PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

THE CASE OF EQUALITY IN THE MATRIX-VALUED TRIANGLE INEQUALITY

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This paper presents an analysis of the case of equality in the matrix-valued triangle inequality. There is complete analogy with the case of equality in the usual scalar triangle inequality.

In order to describe our assertion more precisely, let A and B be n-square complex matrices, and by $|A|$ denote the positive semidefinite Hermitian matrix

$$|A| = (AA^*)^{1/2} ,$$

where A^* is the adjoint of A. It has been speculated several times in the literature that this inequality should "naturally" hold:

$$|A + B| \leqq |A| + |B| ,$$

where the inequality sign signifies that the right hand side minus the left hand side is positive semidefinite. This inequality is false, however, as easy 2×2 examples show. Nevertheless, there is a valid matrix valued triangle inequality. It was discovered in [1], and takes the form

$$|A + B| \leqq U|A|U^* + V|B|V^* \qquad (1)$$

for appropriately chosen unitary matrices U and V (dependent upon A and B). However, no analysis of a "case of equality" for (1) was given in [1], and the purpose of this note is to supply such an analysis. Specifically, we have:

THEOREM 1. *The inequality sign in* (1) *must be equality if A and B have polar decompositions with a common unitary factor.*

THEOREM 2. *Suppose A and B are such that inequality* (1) *can hold only with the equality sign. Then A and B have polar factorizations with a common unitary factor.*

Proof of Theorem 1. We have $A = WH$ and $B = WK$, where W is unitary and H, K are positive semidefinite Hermitian. From (1) we easily deduce that

$$H + K \leqq U_1HU_1^* + V_1KV_1^* ,$$

where U_1, V_1 are unitary. Thus the matrix $U_1HU_1^* + V_1KV_1^* - (H + K)$ is positive semidefinite; but its trace is zero, so it can only be zero.

Proof of Theorem 2. We have to refer to the proof of the matrix triangle inequality in [1]. Let $C = A + B$. After multiplying C, A, and B by a unitary factor to make C positive semidefinite, and renaming the resulting matrices as C, A, B, again, the proof considers the expression

$$C = \frac{1}{2}(A + A^*) + \frac{1}{2}(B + B^*)\ ,$$

then uses $1/2(A + A^*) \leqq U|A|U^*$ for an appropriate unitary U, and a similar fact for B. The hypothesis in the theorem implies that we must have $1/2(A + A^*) = U|A|U^*$ (so that $1/2(A + A^*)$ is necessarily positive semidefinite). Squaring and taking traces, we get

$$\operatorname{tr}\left(\frac{A + A^*}{2}\right)^2 = \operatorname{tr} AA^* = \frac{\operatorname{tr} AA^* + \operatorname{tr} A^*A}{2}\ .$$

Hence

$$0 = \operatorname{tr}(A - A^*)(A^* - A)\ ,$$

so that $||A - A^*||^2 = 0$. Therefore A is Hermitian. Since $1/2(A + A^*)$ is semidefinite, A is semidefinite Hermitian. Similarly, so is B. That is to say: after multiplying the original A, B, C by a unitary matrix to make C semidefinite, A and B then also become semidefinite. This completes the proof.

REFERENCE

1. R. C. Thompson, *Convex and concave functions of singular values of matrix sums*, Pacific J. Math., **16** (1976), 285-290.

Received September 6, 1978. The preparation of this paper was supported in part by the U.S. Air Force, Grant 77-3166.

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