sum_{k \in B_n} a_{nk} \geqq \frac{1}{2} + \varepsilon ,$$

so (1) holds. Also

$$\frac{\nu(n)}{\mu(n)} \sim \frac{\log \varepsilon}{\log \left(\frac{1}{2} + 2\varepsilon\right)} ,$$

so $\nu(n) = O(\mu(n))$, and by property (5), we see that (2) holds. Hence, the assertion follows from Theorem 1.

As another application of Theorem 1, we shall give new proofs of the classical Tauberian theorems for the summability methods of Borel and Euler-Knopp. (See [2] and [3].) As above, we replace the continuous variable with a discrete one and work with the Borel matrix B [7, p. 56], which is given by $b_{nk} = e^{-n} n^k / k!$. The matrix B includes the Borel exponential method, so the next result also applies to the latter method.

THEOREM 5. *If x is a bounded sequence that is summable by the Borel matrix and $(\Delta x)_k = o(k^{-1/2})$, then x is convergent.*

Proof. It is sufficient to determine μ and ν satisfying (1) and $\{\nu(n) - \mu(n)\}\{\mu(n)\}^{-1/2} = O(1)$; from this, Lemma 1 will imply that (2) holds, and Theorem 1 then yields the conclusion. To simplify notation, let $u_k = n^k / k!$. By a simple induction argument, one verifies that if $0 \le i \le \mu < n$, then $u_{\mu-i} < (\mu/n)^i u_\mu$. Thus

$$\sum_{k=0}^\mu u_k < \sum_{i=0}^\mu \left(\frac{\mu}{n}\right)^i u_\mu < \frac{n u_\mu}{n - \mu},$$

and therefore,

$$\sum_{k=0}^\mu b_{nk} < \frac{e^{-n} n u_\mu}{n - \mu} < \frac{n}{(n - \mu)\sqrt{2\pi n}}.$$

It follows that $\sum_{k=0}^\mu b_{nk} \le \alpha < 1/4$, if μ is defined by

$$\mu(n) = \left[n - \left(\frac{n}{2\pi\alpha^2}\right)^{1/2} \right].$$

The selection of ν is made from a similar calculation: if $i > 0$ and $\nu > n$, then $u_{\nu+i} < (n/\nu)^i u_\nu$, which yields

$$\sum_{k>\nu} u_k < \frac{n u_\nu}{\nu - n},$$

and

$$\sum_{k>\nu} b_{nk} < \frac{n}{(\nu - n)\sqrt{2\pi n}}.$$

As before, we want the sum to be no greater than α, so we define $\nu(n)$ to be the least integer that is not exceeded by

$$n + \left(\frac{n}{2\pi\alpha^2}\right)^{1/2}.$$

It is now easy to see that $\{\nu(n) - \mu(n)\}\{\mu(n)\}^{-1/2} = O(1)$, and $\sum_{k \in B_n} b_{nk} \geq 1 - 2\alpha > 1/2$, for every n, so the proof is complete.

Since the matrix B includes the Borel exponential method, which in turn includes the Euler-Knopp means [2, p. 183], the next result follows immediately from Theorem 5.

THEOREM 6. *If x is a bounded sequence that is summable by a regular Euler-Knopp matrix and $(\Delta x)_k = o(k^{-1/2})$, then x is convergent.*

4. Tauberian theorems for Nörlund and Taylor means. In the final section we shall apply the dominant block technique to prove some new Tauberian theorems for several well-known matrix methods. We first obtain a simple theorem for the Nörlund means N_p, which are given by

$$N_p(n, k) = \begin{cases} p_{n-k}/P_n, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

where p is a nonnegative sequence such that $p_0 > 0$, and $P_n = \sum_{i=0}^{n} p_i$.

THEOREM 7. *If the sequence x is summable by N_p, where p is in l^1, and $(\Delta x)_k = o(1)$, then x is convergent.*

Proof. For such p, N_p is regular since $\lim_n p_n/P_n = 0$. Let M be the least positive integer such that $P_M \geq (2/3) \sum_{i=0}^{\infty} p_i$. If $\nu(n) = n$ and $\mu(n) = \max\{0, n - M - 1\}$, then N_p is $\{B_n\}$-dominated and $L_n \leq M + 1$ for every n. Thus $(\Delta x)_k = o(1)$ implies (2); so by Theorem 2, $(\Delta x)_k = o(1)$ is a Tauberian condition for N_p.

We remark that $(\Delta x)_k = o(1)$ is a Tauberian condition for any matrix that can be block dominated with L_n bounded. In particular, if all the nonzero entries of A lie on a finite number of diagonals, then $(\Delta x)_k = o(1)$ is a Tauberian condition for A.

In [4] Iyengar stated the following theorem, which was first proved by Varshney [10].

THEOREM 8. *Let p be a positive sequence such that $\{p_{n+1}/p_n\}$ increases to 1 and $P_n \to \infty$. If x is summable by N_p and $(\Delta x)_k = O(M^{-1})$, where $M = M(n)$ is a positive integer defined by $P_M \leq \beta P_n < P_{n+1}$, for some β in $(0, 1)$, then x is convergent.*

The next result that we shall prove is similar to Iyengar's theorem. Although we must assume a stronger Tauberian condition, the theorem covers a larger class of Nörlund means.

THEOREM 9. *Let p be a nonincreasing sequence and suppose $1/2 <$

$\beta < 1$. *If x is summable by N_p and $(\Delta x)_n = o(M^{-1})$, where $M = M(n)$ is the least integer such that $P_M \geqq \beta P_n$, then x is convergent.*

Proof. Define $\nu(n) = n$ and $\mu(n) = n - M - 1$. Then

$$\sum_{k \in B_n} N_p(n, k) = P_n^{-1} \sum_{i=0}^{M} p_i \geqq \beta ,$$

and $L_n = M + 1$. It is not difficult to verify that M is nonincreasing and $\lim_n \mu(n) = \infty$. Therefore, we shall be able to apply Lemma 1 using $d_k = M(k)^{-1}$, provided that $L_n M(\mu)^{-1} = O(1)$. To prove that this boundedness holds, we first note that because p is nonincreasing, $M(k)/k$ is also nonincreasing. Therefore,

$$\frac{L_n}{M(\mu)} \sim \frac{M(n)/n}{M(\mu)/\mu} \cdot \frac{n}{\mu} \leqq \frac{n}{\mu} = \left\{ 1 - \left(\frac{M(n)}{n} - \frac{1}{n} \right) \right\}^{-1} ,$$

for sufficiently large n. The fact that $M(n)/n$ is nonincreasing implies that $\limsup_n \{M(n)/n\} < 1$, which establishes the boundedness of $L_n/M(\mu)$. Therefore, Lemma 1 guarantees that (2) holds, so Theorem 2 may be applied to complete the proof.

Finally we shall apply Theorem 1 to the Taylor means to get a Tauberian theorem of the usual order type. Recall that the Taylor matrix [7, p. 60] of order r is given by

$$t_{nk} = \begin{cases} \binom{k}{n} r^{k-n} (1 - r)^{n+1} , & \text{if } k \geqq n , \\ \\ 0 , & \text{if } k < n . \end{cases}$$

Note that if $|r| < 1$, then the row sums of T_r are identically equal to 1 because $(1 - r)^{-n-1} = \sum_{k \geqq n} \binom{k}{n} r^{k-n}$.

THEOREM 10. *If $0 < r < 1/2$ and x is a bounded sequence such that $T_r x$ is convergent and $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

Proof. Since T_r is an upper triangular matrix, we define $\mu(n) = n$ and wish to define $\nu = \nu(n)$ so that

$$(7) \qquad\qquad \limsup_n \sum_{k > \nu} t_{nk} \leqq \alpha < 1/2 .$$

In order to get a convenient estimate for $\sum_{k > \nu} t_{nk}$, we consider the MacLaurin series for the function

$$f(r) = (1 - r)^{-n-1} = \sum_{k \geqq n} \binom{k}{n} r^{k-n} = \sum_{j=0}^{\infty} \binom{n + j}{n} r^j .$$

Thus $\sum_{k > \nu} t_{nk}$ is $(1 - r)^{n+1}$ times the remainder term $R_{\nu-n}(r)$ of the

last series. By straightforward calculation the Lagrange form of this remainder is found to be

$$R_{\nu-n}(r) = \binom{\nu}{n}(1 - \xi_{\nu-n})^{-\nu-1}r^{\nu-n} \, ,$$

where $0 < \xi_{\nu-m} < r$. Since $(1 - \xi_{\nu-n})^{-\nu-1} < (1 - r)^{-\nu-1}$, we have

$$R_{\nu-n}(r) < \binom{\nu}{n}(1 - r)^{-\nu-1}r^{\nu-n} \, ,$$

which yields

$$\sum_{k>\nu} t_{nk} = (1 - r)^{n+1}R_{\nu-n}(r)$$

$$(8) \qquad\qquad < \binom{\nu}{n}(1 - r)^{n-\nu}r^{\nu-n}$$

$$= \binom{\nu}{n}\rho^{\nu-n} \, ,$$

where $\rho = r(1 - r)^{-1}$. Note that $\rho < 1$ because $r < 1/2$. Suppose that $\nu(n) = cn$, where c is a positive integer to be specified below. Using Stirling's formula, we find that

$$\binom{\nu}{n} = \frac{(cn)!}{(cn - n)!\,n!} \sim \frac{(cn)^{cn}}{(cn - n)^{cn-n}n^n}\left\{\frac{c}{2\pi n(c - 1)}\right\}^{1/2} \, ,$$

so

$$(9) \qquad\qquad \binom{\nu}{n}\rho^{\nu-n} \sim \left\{\frac{\rho c^{c/(c-1)}}{c - 1}\right\}^{cn-n}\left\{\frac{c}{2\pi n(c - 1)}\right\}^{1/2} \, .$$

Combining (8) and (9), we see that (7) will hold if

$$(10) \qquad\qquad \frac{\rho c^{c/(c-1)}}{c - 1} < 1 \, ,$$

which is equivalent to

$$(11) \qquad\qquad \rho < c^{1/(1-c)} - c^{c/(1-c)} \, .$$

It is an elementary exercise to show that the right-hand member of (11) approaches 1 as $c \to \infty$. Since $\rho < 1$, we can choose c sufficiently large to guarantee that (10) holds. Hence, (7) is satisfied by the definition $\nu(n) = cn$. Also, $(\Delta x)_k = o(k^{-1})$ by hypothesis, so we infer from (5) that (2) holds. Therefore, the hypotheses of Theorem 1 are satisfied, so we conclude that x is convergent.

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