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THE OBSTRUCTION OF THE FORMAL MODULI SPACE IN THE NEGATIVELY GRADED CASE

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Consider a semigroup ring $B_H = k[t^h / h \in H]$ where t is a transcendental over an algebraically closed field k of characteristic 0. Let $T^1(B)$ denote $T^1(B/k, B)$ where $T^1(B/k, -)$ is the upper cotangent functor of Lichtenbaum and Schlessinger. Then $T^1(B)$ is a graded k-vector space of finite dimension and B is said to be negatively graded if $T^1(B)_+ = 0$. It is known that a versal deformation T/S of B/k exists in the sense of Schlessinger, where (S, m_S) is a complete noetherian local k-algebra. We say that the formal moduli space is unobstructed if S is a regular local ring. In this paper we restrict our attention to the negatively graded semigroup rings. In this case we compute the dimension of $T^1(B)$ and are thus able to determine which formal moduli spaces are unobstructed.

Let U denote the (open) subset of Spec (S) consisting of all points with smooth fibres. In a previous paper [5] we computed the dimension of U. We always have inequalities:

$$\dim U \leqq (\text{Krull}) \dim S \leqq [m_S/m_S^2 \colon k] \, .$$

Consequently S is a regular local ring if and only if $\dim U = [m_S/m_S^2 \colon k] = [T^1(B) \colon k]$. In the general case the difference $[T^1(B) \colon k] - \dim U$ gives some indication of the extent of the obstruction.

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2. Preliminaries and notation.

(2.1) Let H be a subsemigroup of the additive subgroup N of nonnegative integers. H is called a *numerical semigroup* if the greatest common divisor of the elements of H is 1, so that only finitely many positive integers are missing from H. Such elements are called the *gaps* of H and the number of gaps is called the *genus* of H, denoted by $g(H)$. The least positive integer c such that $c + N \subset H$ is called the *conductor* of H, denoted by $c(H)$. The least positive integer m in H is called the *multiplicity* of H and is denoted by $m(H)$. Throughout this paper H will denote a numerical semigroup, k an algebraically closed field of characteristic 0.

Let B_H denote the k-subalgebra of the polynomial ring $k[t]$ generated by the monomials t^h, $h \in H$. B_H is called the *semigroup ring* of H.

When no possible confusion can arise we simply write B for B_H, g for $g(H)$, c for $c(H)$ and m for $m(H)$.

(2.2) We now construct a generating set called the *standard basis* for H, denoted S_H. Let $m = m(H)$. For $0 \leqq j \leqq m-1$ choose a_j to be the least positive integer in H such that $a_j \equiv j \pmod{m}$.

For $1 \leqq j \leqq k \leqq m-1$, set

$$f_{j,k} = X_j X_k - X_0^{e(j,k)} X_{r(j,k)}$$

where $0 \leqq r(j,k) \leqq m-1$ and $a_j + a_k = e(j,k)m + a_{r(j,k)}$. Set $I = I_H$ equal to the ideal of $P = k[X_0, \cdots, X_{m-1}]$ generated by $\{f_{j,k}\}_{1 \leqq j \leqq k \leqq m-1}$ where P is a polynomial algebra over k.

PROPOSITION 2.3. *If we define a k-algebra map $\varphi\colon k[X_0, \cdots, X_{m-1}] \to B$ by $\varphi(X_j) = t^{a_j}$ for $0 \leqq j \leqq m-1$ then $0 \to I \to P \to B \to 0$ is exact. Furthermore, if we assign the weight a_j to X_j in P, then φ is a homomorphism (of degree 0) of graded k-algebras and I is homogeneous.*

We will not attempt to give a precise definition of T^* here. For definition and details of T^0, T^1 one can consult [1]; for the full cohomological properties one should consult Rim's article "Formal Deformation Theory" [4] (note that our T^i plays the role of Rim's D^i). We state here some properties of T^* that will facilitate our computations. For proofs of these assertions see [4] and [5].

PROPOSITION 2.4. *Let P be a polynomial algebra over R and let $0 \to I \to P \to A \to 0$ be exact. Then for any A-module M,*

$$\begin{aligned} T^0(A \mid R, M) &\cong \mathrm{Der}_R(A, M)\,, \\ T^1(A \mid R, M) &\cong \mathrm{Coker}\,(\mathrm{Der}_R(P, M) \longrightarrow \mathrm{Hom}_A(I/I^2, M)) \\ &\cong \textit{the set of isomorphism classes of } R\textit{-algebra} \\ &\qquad \textit{extensions of } A \textit{ by } M. \end{aligned}$$

(2.5) In our case, if $B = B_H$ then $T^1(B) = T^1(B \mid k, B)$ becomes a graded k-vector space via the exact sequence of (2.3). We then have

$$\begin{aligned} T^1(B) &= \bigoplus_{-\infty<p<\infty} T^1(B)_p \\ &\cong \bigoplus_{-\infty<p<\infty} \mathrm{Coker}\,(\mathrm{Der}_k\,(P, B)_p \longrightarrow \mathrm{Hom}_B(I/I^2, B)_p)\,, \end{aligned}$$

so that

$$T^1(B)_p \cong \text{the set of isomorphism classes of (degree 0) graded } k\text{-algebra extensions of } B \text{ by } B(p)$$

where $B(p)$ is the graded k-module obtained from B by shifting the degree by p; i.e., $B(p)_n = B_{p+n}$.

Those monomial curves B_H for which $T^1(H)_+ = T^1(B_H)_+ = 0$ are the so called *negatively graded semigroup rings* of Pinkham [3]. In [5] we completely classified these and described a method for computing $T^1(H)_p$. We now recall these results and set up some notation which will be used in § 3.

(2.6) Let $S_H = \{a_0 = m, a_1, \cdots, a_{m-1}\}$ denote the standard basis for H (as in 2.2). For each integer p let $G_p = \{a \in S_H \mid a + p \notin H\}$ and let $R_p = \{f_{j,k} \in I_H \mid a_j + a_k + p \notin H\}$. By abuse of notation associate with each $f_{j,k}$ of R_p a vector $f_{j,k} = (f^0_{j,k}, \cdots, f^{m-1}_{j,k})$ of k^m where the lth component is given by

$$\begin{aligned} f^l_{j,k} &= -e(j, k) && \text{if } l = 0 \text{ and } r(j, k) \neq 0\,, \\ &= -(e(j, k) + 1) && \text{if } l = 0 = r(j, k)\,, \\ &= -1 && \text{if } l = r(j, k) \neq 0\,, \\ &= \;\;2 && \text{if } l = j = k\,, \\ &= \;\;1 && \text{if } l = j \text{ or } l = k \text{ and } j \neq k\,, \\ &= \;\;0 && \text{otherwise}\,. \end{aligned}$$

Again by abuse, let R_p denote the vector subspace of k^m spanned by those $f_{j,k}$ in R_p. We note that if $a_l \notin G_p$ then $f^l_{j,k} = 0$ for all $f_{j,k} \in R_p$. Thus if $G_p \neq \emptyset$, $\dim R_p \leqq \# G_p - 1$.

PROPOSITION 2.7. *In the notation above,*

$$\dim T_p = \dim T^1(H)_p = \max\{0, \# G_p - \dim R_p - 1\}\,.$$

(2.8) We say that H is an *ordinary* semigroup of multiplicity m, denoted by H_m, if $H = \{0, m, m+1, m+2, \cdots\}$. We say that H is *hyperordinary* if $H = mN + H_{m'}$ where $H_{m'}$ is ordinary and $0 < m < m'$.

THEOREM 2.9. *H is negatively graded if and only if H is of one of the following types:*

(i) *H is ordinary;*

(ii) *H is hyperordinary;*

(iii) *Excluding the above two cases, H is negatively graded of multiplicity m if and only if there exists precisely one gap $m + i$*

between m and 2m; if $i = 1$ then $2m + 1 \notin H$ (or H would be hyperordinary).

If $2 \leqq i \leqq m - 1$ then

$$H_{m,i} = \{0, m, m + 1, \cdots, \widehat{m + i}, m + i + 1, m + i + 2, \cdots\} .$$

If $i = 1$ we have

$$H_{m,1} = \{0, m, m + 2, \cdots, 2m, \widehat{2m + 1}, 2m + 2, 2m + 3, \cdots\} .$$

3. A Dimension formula for $T^1(H)$. We now compute the dimension of the tangent space $T^1(H)$ for the negatively graded semigroup rings. We first deal with the ordinary and hyperordinary cases and finally with those of the third type.

For these semigroups $T^1(H) = T^1(H_-)$. Recall the notation of (2.6) and let $a = a(H)$ denote the least positive integer in $H - m(H)N$, let $c = c(H)$. Then $p \leqq 2a - c$ entails $R_{-p} = \varnothing$ since for $f_{j,k} \in I$ we have $a_j + a_k - p \geqq 2a - p \geqq c$ so that $a_j + a_k - p \in H$. Thus by Proposition 2.7 $\dim T^1(H)_{-p} = \max\{0, \# G_{-p} - 1\}$.

Throughout these computations $[r]$ = the greatest integer $\leqq r$; $\{r\}$ = the least integer $\geqq r$; $\delta_{r,s}$ denotes the Kronecker delta, i.e., $\delta_{r,s} = 1$ if $r = s$ and 0 otherwise. Once a semigroup H is fixed we let $T_{-l} = T^1(H)_{-l}$. By dim( ) we mean dimension as a k-vector space.

Now assume H is ordinary or hyperordinary so that $H = mN + \{pm + i, pm + i + 1, pm + i + 2, \cdots\}$ where $p \geqq 1$ and $1 \leqq i \leqq m - 1$. Then $a(H) = pm + i$.

PROPOSITION 3.1. *Let $H = mN + \{pm + 1, pm + 2, \cdots\}$. Then*

$$\begin{aligned}
\dim T_{-l} &= l - 1 && \text{if } 1 \leqq l \leqq m - 1 ,\\
&= m - 2 && \text{if } l = m \text{ or } m + 1 \leqq l \leqq pm + 2 \\
& && \quad \text{and } m \nmid l ,\\
&= m - 1 && \text{if } m + 1 \leqq l \leqq pm + 2 \text{ and } m \mid l ,\\
&= (p + 1)m - l + \delta_{l,(p+1)m} && \text{if } pm + 3 \leqq l \leqq (p + 1)m ,\\
&= \delta_{m,2} && \text{if } (p + 2)m \leqq l \leqq (2p + 1)m \\
& && \quad \text{and } m \mid l ,\\
&= 0 && \text{otherwise} .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\dim T^1(H) &= (p - 1)(m - 1)^2 + m(m - 1) - 1 && \text{if } m \geqq 3 ,\\
&= 2p && \text{if } m = 2 .
\end{aligned}$$

Proof. Note that $2a(H) - c(H) = pm + 2$ so that for $1 \leqq l \leqq$

$pm+2$ we have $\dim T_{-l} = \# G_{-l} - 1$.

Suppose $l > (p+1)m$ and set $q = l - [l/m]m + \delta_{l,[l/m]m} m$. If $q = 1$ then $R_{-l} \supseteqq \{f_{1,1}, \cdots, f_{1,m-1}\}$; if $q = 2 \leqq m-1$ then $R_{-l} \supseteqq \{f_{1,2}, \cdots, f_{1,m-1}, f_{2,2}\}$; if $3 \leqq q \leqq m$ then $R_{-l} \supseteqq \{f_{1,1}, \cdots, \hat{f}_{1,q-1}, \cdots, f_{1,m-1}, f_{2,q-1}\}$. Finally if $q = 2 = m$ we see that $R_{-l} = \varnothing$ for $2(p+2) \leqq l \leqq 2(2p+1)$ while $R_{-l} = \{f_{1,1}\}$ for $l > 2(2p+1)$. Our assertions follow.

Then assume $pm+3 \leqq l \leqq (p+1)m$ and set $q = l - pm$ so that $G_{-l} = S_H - \{a_q\}$ if $q < m$ while $G_{-(p+1)m} = S_H$. Then $R_{-l} = \{f_{j,k} \mid a_j + a_k < pm + a_q\} = \{f_{j,k} \mid j + k \leqq q - 1\}$.

Set $R'_{-l} = \{f_{1,1}, \cdots, f_{1,q-2}\}$. Then R'_{-l} generates R_{-l} for if $j+k \leqq q-1$ and $j \geqq 2$ we have (as vectors) $f_{j,k} = f_{1,j+k-1} + \cdots + f_{1,j} - (f_{1,k-1} + \cdots + f_{1,1})$. Since rank $R'_{-l} = q-2$ we have $\dim T_{-l} = (p+1)m - l + \delta_{l,(p+1)m}$.

Summing up the various components we see that

$$\begin{aligned} \dim T^1(H) &= (p-1)(m-1)^2 + m(m-1) - 1 && \text{if } m \geqq 3\,, \\ &= 2p && \text{if } m = 2\,. \end{aligned}$$

Now suppose $H = mN + \{pm+i, pm+i+1, \cdots\}$ where $2 \leqq i \leqq m-1$. Then $c(H) = a(H) = pm + i = a_i$. We treat the cases $2i \leqq m$ and $2i > m$ separately but as the proofs are analagous we only give the former.

PROPOSITION 3.2. *Suppose that* $H = mN + \{pm+i, pm+i+1, \cdots\}$ *where* $2 \leqq i \leqq m/2$. *Then*

$$\begin{aligned} \dim T^1(H)_{-l} &= l && \textit{if } 1 \leqq l \leqq i-1\,, \\ &= l-1 && \textit{if } i \leqq l \leqq m-i\,, \\ &= l-2 && \textit{if } m-i+1 \leqq l \leqq m\,, \\ &= m-2 && \textit{if } m+1 \leqq l \leqq pm+i \textit{ and } m \nmid l\,, \\ &= m-1 && \textit{if } m+1 \leqq l \leqq pm+i \textit{ and } m \mid l\,, \\ &= m-2(l-pm-i) - \delta_{l,pm+i+1} && \textit{if } pm+i+1 \leqq l \leqq pm+2i-1\,, \\ &= m-2(l-pm-i)+1+\delta_{l,(p+1)m} + \delta_{l,(p+1)m+1} && \textit{if } pm+2i \leqq l \leqq pm+2i+1\,, \\ &= m - \min(2i+1, m-1) - 1 + \delta_{l,(p+1)m} + \delta_{i,2} && \textit{if } l = pm+2i+2\,, \\ &= (p+1)m - l + \delta_{l,(p+1)m} && \textit{if } pm+2i+3 \leqq l \leqq (p+1)m\,, \\ &= 0 && \textit{otherwise}\,. \end{aligned}$$

Consequently,

$$\dim T^1(H) = (p-1)(m-1)^2 + m(m-1) + i(i-2) + \delta_{i,2}\,.$$

Proof. Now $2a(H) - c(H) = a(H) = pm + i$ so for $1 \leqq l \leqq pm + i$ we have $\dim T_{-l} = \sharp G_{-l} - 1$.

For $pm + i + 1 \leqq l \leqq (p+1)m + i - 1$ we set $q = l - [l/m]m + m\cdot\delta_{l,(p+1)m}$. Then $G_{-l} = S_H - \{a_q\}$ if $q \neq m$ and $G_{-(p+1)m} = S_H$. We note that $R_{-l} = \{f_{j,k} \mid a_j + a_k < a_i + l$ and $j + k \not\equiv q (\operatorname{mod} m)\}$. Then $R_{-(pm+i+1)} = \{f_{i,i}\}$ entails $\dim T_{-(pm+i+1)} = m - 3$.

Suppose that $pm + i + 2 \leqq l \leqq pm + 2i - 1$. Then $R_{-l} = \{f_{j,k} \mid j + k \leqq i + q - 1$ and $k \geqq j \geqq i\}$ and is generated by $R'_{-l} = \{f_{i,i}, \cdots, f_{i,q-1}, f_{i+1,i+1}, \cdots, f_{i+1,q-2}\}$. For suppose $f_{j,k} \in R_{-l} - R'_{-l}$ so that $j \geqq i + 2$, $k \leqq q - 3$. Then $i + 2 < j + k - i \leqq q - 1$ and as vectors $f_{j,k} = \Delta_{j+k} - \Delta_j - \Delta_k$ where $\Delta_r = \sum_{s=i+1}^{r-i-1} (f_{i,s+1} - f_{i+1,s})$.

As for independence, we observe that $f_{i,i}, \cdots, f_{i,m-1}, f_{i+1,i+1}, \cdots, f_{i+1,2i-1}$ are independent. This is more readily seen by substituting the vectors

$$v_r = f_{i,r+1} - f_{i+1,r} \text{ if } i + 1 \leqq r \leqq 2i - 2$$

and

$$\begin{aligned} v_{2i-1} &= f_{i,i} + f_{i,2i} - f_{i+1,2i-1} \text{ if } 2i < m\,, \\ &= -f_{i+1,2i-1} \text{ if } 2i = m \end{aligned}$$

for the last $i - 1$ vectors.

Thus $\dim R_{-l} = 2(l - pm - i) - 2$ and $\dim T_{-l} = m - 2(l - pm - i)$ for $pm + i + 2 \leqq l \leqq pm + 2i - 1$.

We wish to consider those integers l between $pm + 2i$ and $(p+1)m + i - 1$.

Suppose $pm + 2i \leqq l \leqq pm + 2i + 1$ and let $q = l - pm$. Then $R'_{-l} = \{f_{i,i}, \cdots, \hat{f}_{i,q-i}, \cdots, f_{i,\min(q-1,m-1)}, f_{i+1,i+1}, \cdots, f_{i+1,q-2}\}$ generates R_{-l} as above and has rank $2(q - i) - 3 - \delta_{l,(p+1)m+1}$.

Let $l = pm + 2i + 2$ and set $q = 2i + 2$. If $i = 2$ so that $q = 6$ then $R_{2-l} = \{f_{1,2}, f_{2,2}, f_{2,3}\}$ if $m = 4$ and $R_{-l} = \{f_{2,2}, f_{2,3}, \hat{f}_{2,4}, \cdots, f_{2}, \min(5, m-1), f_{3,4}\}$ if $m \geqq 5$. In either case rank $R_{-l} = \sharp R_{-l} - 1$ as we note that

$$\begin{aligned} f_{1,2} &= f_{2,2} - f_{2,3} && \text{if } m = 4\,, \\ f_{3,4} &= f_{2,3} - f_{2,2} && \text{if } m = 5\,, \\ f_{3,4} &= f_{2,3} - f_{2,2} + f_{2,5} && \text{if } m \geqq 6\,. \end{aligned}$$

So we have $\dim R_{-l} = \min(q - 1, m - 1) - 2 + \delta_{m,4}$. If $i \geqq 3$ then set $R'_{-l} = \{f_{i,i}, \hat{f}_{i,i+1}, \cdots, f_{i,\min(q-1,m-1)}, f_{i+1,i+2}, \cdots, f_{i+1,2i-1}, f_{i+2,i+2}\}$. Note that $(f_{i+1,i+1} - f_{i,i+2}) = f_{i+1,i+3} - f_{i+2,i+2} + f_{i+1,i+2} - f_{i,i+3}$ and if $2i < m$

we have $f_{i+1,2i} = f_{i,i+1} - f_{i,i} + (1 - \delta_{2i+1,m})f_{i,2i+1}$. So R'_{-l} generates R_{-l} as above and has rank $\min(q-1, m-1) - 1$.

Now assume that $l > pm + 2i + 2$. If $l \leqq (p+1)m$ set $q = l - pm$ and let $R'_{-l} = \{f_{i,i}, \cdots, \hat{f}_{i,q-i}, \cdots, f_{i,q-1}\} \cup B_{-l}$ where

$$\begin{aligned} B_{-l} &= \{f_{i+1,i+1}, \cdots, f_{i+1,2i-1}\} \text{ if } q > 3i \\ &= \{f_{i+1,i+1}, \cdots, \hat{f}_{i+1,q-i-1}, \cdots, f_{i+1,2i-1}, f_{i+2,q-i-1}\} \\ &\qquad \text{if } 2i + 3 \leqq q \leqq 3i\,. \end{aligned}$$

Observe that if $f_{i+1,j} \in R_{-l}$ and $j \geqq 2i$, setting $t = [j/i]$ we have $f_{i+1,j} = (1 - \delta_{j,m-1})f_{i,j+1} - [f_{i,j-i} + f_{i,j-2i} + \cdots + f_{i,j-(t-1)i}] + [f_{i,j-i+1} + f_{i,j-2i+1} + \cdots + f_{i,j-(t-1)i+1}] + (1 - \delta_{j,ti})[f_{i+1,j-(t-1)i} - f_{i,j-(t-1)i+1}]$. Similarly if $i = 2$ then $f_{i+2,q-i-1} = f_{4,q-3}$ is in the span of R'_{-l}. Finally note that $(f_{i,q-i} - f_{i+1,q-i-1}) = (f_{i+1,q-i} - f_{i+2,q-i-1}) + (f_{i,i+2} - f_{i+1,i+1})$ so that R'_{-l} generates R_{-l} as above. Hence $\dim R_{-l} = q - 2$.

If $(p+1)m + i - 1 \geqq l > (p+1)m$ (and $l > pm + 2i + 2$) set $q = l - pm$ so that $i + 3 \leqq q - i \leqq m - 1$. Set

$$R'_{-l} = \{f_{i,i}, \cdots, \hat{f}_{i,q-i}, \cdots, f_{i,m-1}\} \cup B_{-l}$$

where

$$\begin{aligned} B_{-l} &= \{f_{i+1,i+1}, \cdots, f_{i+1,2i-1}\} \text{ if } q > 3i\,, \\ &= \{f_{i+1,i+1}, \cdots, \hat{f}_{i+1,q-i-1}, \cdots, f_{i+1,2i-1}, f_{i+2,q-i-1}\} \text{ if } 2i+3 \leqq q \leqq 3i\,. \end{aligned}$$

Then R'_{-l} generates R_{-l} as it has maximal rank $m - 2$. Hence $T_{-l} = 0$.

Finally suppose that $l \geqq (p+1)m + i$ (and $l > pm + 2i + 2$) and set $q = l - [l/m]m$. If $1 \leqq q \leqq i - 1$ so that $l \geqq (p+2)m$ then

$$R_{-l} \supseteqq \{f_{1,1}, \cdots, \hat{f}_{1,q-1}, \cdots, f_{1,m-1}, f_{2,q-1}\}\,.$$

If $i \leqq q \leqq 2i - 1$ then

$$R_{-l} \supseteqq \{f_{i,i}, \cdots, f_{i,m-1}, f_{i+1,i+1}, \cdots, f_{i+1,2i-1}\}\,.$$

If $2i \leqq q \leqq m - 1$ then

$$R_{-l} \supseteqq \{f_{i,i}, \cdots, \hat{f}_{i,q-i}, \cdots, f_{i,m-1}\} \cup B_{-l}$$

where

$$\begin{aligned} B_{-l} &= \{f_{i+1,i+1}, \cdots, f_{i+1,2i-1}, f_{i+1,q}\} \text{ if } q \leqq 2i + 1 \text{ or } q > 3i\,, \\ &= \{f_{i+1,i+2}, \cdots, f_{i+1,2i-1}, f_{i+1,q}, f_{i+2,i+2}\} \text{ if } q = 2i + 2\,, \\ &= \{f_{i+1,i+1}, \cdots, \widehat{f_{i+1,q-i-1}}, \cdots, f_{i+1,2i-1}, f_{i+1,q}, f_{i+2,q-i-1}\} \\ &\qquad \text{if } 2i + 3 \leqq q \leqq 3i\,. \end{aligned}$$

If $q = 0$ so that $l \geqq (p+2)m$ then

$$R_{-l} \supseteqq \{f_{1,1}, \cdots, \widehat{f_{1,i-1}}, \cdots, \widehat{f_{1,m-1}}, f_{i,m-i+1}, f_{i+1,m-1}\}\ .$$

In all cases $\dim R_{-l} = m - 1$ so that $T_{-l} = 0$.

PROPOSITION 3.3. *Suppose $H = mN + \{pm + i, pm + i + 1, \cdots\}$ where $i \geqq 2$ and $2i > m$. Then*

$$\begin{aligned}
\dim T^1(H)_{-l} &= l && \text{if } 1 \leqq l \leqq m-i\ ,\\
&= l - 1 && \text{if } m-i+1 \leqq l \leqq i-1\ ,\\
&= l - 2 && \text{if } i \leqq l \leqq m\ ,\\
&= m - 2 && \text{if } m+1 \leqq l \leqq pm+i \text{ and } m \nmid l\ ,\\
&= m - 1 && \text{if } m+1 \leqq l \leqq pm+i \text{ and } m \mid l\ ,\\
&= m-2(l-pm-i)-\delta_{l,pm+i+1}+\delta_{l,(p+1)m} && \text{if } pm+i+1 \leqq l \leqq (p+1)m\ ,\\
&= pm+2i-l+\delta_{l,(pm+2i)} && \text{if } (p+1)m+1 \leqq l \leqq pm+2i\ ,\\
&= 1 && \text{if } l = pm+2i+2 \text{ and } i=2\ ,\\
&= 0 && \text{otherwise.}
\end{aligned}$$

Consequently, $\dim T^1(H) = (p-1)(m-1)^2 + m(m-1) + i(i-2) + \delta_{i,2}$.

COROLLARY 3.4. *Suppose H is ordinary or hyperordinary of multiplicity m and $a(H) = pm + i$. Then*

$$\begin{aligned}
\dim T^1(H) &= (p-1)(m-1)^2 + m(m-1) + i(i-2) + \delta_{i,2} \text{ if } m \geqq 3\ ,\\
&= 2p \text{ if } m = 2\ .
\end{aligned}$$

We finally deal with those negatively graded semigroups of the third type so that there is precisely one gap $m + i$ between m and $2m$. Recall that if $i = 1$ then $2m + 1 \notin H$. In any case, $a_j = m + j$ for $j \neq i$ while $a_i = a_j + a_k$ whenever $j + k = i + \delta_{i,1}m$. Again we deal with a series of cases governed by the relation of i and m. As the proofs are similar we only give the proof in case $2 \leqq i \leqq m - 1 \leqq 2i$.

PROPOSITION 3.5. *Let $H = H_m - \{m + 1, 2m + 1\}$ where H_m is ordinary and $m \geqq 3$. Then*

$$\begin{aligned}
\dim T^1(H)_{-l} &= l - \left[\frac{l+1}{2}\right] + \delta_{l,1} && \text{if } 1 \leqq l \leqq m-2\ ,\\
&= l - \left[\frac{l+1}{2}\right] - 1 && \text{if } m-1 \leqq l \leqq m+1\ ,
\end{aligned}$$

$$\begin{aligned}
&= l - \left[\frac{l+1}{2}\right] - 3 + \delta_{l,m+2} && \textit{if } m+2 \leqq l \leqq m+4 \\
& && \textit{and } l \leqq 2m-2\ , \\
&= m - \left[\frac{l+1}{2}\right] + \delta_{l,m+6} && \textit{if } m+5 \leqq l \leqq 2m-2\ , \\
&= \delta_{m,5} + \delta_{m,7} && \textit{if } l=2m-1\ , \\
&= 1 + \delta_{m,4} + \delta_{m,6} && \textit{if } l = 2m\ , \\
&= \delta_{m,3} + \delta_{m,5} && \textit{if } l=2m+1\ , \\
&= \delta_{m,4} && \textit{if } l=2m+2 \textit{ or } 3m+2\ , \\
&= \delta_{m,3} + \delta_{m,4} && \textit{if } l=3m\ , \\
&= \delta_{m,3} && \textit{if } l=3m+1,\ 4m \textit{ or } 5m\ , \\
&= 0 && \textit{otherwise.}
\end{aligned}$$

Consequently,

$$\dim T^1(H) = \frac{m(m-1)}{2} + 2 + 3\delta_{m,3} + 2\delta_{m,4}\ .$$

PROPOSITION 3.6. *Suppose $H = H_m - \{m + i\}$ where H_m is ordinary and $2 \leqq i \leqq (m-2)/2$. Then*

$$\begin{aligned}
\dim T^1(H)_{-l} &= l && \textit{if } 1 \leqq l \leqq i\ , \\
&= l - 1 && \textit{if } i+1 \leqq l \leqq m-i-1\ , \\
&= l - 2 - \left[\frac{l+i-m}{2}\right] - \delta_{l,m+1} && \textit{if } m-i \leqq l \leqq m+1\ , \\
&= 2m - l - \left[\frac{l+i-m}{2}\right] + \delta_{l,m+i} \\
&\qquad + \delta_{l,m+i+1} && \textit{if } m+2 \leqq l \leqq m+i+1\ , \\
&= 2m - (l+i) + \delta_{i,2} && \textit{if } m+i+2 \leqq l \leqq m+i+4 \\
& && \quad \textit{and } l \leqq 2m-i\ , \\
&= 2m - (l+i) && \textit{if } m+i+5 \leqq l \leqq 2m-i\ , \\
&= \delta_{m,6} + \delta_{m,7} && \textit{if } l=2m-1 \textit{ and } i=2\ , \\
&= \delta_{m,6} && \textit{if } l=2m \textit{ and } i=2\ , \\
&= 0 && \textit{otherwise.}
\end{aligned}$$

Consequently,

$$\dim T^1(H) = m^2 - (i+1)m + \frac{i(i+1)}{2} + 3\delta_{i,2}\ .$$

PROPOSITION 3.7. *Suppose that $H = H_m - \{m + i\}$ where H_m is ordinary and $2i \geqq m - 1 \geqq i \geqq 2$. Then*

$$
\begin{aligned}
\dim T^1(H)_{-l} &= l && \text{if } 1 \leqq l \leqq m-i-1\,,\\
&= l-1-\left[\frac{l+i-m}{2}\right] && \text{if } m-i \leqq l \leqq i\,,\\
&= l-2-\left[\frac{l+i-m}{2}\right]-\delta_{l,m+1} && \text{if } i+1 \leqq l \leqq m+1\,,\\
&= 2m-l-\left[\frac{l+i-m}{2}\right]+\delta_{l,m+i} \\
&\qquad + \delta_{l,m+i+1} && \text{if } m+2 \leqq l \leqq 2m-i\,,\\
&= i-\left[\frac{l+i-m}{2}\right]+\delta_{l,m+i} && \text{if } 2m-i+1 \leqq l \leqq m+i\,,\\
&= 1 && \text{if } l=m+i+1\,,\\
&= \delta_{m,5} && \text{if } l=m+4 \text{ and } i=2\,,\\
&= \delta_{m,5}+\delta_{m,4} && \text{if } l=2m \text{ and } i=2\,,\\
&= \delta_{m,4}+\delta_{m,3} && \text{if } l=2m+2\,,\\
&= \delta_{m,4}+\delta_{m,3} && \text{if } l=3m \text{ and } i=m-1,\\
&= \delta_{m,3} && \text{if } l=4m\,,\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\dim T^1(H) &= m^2-(i+1)m+\frac{i(i+1)}{2}+2\delta_{m,4} && \text{if } i \geqq 3\,,\\
&= m^2-3m+5+\delta_{m,3} && \text{if } i=2\,.
\end{aligned}
$$

Proof. We note that $2a(H)-c(H)=m-i+1$. Hence for $1 \leqq l \leqq m-i+1$ one has $\dim T_{-l} = \# G_{-l} - 1$. Also note that

$$
\begin{aligned}
G_{-l} &= \{a_0, \cdots, a_{l-1}, a_{l+i}\} && \text{if } 1 \leqq l \leqq m-i-1\,,\\
&= \{a_0, \cdots, a_{l-1}\} && \text{if } m-i \leqq l \leqq i\,,\\
&= \{a_0, \cdots, \widehat{a_i}, \cdots, a_{l-1}\} && \text{if } i+1 \leqq l \leqq m-1\,,\\
&= \{a_1, \cdots, a_{m-1}\} && \text{if } l=m\,,\\
&= S_H - \{a_i, a_{l-m}\} && \text{if } m+1 \leqq l \leqq m+i-1\,,\\
&= S_H - \{a_{l-m}\} && \text{if } m+i \leqq l \leqq 2m-1\,,\\
&= S_H && \text{if } l \geqq 2m \text{ and } l \neq 2m+i\,,\\
&= S_H - \{a_i\} && \text{if } l = 2m+i\,.
\end{aligned}
$$

If $m-i+2 \leqq l \leqq m+1$ then $R_{-l} = \{f_{j,k} \mid a_j + a_k = m+l+i\} = \{f_{j,k} \mid j+k = l+i-m \text{ and } k \neq i\}$. Hence $\dim R_{-l} = [(l+i-m)/2] - \delta_{l,m+1}$.

If $m+2 \leqq l \leqq 2m-i$ set $q = l-m$. Then

$$
\begin{aligned}
R_{-l} &= \{f_{j,k} \mid a_j + a_k = 2m+i+q \text{ or } a_j + a_k < 2m+q\}\\
&= \{f_{j,k} \mid j+k = i+q \text{ and } j, k \neq i \text{ or } j+k \leqq q-1\}.
\end{aligned}
$$

Hence

$$R_{-l} = \text{span}\{f_{1,q+i-1}, \cdots, \hat{f}_{q,i}, \cdots, f_{[q+i/2],\{q+i/2\}}, f_{1,1}, \cdots, f_{1,q-2}\}$$

and $\dim R_{-l} = q + [(q+i)/2] - 3 = l - m + [(l+i-m)/2] - 3$.

If $2m - i + 1 \leqq l \leqq m + i$ then

$$\begin{aligned} R_{-l} &= \{f_{j,k} \mid a_j + a_k = 2m + i + q \text{ or } a_j + a_k < 2m + q\} \\ &= \{f_{j,k} \mid j + k = i + q \text{ and } j, k \neq i \text{ or } j + k \leqq q - 1\} \\ &= \text{span}\{f_{i+q-m+1,m-1}, \cdots, \widehat{f_{q,i}}, \cdots, f_{[(q+i)/2],\{(q+i)/2\}}, f_{1,1}, \cdots, f_{1,q-2}\}. \end{aligned}$$

Hence $\dim R_{-l} = m - 1 - \{(q+i)/2\} + q - 2 = m + [(q+i)/2] - i - 3$.

Suppose $l = m + i + 1 \geqq 2m - i + 1$ so that $2i \geqq m$. Then if $i = m - 1$ we have $l = 2m$ and $R_{-l} = \text{span}\{f_{1,1}, \cdots, f_{1,m-2}\}$ so that $\dim T_{-l} = 1$. If $i \leqq m - 2$ then $R_{-l} = \text{span}\{f_{1,1}, \cdots, f_{1,i-1}, f_{2i+2-m,m-1}, \cdots, f_{i-1,i+2}\}$ and has rank $m - 3$ so again $\dim T_{-l} = 1$.

Now suppose $m + i + 2 \leqq l \leqq 2m - 1$ and set $q = l - m$. If $i = 2$ then $m = 5$ and $R_{-l} = \{f_{1,1}, f_{3,3}\}$ so $\dim T_{-l} = 1$. If $i \geqq 3$, $R_{-l} = \text{span}\{f_{1,1}, \cdots, \widehat{f_{1,i}}, \cdots, f_{1,q-2}, f_{i+q-m+1,m-1}, \cdots, f_{i-1,q+1}, f_{i+1,q-1}, f_{2,i-1}\}$ so that $\dim R_{-l} = m - 2$ and $T_{-l} = 0$.

Assume that $l = 2m > m + i + 1$, so $i \leqq m - 2$. If $i \geqq 3$ then $R_{-l} = \text{span}\{f_{1,1}, \cdots, \widehat{f_{1,i}}, \cdots, f_{1,m-2}, f_{2,i-1}, f_{i+1,m-1}\}$ and $T_{-l} = 0$.

If $i = 2$ and $m = 4$ or 5 then $R_{-l} = \{f_{1,1}, \widehat{f_{1,2}}, \cdots, f_{1,m-2}, f_{3,m-1}\}$ so that $\dim T_{-l} = 1$.

Now suppose $l \geqq 2m + 1$ and set $q = l - [l/m]m$. If $q = 1$ or $q = i$ and $l \geqq 3m + i$ then

$$R_{-l} \supseteqq \{f_{1,1}, \cdots, f_{1,m-1}\}.$$

If $l = 2m + i$ so that $G_{-l} = S_H - \{a_i\}$ then R_{-l} is spanned by:

$$\begin{aligned} &\{f_{1,1}, \cdots, \widehat{f_{1,i-1}}, \widehat{f_{1,i}}, \cdots, f_{1,m-1}, f_{2,i-1}\} \text{ if } i \geqq 3, \\ &\{f_{1,3}, \cdots, f_{1,m-1}\} \text{ if } i = 2 \text{ and } m \leqq 4, \\ &\{f_{1,3}, f_{1,4}, f_{3,3}\} \text{ if } i = 2 \text{ and } m = 5. \end{aligned}$$

Consequently $\dim T_{-l} = \delta_{i,2}(\delta_{m,3} + \delta_{m,4})$. Suppose $q = 2 \leqq i - 1$. Then R_{-l} is spanned by:

$$\begin{aligned} &\{f_{1,2}, \cdots, \widehat{f_{1,i}}, \cdots, f_{1,m-1}, f_{2,2}, f_{2,i-1}\} \text{ if } i \geqq 4, \\ &\{f_{1,2}, \widehat{f_{1,3}}, \cdots, f_{1,m-1}, f_{2,2}, f_{2,m-1}\} \text{ if } i = 3 \text{ and } m \geqq 5, \\ &\{f_{1,2}, f_{2,2}\} \text{ if } i = 3 \text{ and } m = 4 \text{ and } l = 2m + 2, \\ &\{f_{1,2}, f_{1,3}, f_{2,2}\} \text{ if } i = 3, m = 4 \text{ and } l \geqq 3m + 2. \end{aligned}$$

We note that $\dim T_{-(2m+2)} = \delta_{m,3} + \delta_{m,4}$. Now suppose $3 \leqq q \leqq m - 1$ and that $q \neq i$. Then R_{-l} is spanned by:

$$\{f_{1,1}, \cdots, \widehat{f_{1,q-1}}, \cdots, \widehat{f_{1,i}}, \cdots, f_{1,m-1}, f_{2,q-1}, f_{2,i-1}\} \text{ if } i \geqq 3 \text{ and } q \neq i+1$$
$$\{f_{1,1}, \cdots, \widehat{f_{1,i}}, \cdots, f_{1,m-1}, f_{i+1,i+1}\} \text{ if } q = i+1 \leqq m-1\ ,$$
$$\{f_{1,1}, f_{1,2}, \widehat{f_{1,3}}, f_{1,4}, f_{3,3}\} \text{ if } i = 2,\ m = 5,\ q = 4\ .$$

Hence $\dim R_{-l} = m - 1$ and $T_{-l} = 0$. If $q = 0$ so that $l = [l/m]m \geqq 3m$, then R_{-l} is spanned by:

$$\{f_{1,1}, \cdots, f_{1,m-2}, f_{i+1,m-1}\} \text{ if } i \leqq m-2\ ,$$
$$\{f_{1,1}, \cdots, f_{1,m-2}\} \text{ if } i = m-1,\ m \leqq 4 \text{ and } l = 3m\ ,$$
$$\{f_{1,1}, \cdots, f_{1,m-2}, f_{3,m-2}\} \text{ if } i = m-1 \text{ and } m \geqq 5\ ,$$
$$\{f_{1,1}\} \text{ if } i = m-1,\ m = 3 \text{ and } l = 4m\ ,$$
$$\{f_{1,1}, \cdots, f_{1,m-2}, f_{2,2}\} \text{ if } i = m-1,\ m = 4 \text{ and } l \geqq 4m \text{ or } m = 3 \text{ and } l \geqq 5m\ .$$

Hence

$$\dim T_{-3m} = \delta_{m,3} + \delta_{m,4} \cdot \delta_{i,3}$$
$$\dim T_{-4m} = \delta_{m,3}$$
$$\dim T_{-l} = 0 \text{ if } m \mid l \text{ and } l \geqq 5m\ .$$

COROLLARY 3.8. *If H is negatively graded of the third type with $c(H) = m + i + 1 \leqq 2m$ then*

$$\dim T^1(H) = m^2 - (i+1)m + \frac{i(i+1)}{2} + 2\delta_{m,4} \text{ if } i \geqq 3\ ,$$
$$= m^2 - 3m + 6 - \delta_{m,4} - \delta_{m,5} \text{ if } i = 2\ .$$

If $c(H) = 2m + 2$ then

$$\dim T^1(H) = \frac{(m-1)m}{2} + 2 + 3\delta_{m,3} + 2\delta_{m,4}\ .$$

4. The obstruction of the formal moduli space. Let $B = B_H$ be negatively graded and let $T|S$ represent the versal deformation of $B|k$ in the sense of Schlessinger [6]. Then (S, m_S) is a complete noetherian k-algebra with residue field k. T is flat as an S-module and $T \otimes_S k \cong B$.

Pinkham [3] has shown that $T|S$ admit gradings as k-algebras which are compatible with the structure of B as a graded k-algebra. One then has the isomorphism $T^1(B) \cong \mathrm{Hom}_k(m_S/m_S^2, k)$ in the category of graded k-vector spaces. Thus $\dim T^1(B)$ also is the dimension of the tangent space $(m_S/m_S^2)^*$ of the formal moduli space $\mathrm{Spec}\,(S)$.

We say the formal moduli space is *unobstructed* if S is a regular local ring. Now S is regular if and only if Krull-dim $S = \dim(m_S/m_S^2)$ if and only if S is formally smooth over k ([2], Proposition 28. M). Thus the formal moduli space is unobstructed if and only if $\dim T^1(B) =$ Krull-dim S.

Let U denote that open subset of Spec (S) cansisting of all points having smooth fibers, i.e., $U = \{x \in \mathrm{Spec}(S) \mid T(x)$ is smooth over $\kappa(x)\}$ where $T(x) = T \otimes_S \kappa(x)$ and $\kappa(x) = A_x/\mathfrak{p}_x A_x$.

In [5] we showed that U is nonempty (as B can be smoothed) and effectively computed the dimension of U. We note that

$$\dim U \leqq \dim \mathrm{Spec}(S) \leqq \dim T^1(B) .$$

Hence Spec (S) is unobstructed iff $\dim U = \dim T^1(B)$.

We now recall the dimension formula for U and compare dim U to $\dim T^1(B)$.

If H is a numerical semigroup let $\mathrm{End}(H) = \{n \in N \mid n + H^+ \subset H\}$ where $H^+ = H - \{0\}$. Let $\lambda(H) = [\mathrm{End}(H): H]$ so that $1 \leqq \lambda(H) \leqq g(H) = g$.

PROPOSITION 4.1. *If H is negatively graded with $\lambda(H) = \lambda$, $g(H) = g$ and U is as above then*

$$\dim U = 2g + \lambda - 1 .$$

Proof. See [5], proof of Corollary 6.3.

Now suppose that H is ordinary or hyperordinary of multiplicity m with $a(H) = pm + i$ (recall that $a(H) = \inf\{H - mN\}$). Then $g(H) = p(m-1) + i - 1$ and $\lambda(H) = m - 1$ ([5], Proposition 2.2). Thus $\dim U = 2g + \lambda - 1 = (2p+1)(m-1) + 2i - 3$. Combining this with Corollary 3.4 we obtain:

PROPOSITION 4.2. *Suppose that H is ordinary or hyperordinary of multiplicity m with $a(H) = pm + i$. Then*

$$\begin{aligned} \dim T^1(H) - \dim U &= p(m-1)(m-3) + i(i-4) + 3 + \delta_{i,2} \quad \textit{if } m \geqq 3 , \\ &= 0 \quad \textit{if } m = 2 . \end{aligned}$$

Consequently the formal moduli space for B_H is unobstructed iff $m \leqq 3$.

Now suppose H is negatively graded of the third type with $m(H) = m$ and $m + i$ a gap for H. Then $g(H) = m + \delta_{i,1}$ and $\lambda(H) = m - i - \delta_{i,1}(m-2)$ ([5], Proposition 2.2). Hence $\dim U = 2g + \lambda - 1 = 3m - i - 1 - \delta_{i,1}(m-4)$. Combining this with Corollary 3.8 we obtain:

PROPOSITION 4.3. *Suppose that $H = H_m - \{m + i\}$ where H_m is ordinary and $2 \leqq i \leqq m - 1$. Let U be as above. Then*

$$\dim T^1(H) - \dim U = (m-3)^2 - \delta_{m,4} - \delta_{m,5} \quad \text{if } i = 2 ,$$
$$= m^2 - (i+4)m + \frac{(i+1)(i+2)}{2} + 2\delta_{m,4} \quad \text{if } i \geqq 3 .$$

If $H = H_m - \{m + 1, 2m + 1\}$ then

$$\dim T^1(H) - \dim U = \frac{m(m-5)}{2} + 3\delta_{m,3} + 2\delta_{m,4} .$$

Summarizing, the formal moduli space for B_H is unobstructed iff $m \leqq 4$ or $m = 5$ and $i \neq 2$ (i.e., $m + 2 \in H$).

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