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APPROXIMATE IDENTITIES AND POINTWISE CONVERGENCE

H. S. BEAR

We give two additional conditions on an approximate identity (or positive kernel) $\{K_\alpha\}$ which insure that $f*K_\alpha \to f$ a.e. if $f \in L^1$ on the line or circle. Where the convolution defines a function on the disc or a half-plane, as for the Poisson kernels or heat kernels, then the theorem gives automatically the paths toward a boundary point along which pointwise convergence occurs.

1. Introduction. An approximate identity on the line or the circle is a family of bounded nonnegative L^1 functions $\{K_\alpha\}$ such that $\int K_\alpha = 1$ and $\lim_\alpha \int_I K_\alpha = 1$ for all intervals $I = \{x: |x| \leqq \delta\}$, $\delta > 0$. The convolutions $f_\alpha = f*K_\alpha$ of a given function f with the members of an approximate identity provide approximations which converge to f in various ways depending on f. For a finite interval, for example, $f_\alpha \to f$ uniformly if f is continuous; $f_\alpha \to f$ in L^p if $f \in L^p (1 \leqq p < \infty)$; $f_\alpha \to f$ w^* if $f \in L_\infty$ [3, p. 22].

For specific approximate identities (Poisson kernels, heat kernels, the Fejér kernel) one also has $f_\alpha \to f$ pointwise almost everywhere. The proofs of these theorems use additional properties of the several kernels beyond the very general conditions for an approximate identity.

To illustrate, the Poisson kernel for the disc is

$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} . \qquad (0 \leqq r < 1; \ -\pi < \theta \leqq \pi) .$$

If $f \in L^1(-\pi, \pi)$, let $f(r, \theta) = f_r(\theta) = (f*P_r)(\theta)$. Then $f(r, \theta)$ is harmonic in $|z| < 1$, and $f(r, \theta) \to f(\theta)$ a.e. as $r \to 1$. In fact, one actually has the following classical theorem on nontangential approach: $f(r, \theta) \to f(\theta)$ a.e. as $(r, \theta) \to (1, \theta)$ along any nontangential path.

Our purpose here is to prove a theorem of this form: If $f \in L^1$ and $f_\alpha = f*K_\alpha$, then $f_\alpha \to f$ a.e. The hypothesis is that $\{K_\alpha\}$ be an approximate identity (on the circle or line) with two additional assumptions. The first of these is simply a smoothness assumption: K'_α is continuous, and each K_α has a unique maximum, and decreases monotonically away from this maximum in both directions. The second extra assumption (condition (e)) limits the distance from the origin at which K_α can have its maximum.

In the applications, condition (e) determines which translates of a given approximate identity can be added to the family so the result will still be an approximate identity. For the Poisson kernels, condition (e) is equivalent to restricting approach to a boundary point to paths within a Stolz angle. Hence condition (e) is in some sense "best possible."

Applied to the heat kernels our theorem gives apparently new results on the kind of approach toward a boundary point for which heat convolutions converge to the boundary function a.e.

For the Fejér kernel, "approach to a boundary point" is not a relevant idea. However, we do obtain information about how the modulus of continuity of the Cesáro sums of an L^1 function depend on n. The same sort of inference can be made for the other kernels, and is basically what is involved in the results on "non-perpendicular" approach.

There is nothing novel in the proof of the theorem. What is new is the isolation of the simple conditions which make all the standard proofs work, and the fact that translates of an approximate identity again form an approximate identity when suitably indexed. It is this last fact which gives the paths toward boundary points along which pointwise convergence takes place.

2. Proof of the theorem. Let $\{K_\alpha\}$ be a net of nonnegative real functions on $X=(-\infty, \infty)$, or on $X=(-\pi, \pi]$ (the circle). The index α is an element of a set D with a transitive partial ordering $\succ$. In addition we assume that for every $\alpha_1, \alpha_2 \in D$, there is $\alpha_3 \in D$ so that $\alpha_3 \succ \alpha_1$ and $\alpha_3 \succ \alpha_2$. Thus $(D, \succ)$ is a directed set. We write $\alpha \to \infty$ to indicate limits as α runs over D; e.g., $K_\alpha(x_0) \to 0$ as $\alpha \to \infty$.

The net $\{K_\alpha\}$ is an *approximate identity* if (a), (b), (c) below are satisfied, and we will call $\{K_\alpha\}$ a *smooth approximate identity* if in addition (d) and (e) are satisfied.

(a) $K_\alpha \in L^1$ and $K_\alpha \geqq 0$ for all α.

(b) $\int K_\alpha = 1$ for all α.

(c) $\int_{|x| \leqq \delta} K_\alpha \to 1$ as $\alpha \to \infty$, for all $\delta > 0$.

(d) K_α' is continuous on X. K_α increases to a unique maximum at x_α, and decreases for $x \geqq x_\alpha$. For the circle, K_α increases from some minimum value along the two complementary arcs to a unique maximum at x_α.

(e) For some constant A, $|x_\alpha| K_\alpha(x_\alpha) \leqq A$ for all α.

We will prove the following theorem.

THEOREM 1. *If $f \in L^1$ on X and $\{K_\alpha\}$ is a smooth approximate identity, then* $f * K_\alpha(x) \to f(x)$ a.e. *as* $\alpha \to \infty$.

The proof depends on a number of simple lemmas. Throughout this section we assume that $\{K_\alpha\}$ is a smooth approximate identity on X, where X is either the line or the circle. The notation will be for the case $X = (-\infty, \infty)$.

LEMMA 1. $\int_{|x|\geqq\delta} K_\alpha \to 0$ *as* $\alpha \to \infty$ *for all* $\delta > 0$.

Proof. This is immediate from (b) and (c).

LEMMA 2. $x_\alpha \to 0$ *as* $\alpha \to \infty$.

Proof. By (c), max $K_\alpha = K_\alpha(x_\alpha) \to \infty$ as $\alpha \to \infty$. Hence $x_\alpha \to 0$ by (e).

LEMMA 3. $\sup\{K_\alpha(s)\colon |s| \geqq \delta\} \to 0$ *as* $\alpha \to \infty$, *for all* $\delta > 0$. *In particular,* $K_\alpha(s_0) \to 0$ *as* $\alpha \to \infty$ *for all* $s_0 \neq 0$.

Proof. Fix $\delta > 0$. Then $|x_\alpha| < \delta$ for all sufficiently "large" α, and $\sup\{K_\alpha(s)\colon |s| \geqq \delta\}$ is either $K_\alpha(\delta)$ or $K_\alpha(-\delta)$. Suppose the lemma is false, and to be specific that there are arbitrarily large values of α for which $K_\alpha(\delta) \geqq p > 0$. Then there are arbitrarily large values of α such that $|x_\alpha| \leqq \delta/2$ and $K_\alpha(\delta) \geqq p$ by Lemma 2. For all such α, $\int_{|x|\geqq\delta/2} K_\alpha \geqq 1/2\,\delta p$, which contradicts Lemma 1.

The next lemma is relevant only if $X = (-\infty, \infty)$.

LEMMA 4. $\lim_{|s|\to\infty} K_\alpha(s) = 0$ *for all* α.

Proof. This follows from (d) and the fact that $K_\alpha \in L^1$.

LEMMA 5. *For all* $\delta > 0$, $\int_{|x|\geqq\delta} K'_\alpha \to 0$ *as* $\alpha \to \infty$.

Proof. We consider the interval $[\delta, \infty)$. The proof for the other cases is similar. For each α, $K'_\alpha \in L^1$ by (d) and Lemma 3, and

$$\int_\delta^\infty K'_\alpha = \lim_{s\to\infty} K_\alpha(s) - K_\alpha(\delta) = -K_\alpha(\delta)\ .$$

As $\alpha \to \infty$, $K_\alpha(\delta) \to 0$.

The next lemma uses condition (e) in an essential way. Condition (e) determines the paths toward boundary points (i.e., points x_0 of X) along which the convolutions $f*K_\alpha$ will approach $f(x_0)$ a.e.

LEMMA 6. *There is a constant B and $\alpha_0 \in D$ such that $\int |xK'_\alpha(x)|\, dx \leqq B$ for all $\alpha \succ \alpha_0$.*

Proof. Fix $M > 0$, and pick α_0 so that $|x_\alpha| < M$ for $\alpha \succ \alpha_0$. Now we consider only $\alpha \succ \alpha_0$, and show that $\int_{-M}^{M} |xK'_\alpha(x)|\, dx \leqq B$, where B is independent of M. Fix α, and assume $0 \leqq x_\alpha < M$; the same sort of argument works if $-M < x_\alpha \leqq 0$.

$$\begin{aligned}\int_{-M}^{M} |xK'_\alpha(x)|\, dx &= \int_{-M}^{0} -xK'(x)dx + \int_0^{x_\alpha} xK'_\alpha(x)dx - \int_{x_\alpha}^{M} xK'(x)dx \\ &= -xK_\alpha(x)\Big]_{-M}^{0} + \int_{-M}^{0} K_\alpha(x)dx \\ &\quad + xK_\alpha(x)\Big]_0^{x_\alpha} - \int_0^{x_\alpha} K_\alpha(x)dx \\ &\quad - xK_\alpha(x)\Big]_{x_\alpha}^{M} + \int_{x_\alpha}^{M} K_\alpha(x)dx \\ &\leqq (-M)K_\alpha(-M) + x_\alpha K_\alpha(x_\alpha) - MK_\alpha(M) + x_\alpha K_\alpha(x_\alpha) + 1 \\ &\leqq 2x_\alpha K_\alpha(x_\alpha) + 1 \leqq 2A + 1 \equiv B\,.\end{aligned}$$

LEMMA 7. *If $f \in L^1$, then for almost all x,*

$$\lim_{s\to x} \frac{1}{s-x}\int_x^s (f(u) - f(x))du = 0\,. \tag{1}$$

Proof. If $F(x) = \int_0^x f(u)du$, then $F'(x) = f(x)$ a.e.; i.e., $(F(s) - F(x))/(s-x) \to f(x)$ a.e., which is the same as (1).

COROLLARY. *The limit* (1) *holds whenever f is continuous at x.*

THEOREM 1. *If $f \in L^1$, and $\{K_\alpha\}$ is a smooth approximate identity, then $(f * K_\alpha)(x) \to f(x)$* a.e. *as $\alpha \to \infty$; specifically, the limit holds for all x for which* (1) *holds, and so in particular for x where f is continuous.*

Proof. Fix x, and let $\delta > 0$.

$$\begin{aligned}f * K_\alpha(x) - f(x) &= \int_{-\infty}^{\infty} K_\alpha(x-s)(f(s) - f(x))ds \\ &= \int_{-\infty}^{x-\delta} K_\alpha(x-s)f(s)ds - f(x)\int_{-\infty}^{x-\delta} K_\alpha(x-s)ds \\ &\quad + \int_{x+\delta}^{\infty} K_\alpha(x-s)f(s)ds - f(x)\int_{x+\delta}^{\infty} K_\alpha(x-s)ds \\ &\quad + \int_{x-\delta}^{x} K_\alpha(x-s)(f(s) - f(x))dx + \int_x^{x+\delta} K_\alpha(x-s)(f(s) - f(x))ds\,.\end{aligned}$$

Let $J_1, J_2, \cdots, J_6$ be the six integrals above, in the order in which they occur.

By Lemma 1, $J_2 \to 0$ and $J_4 \to 0$ as $\alpha \to \infty$.

J_1 and J_3 are similar to each other, and we estimate J_1:

$$|J_1| \leqq ||f||_1 \sup \{K_\alpha(t): t \geqq \delta\} .$$

Hence $J_1 \to 0$ and $J_3 \to 0$ as $\alpha \to \infty$ by Lemma 3.

Finally we show that $J_5 \to 0$ for every x for which (1) holds, and a similar argument holds for J_6.

Let x be a number for which (1) holds, and let

$$\beta(s) = \int_x^s (f(s) - f(x))ds ,$$

so that $\beta(s)/(s - x) \to 0$ as $s \to x$. Then

$$\begin{aligned} J_5 &= \int_{x-\delta}^x K_\alpha(x - s)d\beta(s) \\ &= K_\alpha(0)\beta(x) - K_\alpha(\delta)\beta(x - \delta) + \int_{x-\delta}^x \beta(s)K'_\alpha(x - s)ds . \end{aligned}$$

Observe that $\beta(x) = 0$, that β is continuous, and that $K_\alpha(\delta) \to 0$ for any fixed δ, as $\alpha \to \infty$. Now we estimate the final integral.

$$\begin{aligned} \int_{x-\delta}^x \beta(s)K'_\alpha(x - s)ds &\leqq \int_{x-\delta}^x \left|\frac{\beta(s)}{s - x}\right| |(x - s)K'_\alpha(x - s)|\, ds \\ &\leqq \max_{x-\delta \leqq s \leqq x} \left|\frac{\beta(s)}{s - x}\right| \cdot \int_0^\delta |tK'_\alpha(t)|\, dt \qquad (2) \\ &\leqq B \max_{x-\delta \leqq s \leqq x} \left|\frac{\beta(s)}{s - x}\right| , \end{aligned}$$

where B is the constant of Lemma 6. Choose δ so the right side of (2) is less than ε. Then pick α_0 so that $|K_\alpha(\delta)\beta(x - \delta)| < \varepsilon$ if $\alpha \succ \alpha_0$. Hence $|J_5| < 2\varepsilon$ if $\alpha \succ \alpha_0$, and we conclude that $J_5 \to 0$ as $\alpha \to \infty$.

In all the applications we take as our net $\{K_\alpha\}$ the translates of some standard approximate identity $\{L_\beta\}$. Hence we state the following simple observation as a lemma.

LEMMA 8. *Let $\{L_\beta\}$ be a smooth approximate identity, and let*

$$K_{(\beta,t)}(s) = L_\beta(s - t) .$$

If the indices (β, t) are so ordered that $(\beta, t) \to \infty$ implies $\beta \to \infty$ and $t \to 0$, then $\{K_{(\beta,t)}\}$ satisfies (a), (b), (c), *and* (d).

For the applications, we so order the pairs (β, t) that condition

(e) also holds. The restrictions imposed by (e) determine the paths along which pointwise convergence occurs.

3. **Applications.** In this section we apply Theorem 1 to the Poisson kernels for the disc and half-plane, to heat kernels for the half-plane and the first quadrant, and to the Fejér kernel.

(A) *The Poisson kernel for the disc.*
Let

$$P_r(s) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos s}, \tag{3}$$

where $-\pi < s \leqq \pi$ and $r < 1$. The kernels $\{P_r\}$ are a smooth approximate identity as $r \to 1$ [2, p. 102], [3, p. 32]. Let

$$K_{(r,\theta)}(s) = P_r(s-\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(s-\theta)}.$$

Here $\alpha = (r, \theta)$ for $0 \leqq r < 1$ and $-\pi < \theta \leqq \pi$. The α's are ordered as follows: fix A and let

$$(r, \theta) \succ (r', \theta') \text{ iff } r \geqq r', |\theta| \leqq A(1-r), |\theta'| \leqq A(1-r').$$

Clearly $(r, \theta) \to \infty$ implies $r \to 1$ and $\theta \to 0$, so $\{K_{(r,\theta)}\}$ satisfies (a)-(d). The polar curve $\theta = A(1-r)$ has limiting slope $\pm A$ as $r \to 1$. Hence $(r, \theta) \to \infty$ implies that $(r, \theta) \to (1, 0)$ between the lines through $(1, 0)$ with slopes $\pm A$. If $F(r, \theta) \to L$ as $(r, \theta) \to \infty$, then $\lim_{r\to 1} F(r, \theta) = L$ and the limit is uniform in θ for (r, θ) within the given angle at $(1, 0)$. This is formally stronger than the usual statement "$F(r, \theta) \to L$ along any path to $(1, 0)$ within an angle." Actually, the two statements are equivalent, and we will write "$(r, \theta) \to (1, 0)$ in an angle" for $(r, \theta) \to \infty$.

If $\alpha = (r, \theta)$, then x_α of condition (e) is given by $x_\alpha = \theta$, and condition (e) is satisfied as follows:

$$|x_\alpha| K_\alpha(x_\alpha) = \frac{|\theta|}{2\pi}\frac{1-r^2}{(1-r)^2} = \frac{1+r}{2\pi}\frac{|\theta|}{1-r} \leqq \frac{A}{\pi}.$$

THEOREM A. *If* $f \in L^1(-\pi, \pi)$, *then*

$$f * K_{(r,\theta)}(s_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)f(s)}{1+r^2-2r\cos(s_0-s-\theta)}ds \tag{4}$$

$$\longrightarrow f(s_0) \text{ a.e. } \textit{as } (r, \theta) \longrightarrow (1, 0) \textit{ in an angle}.$$

To get the usual statement, replace $s_0 - \theta$ by θ in the right side of (4) and let

$$F(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - r^2)f(s)}{1 + r^2 - 2r\cos(\theta - s)}\, ds\ .$$

Then F is harmonic in the disc and the classical theorem is:

THEOREM A′. *If $f \in L^1(-\pi, \pi)$, then for almost all s_0, $F(r, \theta) \to f(s_0)$ as $(r, \theta) \to (1, s_0)$ within an angle at $(1, s_0)$.*

To see that condition (e) is an appropriate hypothesis, consider the function $g(z) = \exp(z + 1)/(z - 1)$, with $g(1) = 0$. By [1, Theorem 3.2], (1) holds for this g at $\theta = 0$, since $g(r) \to 0$ as $r \to 1$. Observe that $|g(z)| = \exp(-P(r, \theta))$, where $z = re^{i\theta}$. On the circle $r = \cos\theta$, $P(r, \theta) = 1/2\pi$, so $|g(z)|$ is a constant different from $g(1) = 0$. If $f(\theta) = \operatorname{Re} g(e^{i\theta})$ and $F(r, \theta)$ is the Poisson integral of f, then $F(r, \theta) \not\to f(0)$ as $(r, \theta) \to (1, 0)$ along the tangential path $r = \cos\theta$, even though (1) holds for f at $\theta = 0$.

(B) *The Poisson kernel for the half-plane.*
Let

$$P_y(s) = \frac{y}{\pi}\frac{1}{y^2 + s^2}\ . \tag{5}$$

Here $-\infty < s < \infty$, and $y > 0$, and $\{P_y\}$ is an approximate identity as $y \to 0+$ [3, p. 123]. Let

$$K_{(x,y)}(s) = P_y(s - x) = \frac{y}{\pi}\frac{1}{y^2 + (s - x)^2}\ ,$$

where $-\infty < x < \infty$ and $y > 0$. Order the pairs (x, y) as follows:

$$(x, y) \succ (x', y') \text{ iff } y \leqq y'\ ,\quad |x| \leqq Ay\ ,\quad |x'| \leqq Ay'\ .$$

Hence $(x, y) \to \infty$ means $y \to 0+$ and (x, y) stays between the lines $y = \pm Ax$. We will indicate such a limit by "$(x, y) \to (0, 0)$ in an angle."

If $\alpha = (x, y)$, then $x_\alpha = x$, and condition (e) becomes

$$|x_\alpha|\, K_\alpha(x_\alpha) = |x| \cdot \frac{1}{\pi y} = \frac{1}{\pi}\frac{|x|}{y} \leqq \frac{A}{\pi}\ .$$

Theorem 1 in this case is

THEOREM B. *If $f \in L^1(-\infty, \infty)$, then*

$$f * K_{(x,y)}(s_0) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (s_0 - s - x)^2}\, ds \longrightarrow f(s_0) \text{ a.e. as } (x, y) \longrightarrow (0, 0) \text{ in an angle}\ . \tag{6}$$

Replace $s_0 - x$ by x in the right side of (6), and let

$$F(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (s - x)^2}\, ds .$$

Then F is harmonic in the upper half-plane, and we have:

THEOREM B′. *If $f \in L^1(-\infty, \infty)$, then for almost all s_0, $F(x, y) \to f(s_0)$ as $(x, y) \to (s_0, 0)$ within an angle at $(s_0, 0)$.*

(C) *Heat kernel for the upper half-plane.*
Let

(7)
$$k_t(s) = \frac{1}{\sqrt{4\pi t}}\, e^{-s^2/4t} ,$$

where $-\infty < s < \infty$, and $t > 0$. The family $\{k_t\}$ is an approximate identity as $t \to 0+$ [4, p. 31], and $k_t(s)$ satisfies the heat equation $\partial^2 u/\partial s^2 = \partial u/\partial t$. Let

$$K_{(x,t)}(s) = k_t(s - x) = \frac{1}{\sqrt{4\pi t}}\, e^{-(s-x)^2/4t} .$$

Order the indices (x, t) as follows:

$$(x, t) \succ (x', t') \text{ iff } t \leqq t', |x| \leqq A\sqrt{t}, |x'| \leqq A\sqrt{t'} .$$

Then $(x, t) \to \infty$ means $t \to 0+$ and (x, t) lies over the parabola $t = x^2/A$. We will write "$(x, t) \to (0, 0)$ over a parabola" for $(x, t) \to \infty$. If $\alpha = (x, t)$, then $x_\alpha = x$, and condition (e) is satisfied as follows:.

$$|x_\alpha| K_\alpha(x_\alpha) = |x| \frac{1}{\sqrt{4\pi t}} \leqq A/\sqrt{4\pi} .$$

THEOREM C. *If $f \in L^1(-\infty, \infty)$, then*

(8)
$$f * K_{(x,t)}(s_0) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(s_0-s-x)^2/4t} f(s)ds$$
$$\longrightarrow f(s_0) \text{ a.e. } as\ (x, t) \to (0, 0)\ over\ a\ parabola .$$

Replace $s_0 - x$ by x on the right side of (8), and let

$$F(x, t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} f(s)ds .$$

Then $F(x, t)$ satisfies the heat equation in the upper half-plane, and:

THEOREM C′. *If $f \in L^1(-\infty, \infty)$, then for almost all s_0, $F(x, t) \to f(s_0)$ as $(x, t) \to (s_0, 0)$ over any parabola $t = A(x - s_0)^2$.*

(D) *The heat kernel for the first quadrant.*

For a function f in $L^1(t \geqq 0)$ we want to obtain an extension $f(x, t)$ which satisfies the heat equation in the first quadrant. Since Theorem 1 treats a convolution on $(-\infty, \infty)$ rather than $(0, \infty)$, we first prove a lemma which puts Theorem 1 in accessible form.

LEMMA 9. *If $\{K_\alpha\}$ is a smooth approximate identity on $(-\infty, \infty)$, and $K_\alpha \equiv 0$ on $(-\infty, b_\alpha)$, and $f \in L^1(0, N)$ for every N, then*

$$\int_0^{x-b_\alpha} K_\alpha(x-s)f(s)ds \longrightarrow f(x) \text{ a.e. as } \alpha \longrightarrow \infty . \tag{9}$$

Proof. Extend f to the whole line by letting $f \equiv 0$ on $(-\infty, 0)$. Then for $x \geqq b_\alpha$,

$$\begin{aligned} f * K_\alpha(x) &= \int_{-\infty}^{\infty} K_\alpha(x-s)f(s)ds \\ &= \int_0^{x-b_\alpha} K_\alpha(x-s)f(s)ds . \end{aligned}$$

As $\alpha \to \infty$, $\limsup b_\alpha \leqq 0$ by Lemma 2. Hence (9) holds for almost all x in $(0, \infty)$.

For $x > 0$, the kernel h_x is defined as follows:

$$h_x(s) = \begin{cases} \dfrac{1}{\sqrt{4\pi}} \dfrac{x}{s^{3/2}} e^{-x^2/4s} & \text{if } s > 0 \\ 0 \quad \text{if } s \leqq 0 . \end{cases} \tag{10}$$

The function $h_x(s)$ satisfies the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial s$ everywhere except $(0, 0)$ [4, p. 16]. For fixed $x > 0$, $h_x(s) \equiv 0$ for $s \leqq 0$, and h_x has a unique maximum at $s = x^2/6$. The maximum value is

$$h_x(x^2/6) = \frac{1}{\sqrt{4\pi}} (6/e)^{3/2} \frac{1}{x^2} .$$

The functions $\{h_x\}$ form an approximate identity as $x \to 0+$ [4, p. 71]. Let

$$K_{(x,t)}(s) = h_x(s-t) = \begin{cases} \dfrac{1}{\sqrt{4\pi}} \dfrac{x}{(s-t)^{3/2}} e^{-x^2/4(s-t)} & \text{if } s > t \\ 0 \quad \text{if } s \leqq t . \end{cases}$$

Note that $K_{(x,t)} \equiv 0$ on $(-\infty, t)$, so the b_α of Lemma 9 is: $b_{(x,t)} = t$. Order the points (x, t) of the right half-plane as follows: fix $B > 0$, and let

$$(x, t) \succ (x', t') \text{ iff } x \leqq x' , \ |t| \leqq Bx^2 , \ |t'| \leqq B(x')^2 .$$

Hence $(x, t) \to \infty$ iff $x \to 0+$ and (x, t) lies in the right half-plane between the parabolas $t = \pm Bx^2$.

The function $K_{(x,t)}$ has its unique maximum where $s - t = x^2/6$. Hence for $\alpha = (x, t)$, $x_\alpha = t + x^2/6$, and the left side of the condition (e) becomes

$$|x_\alpha| K_\alpha(x_\alpha) = |t + x^2/6| \frac{1}{\sqrt{4\pi}} \left(\frac{6}{e}\right)^{3/2} \frac{1}{x^2} . \tag{11}$$

If $t \geqq - x^2/6$, then (11) becomes, with $c = (6/e)^{3/2}/\sqrt{4\pi}$,

$$|x_\alpha| K_\alpha(x_\alpha) = c\left(\frac{t}{x^2} + \frac{1}{6}\right) \leqq c\left(B + \frac{1}{6}\right) .$$

If $t < - x^2/6$, then (11) becomes

$$|x_\alpha| K_\alpha(x_\alpha) = c\left(-\frac{t}{x^2} - \frac{1}{6}\right) \leqq cB .$$

Hence (e) holds with constant

$$A = c\left(B + \frac{1}{6}\right) = \frac{1}{\sqrt{4\pi}} \left(\frac{6}{e}\right)^{3/2} \left(B + \frac{1}{6}\right) .$$

THEOREM D. *If $f \in L^1(0, N)$ for every N, then for almost all s_0,*

$$f * K_{(x,t)}(s_0) = \int_0^{s_0 - t} \frac{1}{\sqrt{4\pi}} \frac{x}{(s_0 - s - t)^{3/2}} e^{-x^2/4(s_0 - s - t)} f(s) ds$$

$$\longrightarrow f(s_0) \text{ as } (x, t) \longrightarrow (0, 0) \text{ between parabolas} .$$

Now replace $s_0 - t$ by t, and let

$$F(x, t) = \frac{1}{\sqrt{4\pi}} \int_0^t \frac{x}{(t - s)^{3/2}} e^{-x^2/4(t-s)} f(s) ds .$$

Then $F(x, t)$ satisfies the heat equation for $x > 0, t > 0$, and (cf. [4, p. 78]):

THEOREM D′. *If $f \in L^1(0, N)$ for any N, then for almost all s_0, $F(x, t) \to f(s_0)$ as $(x, t) \to (0, s_0)$ between any parabolas $t = s_0 \pm Bx^2$.*

(E) *The Fejér kernel.*

The Fejér kernel is defined by

$$L_n(s) = \begin{cases} \dfrac{1}{n} \left| \dfrac{\sin\left(\dfrac{ns}{2}\right)}{\sin\left(\dfrac{s}{2}\right)} \right|^2 & \text{if } s \neq 0 \\ n & \text{if } s = 0 . \end{cases}$$

The family $\{L_n\}$ is smooth approximate identity on $[-\pi, \pi]$ and $f*L_n = \sigma_n$, the nth Cesàro sum of the Fourier series for f [2, p. 79], [3, p. 17]. We let $K_{(n,x)}(s) = L_n(s-x)$, and order the pairs (n, x) as follows:

$$(n, x) \succ (n', x') \quad \text{iff} \quad n \geqq n' , \quad n|x| \leqq A, n'|x'| \leqq A .$$

Then $(n, x) \to \infty$ means $n \to \infty$ and $|x| \leqq A/n$. The unique maximum of $K_{(n,x)}$ is $K_{(n,x)}(x) = n$, so condition (e) reads

$$|x_\alpha| K_\alpha(x_\alpha) = |x| \cdot n \leqq A .$$

Hence $\{K_{(n,x)}\}$ is a smooth approximate identity and

THEOREM E. *If* $f \in L^1(-\pi, \pi)$, *then*

$$\sigma_n(s_0 - x) = f*K_{(n,x)}(s_0) \longrightarrow f(s_0) \text{ a.e. } as\ n \longrightarrow \infty ,$$

uniformly in x *if* $|x| \leqq A/n$.

Let s_0 be a point where the limit above exists, and let $\varepsilon > 0$. Then there is N so that $|\sigma_n(s_0 - x) - \sigma_n(s_0)| < \varepsilon$ if $n \geqq N$ and $|x| \leqq 1/n$. Hence we have an estimate of the modulus of continuity of σ_n, for large n, and this estimate does not depend on f, so long as (1) holds at the point in question.

The kind of continuity statement made above for the Cesàro sums can be made for convolutions with any smooth approximate identity. For example, let $f_r(\theta) = (f*P_r)(\theta)$ where P_r is the Poisson kernel. For $\varepsilon > 0$, there is r_ε so that $|f_r(s_0) - f_r(s_0 + \theta)| < \varepsilon$ for all $r \geqq r_\varepsilon$ and all θ such that $\theta \leqq A(1-r)$. We have an explicit estimate of how the continuity of f_r at s_0 depends on r. The estimate is independent of f, so long as (1) holds at s_0.

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UNIVERSITY OF HAWAII
HONOLULU, HI 96822