PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

BANACH SPACES WITH POLYNOMIAL NORMS

BRUCE REZNICK

A Banach space X is said to be in the class $\mathscr{P}_{2n}$ if, for all elements x and y, $||x + ty||^{2n}$ is a polynomial in real t. These spaces generalize L_{2n} and are precisely those Banach spaces in which linear identities can occur. We shall discuss further properties of $\mathscr{P}_{2n}$ spaces, often in terms of the permissible polynomials $p(t) = ||x + ty||^{2n}$. For each n, the set of such polynomials forms a cone. All spaces in $\mathscr{P}_2$ are Hilbert spaces. If X is a two-dimensional real space in $\mathscr{P}_4$, then it is embeddable in L_4. This is not necessarily true for spaces with more dimensions or for $\mathscr{P}_{2n}$, $n \geqq 3$. The question of embeddability is equivalent to the classical moment problem. All spaces in $\mathscr{P}_{2n}$ are uniformly convex and uniformly smooth and thus reflexive. They obey generally weaker versions of the Hölder and Clarkson inequalities. Krivine's inequalities, shown to determine embeddability into L_p, $p \neq 2n$, fail in the even case.

1. Introduction. Throughout, we shall consider real Banach spaces, and, except where indicated, $L_{2n}(Y, \mu)$ with real-valued functions and real scalars. The phrase "X is embeddable in L_{2n}" is an abbreviation for "X is isometrically isomorphic to a subspace of $L_{2n}(Y, \mu)$ for some (Y, μ)." Although $\mathscr{P}_{2n}$ was introduced and motivated in [11], that paper and this one are largely independent.

2. Norm functions. Suppose $X = \langle x_1, \cdots, x_m \rangle$ is the real vector space spanned by the x_i's and ϕ is a real function of m real variables. Under what circumstances does $||\Sigma u_i x_i|| = \phi(u_1, \cdots, u_m)$ make $(X, ||\cdot||)$ a Banach space? For $\boldsymbol{u} = (u_1, \cdots, u_m)$, let $\phi(\boldsymbol{u}) = \phi(u_1, \cdots, u_m)$. From the standard definition of the norm, it is evident that conditions (A), (B) and (C) are necessary and sufficient. (Here, t is an arbitrary real.)

(A) $\phi(\boldsymbol{u}) \geqq 0$ and $\phi(\boldsymbol{v}) = 0$ implies $\phi(\boldsymbol{u}) \equiv \phi(\boldsymbol{u} + t\boldsymbol{v})$

(B) $\phi(t\boldsymbol{u}) = |t|\phi(\boldsymbol{u})$

(C) $\phi(\boldsymbol{u}) + \phi(\boldsymbol{v}) \geqq \phi(\boldsymbol{u} + \boldsymbol{v})$.

Condition (C) is cumbersome to verify; the following lemma simplifies matters.

LEMMA 1. *Conditions* (A), (B) *and* (C) *are equivalent to* (A), (B) *and* (D).

(D) *$\psi(t) = \phi(\boldsymbol{u} + t\boldsymbol{v})$ is a convex function in t for all $\boldsymbol{u}$ and $\boldsymbol{v}$.*

Proof. Assume (A), (B) and (C) and fix $\boldsymbol{u}$ and $\boldsymbol{v}$. Then for $0 \leqq$

$\lambda \leqq 1$, $\lambda\psi(t_0) + (1-\lambda)\psi(t_1) = \lambda\phi(\boldsymbol{u}+t_0\boldsymbol{v}) + (1-\lambda)\phi(\boldsymbol{u}+t_1\boldsymbol{v}) = \phi(\lambda\boldsymbol{u}+\lambda t_0\boldsymbol{v}) + \phi((1-\lambda)\boldsymbol{u} + (1-\lambda)t_1\boldsymbol{v}) \geqq \phi(\boldsymbol{u} + (\lambda t_0 + (1-\lambda)t_1)\boldsymbol{v}) = \psi(\lambda t_0 + (1-\lambda)t_1)$. Conversely, assume (A), (B) and (D), then $\phi(\boldsymbol{u}) + \phi(\boldsymbol{v}) = \psi(0) + \psi(1) \geqq 2\psi(1/2) = \phi(\boldsymbol{u} + \boldsymbol{v})$.

Observe that it is sufficient to check ϕ on all two-dimensional subspaces of X. For a discussion of a different condition on two-dimensional subspaces, see Dor [2]. We shall consider spaces X in $\mathscr{P}_{2n}$ for which $p(t) = ||x + ty||^{2n}$ is a polynomial in t of degree $2n$. When p is given in this way, we shall tacitly assume that $||sx + ty||^{2n} = s^{2n}p(t/s)$ for $s \neq 0$ and $||y||^{2n} = \lim_{t\to\infty} t^{-2n}p(t)$; that is, (B) is implicit.

THEOREM 1. *Suppose p is a nonnegative polynomial of degree $2n$. Let $X = \langle x, y\rangle$ and define $||\cdot||$ on X by $p(t) = ||x + ty||^{2n}$. Then $(X, ||\cdot||)$ is a Banach space if and only if $2np(t)p''(t) - (2n-1)(p'(t))^2 \geqq 0$ for all t.*

Proof. With $||sx + ty||^{2n}$ defined as above, we need verify (A) and (D). Suppose $(X, ||\cdot||)$ is a Banach space, then $\psi(t) = ||x + ty|| = p(t)^{1/2n} = f(t)$ is convex. If x and y are linearly dependent then $||x + ty|| = |a + bt|$, and for $p(t) = (a + bt)^{2n}$, $2npp'' = (2n - 1)(p')^2$. If x and y are linearly independent, then $f(t) > 0$ and f is convex if and only if $f''(t) = (2n)^{-2}(f(t))^{1-4n}(2np(t)p''(t) - (2n - 1)(p'(t))^2) \geqq 0$.

On the other hand, suppose $2np(t)p''(t) - (2n - 1)(p'(t))^2 \geqq 0$ and $||\cdot||$ is defined as above. If $||sx + ty|| = 0$ for $(s, t) \neq (0, 0)$ then either $p(t_0) = 0$ or $\lim t^{-2n}p(t) = 0$. As the hypothesized condition is translation-invariant, assume $t_0 = 0$ in the first case. Since $p(t) \geqq 0$ we have $p'(0) = 0$; let $p(t) = a_kt^k + o(t^k)$, $a_k \neq 0$, $k \geqq 2$, for small t. Then $2np(t)p''(t) - (2n - 1)(p'(t))^2 = -a_k^2k(2n - k)t^{2k-2} + o(t^{2k-2})$ hence $k = 2n$, $p(t) = a_{2n}t^{2n}$ and $(X, ||\cdot||)$ is a valid one-dimensional space. In the second case, let $p(t) = a_kt^k + o(t^k)$ for $k < 2n$, $a_k \neq 0$ and t large. Then $k = 0$ and $(X, ||\cdot||)$ is again one-dimensional.

Now suppose $p(t) > 0$. Let $\boldsymbol{u} = dx + by$, $\boldsymbol{v} = cx + ay$ be given; (D) will be satisfied provided $\psi(t)$ is convex, where

$$\psi^{2n}(t) = ||dx + by + t(cx + ay)||^{2n} = |ct + d|^{2n}p((at + b)/(ct + d)) .$$

(If $c = d = 0$, then ψ is a constant and so convex). Note that ψ^{2n} is again a positive polynomial of degree $2n$ so that ψ'' is continuous. It suffices, therefore, to check that $\psi''(t) \geqq 0$ for $t \neq -d/c$. As above, $\psi''(t) \geqq 0$ provided $2n\psi(t)\psi''(t) - (2n - 1)(\psi'(t))^2 \geqq 0$. A computation shows that this expression equals $(ad - bc)^2(ct + d)^{4n-4}(2np(u)p''(u) - (2n - 1)(p'(u))^2)$, where $u = (at + b)/(ct + d)$. Thus, if $2npp'' - (2n - 1)(p')^2 \geqq 0$ then every ψ is convex and $(X, ||\cdot||)$ is a Banach space.

It follows from Theorem 1 that the two-dimensional spaces in $\mathscr{P}_{2n}$ are characterized by $p(t) = \|x + ty\|^{2n}$, and that a study of such polynomials is appropriate. Note also that generators may be chosen to make any computations easier; in general, (D) must be separately verified for each two-dimensional subspace.

3. The cone P_{2n}. Let P_{2n} consist of all polynomials p of degree $2n$ for which $p(t) \geqq 0$ and $C_{2n}(p(t)) = 2np(t)p''(t) - (2n-1)(p'(t))^2 \geqq 0$. If $p(t) = \Sigma\binom{2n}{k}a_k t^k$, then

$$C_{2n}(p(t)) = 4n^2(2n-1)\left(\left(\Sigma\binom{2n}{k}a_k t^k\right)\left(\Sigma\binom{2n-2}{k}a_{k+2}t^k\right) - \left(\Sigma\binom{2n-1}{k}a_{k+1}t^k\right)^2\right).$$

We shall omit the subscript $2n$ when it is superfluous. As defined, $C_{2n}(p)$ is a polynomial with nominal degree $4n-2$; the coefficients for t^{4n-2} and t^{4n-3} actually vanish identically.

THEOREM 2. *The set P_{2n} is a closed cone.*

Proof. Suppose p is in P_{2n}. Then $C(p) \geqq 0$ and for $\lambda \geqq 0$, $\lambda p \geqq 0$ and $C(\lambda p) = \lambda^2 C(p)$ so λp is in P_{2n}. If p_i is in P_{2n}, then $p_1 + p_2 \geqq 0$ and $C(p_1 + p_2) = C(p_1) + C(p_2) + 2np_1''p_2 + 2np_1p_2'' - (4n-2)p_1'p_2'$. Since $p_ip_i'' \geqq 0$ we have $(2np_ip_i'')^{1/2} \geqq (2n-1)^{1/2}|p_i|$ so that $2np_1''p_2 + 2np_1p_2'' - (4n-2)p_1'p_2' = 2n((p_1''p_2)^{1/2} - (p_1p_2'')^{1/2})^2 + 4n(p_1p_1'')^{1/2}(p_2p_2'')^{1/2} - (4n-2)|p_1p_2| + (4n-2)(|p_1p_2| - p_1p_2) \geqq 0$. Thus, P_{2n} forms a cone.

Associate $p(t) = \Sigma\binom{2n}{k}a_k t^k$ with the element $(a_0, \cdots, a_{2n})$ in $\boldsymbol{R}^{2n+1}$ and pull back the usual topology. Convergence is then either pointwise or coefficientwise. If $\{p_m\}$ is a sequence of polynomials in P_{2n} and $p_m \to p$ then $C(p_m(t)) \to C(p(t))$. Hence P_{2n} is closed.

By the proof of Theorem 1, if $p(t)$ is in P_{2n} then so is

$$(ct + d)^{2n}p((at + b)/(ct + d)) .$$

For future reference, observe that, if p_1 and p_2 are in P_{2n} and $C((p_1 + p_2)(t_0)) = 0$ then $C(p_1(t_0)) = C(p_2(t_0)) = 0$, $p_1''(t_0)p_2(t_0) = p_1(t_0)p_2''(t_0)$ and $p_1'(t_0)p_2'(t_0) \geqq 0$.

Since P_{2n} is a cone, it is natural to study its extreme elements. For $q(t) = (bt + c)^{2n}$, $C_{2n}(q) \equiv 0$. Suppose $q = p_1 + p_2$, with p_i in P_{2n}. If $b = 0$, then p_1 and p_2 must both be nonnegative constants. Suppose $b \neq 0$, then we may normalize $b = 1$ so $q(t) = (t + c)^{2n}$, hence $p_1(-c) = p_2(-c) = 0$. As in the proof of Theorem 1, it follows that $p_i(t) = r_i(t+c)^{2n}$ so each p_i is a multiple of q. We have proved that $(bt+c)^{2n}$ is an extreme element in P_{2n}. Since P_{2n} is a cone, $\Sigma(b_k t + c_k)^{2n}$ is in P_{2n}. This is to be expected in light of Theorem 1 applied to the subspace of ℓ_{2n} generated by $(b_1, b_2, \cdots)$ and $(c_1, c_2, \cdots)$.

If $2n = 2$, $C_2(a_2 t^2 + 2a_1 t + a_0) = 4(a_0 a_2 - a_1^2)$ so that $p \geqq 0$ implies $C_2(p) \geqq 0$. Hence the extreme elements of P_2 are precisely $(bt + c)^2$. Surprisingly enough, the same is true for $2n = 4$.

THEOREM 3. *The extreme functions of P_4 are $(bt + c)^4$; indeed, if p is in P_4 then $p(t) = (b_0 t + c_0)^4 + (b_1 t + c_1)^4 + c_2^4$ for some b_i and c_i.*

Proof. Write $p(t) = \Sigma\binom{4}{k} a_k t^k$, then $(48)^{-1}C_4(p(t)) = (a_2 a_4 - a_3^2)t^4 + (2a_1 a_4 - 2a_2 a_3)t^3 + (a_0 a_4 + 2a_1 a_3 - 3a_2^2)t^2 + (2a_0 a_3 - 2a_1 a_2)t + a_0 a_2 - a_1^2$. If $p(t_0) = 0$, then, as before, $p(t) = a_4(t - t_0)^4$. If $C(p(t_0)) = 0$, then with $q(t) = p(t - t_0)$, $C(q(0)) = 0$. As the conclusion is invariant under translation, assume $t_0 = 0$. In this case, since $C(p) \geqq 0$, $a_0 a_2 = a_1^2$ and $a_0 a_3 = a_1 a_2$. As $a_0 = p(0) \neq 0$, let $a_1 = r a_0$, then $a_2 = r^2 a_0$ and $a_3 = r^3 a_0$. If $a_4 = r^4 a_0 + s$ then $C(p(t)) = s a_0 t^2 (rt + 1)^2$, so $s \geqq 0$ and $p(t) = a_0(rt + 1)^4 + st^4$. (In general $p(t) = a_0(r(t - t_0) + 1)^4 + s(t - t_0)^4$.) If the degree of $C(p(t))$ is less than four, then by a similar argument, $p(t) = a_4(t + r)^4 + s$, $s \geqq 0$. Finally, suppose that $C(p(t))$ is a positive quartic and let $p_\lambda(t) = p(t) - \lambda$, then $C(p_\lambda(t)) = C(p(t)) - 4\lambda p''(t)$. Since p'' is quadratic, and $pp'' > 0$, $(4p''(t))^{-1}C(p(t))$ is continuous, goes to infinity quadratically in t, and achieves a minimum $\lambda_0 > 0$ at $t = t_0$. Thus $p(t) - \lambda_0$ is in P_4, $C(p_{\lambda_0}(t_0)) = 0$; hence $p(t) = \lambda_0 + a_0(r(t - t_0) + 1)^4 + s(t - t_0)^4$, which may be rewritten as in the conclusion.

By considering $(ct + d)^4 p((at + b)/(ct + d))$ instead of p, we may replace c_2^4 by $s^4(ct + d)^4$ for any pre-selected c and d. It would be nice if this pattern continued for $2n \geqq 6$; unfortunately, this is not the case.

THEOREM 4. *If $n \geqq 3$ then there exists a polynomial p in P_{2n} which cannot be written $p(t) = \Sigma(b_k t + c_k)^{2n}$.*

Proof. Fix n and let $p(t) = t^{2n} + t^2 + 1$. A computation shows that $C_{2n}(p(t)) = (8n^3 - 20n^2 + 12n)t^{2n} + (8n^3 - 4n^2)t^{2n-2} + (4 - 4n)t^2 + 4n$. Since $n \geqq 3$, each term but $(4 - 4n)t^2$ is positive. For $|t| \leqq 1$, $(4 - 4n)t^2 + 4n \geqq 0$; for $|t| \geqq 1$, $(8n^3 - 4n^2)t^{2n-2} + (4 - 4n)t^2 > (8n^3 - 4n^2 - 4n)t^2 > 0$. Thus $C_{2n}(p(t)) \geqq 0$ and p is in P_{2n}.

Suppose $t^{2n} + t^2 + 1 = \Sigma(b_k t + c_k)^{2n}$; from the coefficient of t^4 and t^2, $0 = \Sigma b_k^4 c_k^{2n-4}$ and $1 = \binom{2n}{2}\Sigma b_k^2 c_k^{2n-2}$. Since $n \geqq 3$, the first implies that $b_k c_k = 0$ for each k, and this contradicts the second.

The coefficient 1 for t^2 is not the best possible. The following proposition provides a sharp estimate.

PROPOSITION 1. *If* $t^{2n} + \alpha t^{2k} + 1$ *is in* P_{2n}, *then*

$$0 \leqq \alpha \leqq 2n(2n-1)c(k, n) ,$$

where $(c(k, n))^n = (2k)^{-k}(2n-2k)^{k-n}(2k-1)^{n-2k}(2n-2k-1)^{2k-n}$.

Outline of proof. Suppose $p_\alpha(t) = t^{2n} + \alpha t^{2k} + 1$ has the largest α, then $C_{2n}(p_\alpha(t)) \geqq 0$ and $C_{2n}(p_\alpha(t_0)) = 0$ for some t_0. Hence the derivative vanishes at t_0 as well. This gives two quadratic equations in α which may be solved simultaneously. After eliminating an extraneous solution, the bound is derived.

We see then that there are extreme functions in P_{2n}, $n \geqq 3$, which are not of the form $(bt + c)^{2n}$.

PROPOSITION 2. *The extreme rays of* P_6 *are generated by*

$$(ct+d)^{2n} f_\lambda((at + b)/(ct + d)) ,$$

where $f_\lambda(t) = t^6 + 6\lambda t^5 + 15\lambda^2 t^4 + 20\lambda^3 t^3 + 15\lambda^2 t^2 + 6\lambda t + 1$, *and* $|\lambda| \leqq 1/2$ *or* $|\lambda| = 1$.

Outline of proof. As in Theorem 3, we consider special cases and then subtract various $(ct + d)^6$'s. Then f_λ are those polynomials for which $C_6(f_\lambda(0)) = 0$ and $C_6(f)$ is at most quartic.

As Proposition 2 is not directly relevant to the rest of this paper and its proof is tedious, we omit the details. The general question of finding the extreme rays of P_{2n} for $n \geqq 4$ remains open.

Let Q_{2n} denote the closure of the cone of polynomials of the form $\sum_{j=1}^{R} (b_j t + c_j)^{2n}$; $Q_{2n} \subseteqq P_{2n}$ with equality if and only if $2n = 2$ or 4. As any $2n + 2$ distinct $2n$th powers are linearly dependent, we may assume that $R \leqq 2n + 1$. Suppose $q(t) = \Sigma \binom{2n}{k} a_k t^k$ is in Q_{2n}. Then $q = \lim q_m$, where $q_m(t) = \sum_{j=1}^{2n+1} (b_j^{(m)} t + c_j^{(m)})^{2n}$. Since $\Sigma (b_j^{(m)})^{2n} \to a_{2n}$ and $\Sigma (c_j^{(m)})^{2n} \to a_0$, we may take $|b_j^{(m)}| < M$, $|c_j^{(m)}| < M$. Thus there exists a convergent subsequence with limit b_j and c_j so that one may write $q(t) = \sum_{j=1}^{2n+1} (b_j t + c_j)^{2n}$ for all q in Q_{2n}. Similar considerations apply for the generalization of Q_{2n} to several variables.

4. Subspaces of L_{2n}. In [11] we showed that $L_{2n}(Y, \mu)$ is in $\mathscr{P}_{2n}$, that is, $\|f + tg\|^{2n} = \int |f + tg|^{2n} d\mu$ is a polynomial in t for all f and g. The converse, as we shall see, is false. Suppose that

$X = \langle x, y \rangle$ is a two-dimensional space in $\mathscr{P}_{2n}$, then $p(t) = \|x + ty\|^{2n}$ is in P_{2n}. Suppose that X is embeddable in $L_{2n}(Y, \mu)$, then $p(t) = \Sigma\binom{2n}{k}a_k t^k = \int (f + tg)^{2n} d\mu = \|f + tg\|^{2n}$. By Hölder's inequality, since $\int f^{2n} d\mu < \infty$ and $\int g^{2n} d\mu < \infty$, $\int f^{2n-k} g^k d\mu < \infty$ so that the integral can be broken up and $a_k = \int f^{2n-k} g^k d\mu$. Let $Y_0 = \{s \in Y: f(s) = 0\}$, $Z = Y - Y_0$; let $d\nu = f^{2n} d\mu$ and $h = gf^{-1}$ on Z. Then we have $a_k = \int h^k d\nu$, $0 \leqq k \leqq 2n - 1$, and $a_{2n} = \int_Z h^{2n} d\nu + \int_{Y_0} g^{2n} d\mu$. If $\Phi(r) = \nu(h^{-1}\{(-\infty, r]\})$, then $a_k = \int_{-\infty}^{\infty} s^k d\Phi$ for $0 \leqq k \leqq 2n - 1$ and $a_{2n} \geqq \int_{-\infty}^{\infty} s^{2n} d\Phi$.

Conversely, suppose there exists a nonnegative measure Φ and a_k's so that $a_k = \int_{-\infty}^{\infty} t^k d\Phi$ and $a_{2n} \geqq \int_{-\infty}^{\infty} t^{2n} d\Phi$. Define (Y, μ) as follows: $Y = \boldsymbol{R} \cup \{p_0\}$, $\mu = \Phi$ on $\boldsymbol{R}$ and $\mu\{p_0\} = a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\Phi$. Let $(f(s), g(s)) = (1, s)$ on $\boldsymbol{R}$ and $(0, 1)$ on $\{p_0\}$. Then $\|f + tg\|^{2n} = \int_{-\infty}^{\infty} (1 + st)^{2n} d\Phi + \left(a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\Phi\right)t^{2n} = \Sigma\binom{2n}{k}t^k \int_{-\infty}^{\infty} s^k d\phi + \left(a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\phi\right)t^{2n} = \Sigma\binom{2n}{k}a_k t^k = p(t)$.

Fortunately, this transforms the embedding problem into the classical moment problem, which has been studied extensively. The complete solution is known, see for example Akhiezer [1] p. 71, and we may combine this solution with the previous discussion to obtain the following theorem.

THEOREM 5. *Let X be a two-dimensional Banach space in $\mathscr{P}_{2n}$ with generators x and y and let $p(t) = \|x + ty\|^{2n} = \Sigma\binom{2n}{k}a_k t^k$. Define the $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ by $b_{ij} = a_{i+j}$ for $0 \leqq i, j \leqq n$. Then X is embeddable in L_{2n} if and only if the matrix B is positive semidefinite. Further, X is embeddable in L_{2n} if and only if p is in Q_{2n}.*

Proof. The positive semidefiniteness of B is equivalent to the solution of the described moment problem. If p is in Q_{2n} then X is embeddable in ℓ_{2n}^{2n+1} in the obvious fashion. If X is embeddable in L_{2n}, then by approximating $d\Phi$ by a sequence of point masses, we see that p is in Q_{2n}.

COROLLARY 6. *If X is two-dimensional space in $\mathscr{P}_4$, then X is embeddable in L_4. There are two-dimensional spaces in $\mathscr{P}_{2n}$, $n \geqq 3$, which are not embeddable in L_{2n}.*

Proof. Combine Theorems 3, 4 and 5.

The case for higher dimensions is less clearcut. Professor J. H. B. Kemperman [6] has pointed out, using techniques from [4] and [5], that the analogous moment problem in more than one variable has a solution which requires knowledge of all polynomials $f(u_1, \cdots, u_p)$ of total degree $2n$ which are nonnegative for all real u_i.

Specifically, one transforms the polynomial $p(t_1, \cdots, t_p) = \|x_0 + t_1x_1 + \cdots + t_px_p\|^{2n}$ for a space $X = \langle x_0, \cdots, x_p\rangle$ into a family of equations $a(m_1, \cdots, m_p) = \int \cdots \int t_1^{m_1} \cdots t_p^{m_p} d\mu$; $m_1 + \cdots + m_p < 2n$, with inequality if $\Sigma m_i = 2n$. Suppose $f(u_1, \cdots, u_p) \geqq 0$ for all real u_i and $f(u_1, \cdots, u_p) = \Sigma b(m_1, \cdots, m_p)u_1^{m_1} \cdots u_p^{m_p}$, where the sum is taken over all m_i, $\Sigma m_i \leqq 2n$. Then certainly $\int \cdots \int f(u_1, \cdots, u_p)d\mu = \Sigma a(m_1, \cdots, m_p)b(m_1, \cdots, m_p) \geqq 0$. It turns out this condition holding for all such f is sufficient for the existence of a measure with the desired property.

Since X is real, it is unreasonable to embed X in an L_{2n} space with complex scalars; one might, however, embed X in an $L_{2n}(Y, \mu)$ space with real scalars but complex-valued functions. This situation is taken care of by the following theorem.

THEOREM 7. *There is an isometry from the space of all complex-functions in $L_{2n}(Y, \mu)$, taken with real scalars, into real $L_{2n}(Z, \nu)$, where (Z, ν) consists of $2n + 1$ copies of (Y, μ).*

Proof. It is well known that ℓ_2^2 is embeddable in any infinite-dimensional Banach space. Let x and y be orthogonal generators of ℓ_2^2 and let $\bar{x}$ and $\bar{y}$ be their isometric images in ℓ_{2n}. Then $(t^2 + u^2)^n = \|tx + uy\|^{2n} = \|t\bar{x} + u\bar{y}\|^{2n} = \Sigma(b_kt + c_ku)^{2n}$; by the remarks at the end of §3, we may say that $(t^2 + u^2)^n = \sum_{k=1}^{2n+1} (b_kt + c_ku)^{2n}$. Define the mapping ϕ from $L_{2n}(Y, \mu)$ with complex-valued functions to $L_{2n}(Z, \nu)$ as follows: if $f = g + ih$ is the decomposition into real and imaginary parts, then $\phi(f) = b_kg + c_kh$ on the kth copy of (Y, μ). For real λ_i, $\phi(\lambda_1f_1 + \lambda_2f_2) = \lambda_1\phi(f_1) + \lambda_2\phi(f_2)$; $\|\phi(f)\|^{2n} = \sum_{k=1}^{2n+1}\int_Y (b_kg + c_kh)^{2n}d\mu = \int_Y (g^2 + h^2)^n d\mu = \int_Y |f|^{2n}d\mu = \|f\|^{2n}$ so ϕ is an isometry.

We may actually choose b_k and c_k by: $b_k + ic_k = a(n)\exp(2\pi ki(n+1)^{-1})$, where $a(n) = 2\left(\binom{2n}{n}(2n + 1)\right)^{-1/2n}$. Hilbert has proved that b_k and c_k may be chosen to be rational; see Ellison [3] p. 11 for an extended discussion. In any case, it suffices to consider embeddings into real L_{2n}.

5. A counterexample. The remaining case for embedding is the three-dimensional one for $\mathscr{P}_4$. We shall construct a three-dimensional space in $\mathscr{P}_4$ which is not embeddable in L_4. Consequently, there are spaces with arbitrarily large dimensions which are not embeddable in L_4. This example is drastically simplified from the one appearing in the author's thesis.

Suppose $X = \langle x, y, z\rangle$ and a polynomial $p(u, v)$ with total degree 4 is given. Let $\|\cdot\|$ be defined on X by $\|x + uy + vz\|^4 = p(u, v)$; $\|tx + uy + vz\|^4$ for $t \neq 1$ is defined in the usual way. In view of Lemma 1, we need check (A), (B) and (D) on every two-dimensional subspace of X. Conditions (A) and (B) will be automatic. A two-dimensional subspace of X is either $\langle y, z\rangle$ or $\langle x + ay + cz, by + dz\rangle$ for some a, b, c, d. Thus, for $f(u, v) = (p(u, v))^{1/4}$, it suffices to show that $\psi(t) = f(a + bt, c + dt)$ is convex for all a, b, c, d. (We consider $\langle y, z\rangle$ separately.) Adopt the usual convention that $f_1(u, v) = (\partial/\partial u)f(u, v)$, $f_{22}(u, v) = (\partial^2/\partial v^2)f(u, v)$, etc. Then $\psi''(t) = (b^2 f_{11} + 2bdf_{12} + d^2 f_{22})(a + bt, c + dt)$. Hence it suffices to show that $f_{11} \geqq 0, f_{22} \geqq 0$ and $f_{11}f_{22} \geqq f_{12}^2$ at all points in the plane. If we can verify this for $f = p^{1/4}$ then $(X, \|\cdot\|)$ will be a Banach space.

THEOREM 8. *For* $X = \langle x, y, z\rangle$, *let* $\|tx + uy + vz\|^4 = t^4 + 6t^2(u^2 + v^2) + (u^2 + v^2)^2$. *Then* $(X, \|\cdot\|)$ *is a Banach space which is not embeddable in* L_4.

Proof. Note that $\|tx + uy + vz\| > 0$ unless $t = u = v = 0$ so that (A) is satisfied. On $\langle y, z\rangle$, $\|uy + vz\| = (u^2 + v^2)^{1/2}$ so $\langle y, z\rangle$ is isometric to ℓ_2^2 and (D) is satisfied. In general, let $f = p^{1/4}$, then $16f_{11} = p^{-7/4}(4pp_{11} - 3p_1^2)$, $16f_{22} = p^{-7/4}(4pp_{22} - 3p_2^2)$ and $16f_{12} = p^{-7/4}(4pp_{12} - 3p_1p_2)$. We must show that $4pp_{ii} - 3p_i^2 \geqq 0$ and that

$$\begin{aligned}(4pp_{11} &- 3p_1^2)(4pp_{22} - 3p_2^2) - (4pp_{12} - 3p_1p_2)^2 \\ &= 4p(4p(p_{11}p_{22} - p_{12}^2) - 3p_1^2p_{22} + 6p_1p_2p_{12} - 3p_2^2p_{11}) \\ &= 4pD(p) \geqq 0 .\end{aligned}$$

For $p(u, v) = \|x + uy + vz\|^4 = 1 + 6(u^2 + v^2) + (u^2 + v^2)^2$ let $w = u^2 + v^2$, then $p = 1 + 6w + w^2$, $p_1 = 4u(3 + w)$, $p_2 = 4v(3 + w)$, $p_{11} = 4(3 + w + 2u^2)$, $p_{12} = 8uv$, $p_{22} = 4(3 + w + 2v^2)$. Hence

$$4pp_{11} - 3p_1^2 = 16(3(1 - u^2)^2 + v^2(19 + 12u^2 + u^4) + v^4(9 + 2u^2) + v^6) \geqq 0$$

and similarly $4pp_{22} - 3p_2^2 \geqq 0$. Further, $p_{11}p_{22} - p_{12}^2 = 48(w + 3)(w + 1)$ and $p_1^2p_{22} - 2p_1p_2p_{12} + p_2^2p_{11} = 64w(w + 3)^3$, hence

$$\begin{aligned}D(p) &= 192(w + 3)(w + 1)(w^2 + 6w + 1) - 192w(w + 3)^3 \\ &= 192(w + 3)(w - 1)^2 \geqq 0 .\end{aligned}$$

Thus $(x, \|\cdot\|)$ is a Banach space.

If X were embeddable in L_4, then for some f, g and h, $t^4 + 6t^2(u^2+v^2) + (u^2+v^2)^2 = \int_Y (tf + ug + vh)^4 d\mu$, so $\int f^4 = \int g^4 = \int h^4 = \int f^2g^2 = \int f^2h^2 = 1$, $\int g^2h^2 = 1/3$. The first five equations imply that $f^2 = g^2$ and $f^{\circ} = h^2$ μ – a.e.; this is contradicted by the sixth. Alternatively, in the spirit of the moment problem, $0 \leqq \int_Y (f^2 - g^2 - h^2)^2 d\mu = -1/3$. Either proof shows that X is not embeddable in L_4.

One can make a lengthy plausibility argument that the set of polynomials $p(t, u, v) = \|tx + uy + vz\|^4$ has 15 degrees of freedom for spaces in $\mathscr{P}_4$ and 14 for spaces in L_4. The last degree of freedom manifests itself here as the coefficient of u^2v^2.

6. **Other properties of $\mathscr{P}_{2n}$.** Since $Q_{2n} \subseteqq P_{2n}$, with strict inclusion for $n \geqq 3$, it is not obvious that spaces in $\mathscr{P}_{2n}$ are necessarily as "nice" as spaces in L_{2n}. For example, $L_{2n}(Y, \mu)$ is uniformly convex and uniformly smooth (see Lindenstrauss and Tzafriri [10] p. 127 for definition) and hence reflexive. Hölder's inequality says that, if $\int f^{2n} = \int g^{2n} = 1$, then $\left|\int f^k g^{2n-k}\right| \leqq 1$ for $0 \leqq k \leqq 2n$. Thus if $q(t) = 1 + \sum_{k=1}^{2n-1}\binom{2n}{k}a_k t^k + t^{2n}$ is in Q_{2n}, then $|a_k| \leqq 1$; indeed, $1 \geqq a_k \geqq r(k)$, where $r(2j) = 0$, $r(2j+1) = -1$. Clarkson's inequality states that $\|f+g\|^{2n} + \|f-g\|^{2n} \geqq 2(\|f\|^{2n} + \|g\|^{2n})$; if $q(t) = \sum_{k=0}^{2n}\binom{2n}{k}a_k t^k$ is in Q_{2n}, then $q(1) + q(-1) \geqq 2(q(0) + a_{2n})$. As a whole, these properties extend to $\mathscr{P}_{2n}$, although numerical constants are generally weaker.

Koehler [7] defined a G_{2n} space to be a Banach space on which a $2n$-fold inner product $\langle x_1, \cdots, x_{2n}\rangle$ is defined, satisfying certain regularity conditions. In [11] it was shown that G_{2n} spaces and $\mathscr{P}_{2n}$ spaces coincide. Koehler [8] proved that G_{2n} spaces are uniformly convex. That is, $\mathscr{P}_{2n}$ spaces are uniformly convex and thus reflexive. To prove uniform smoothness and the other regularity conditions we need the analogue to Hölder's inequality.

THEOREM 9. *If* $p(t) = 1 + \sum_{k=1}^{2n-1}\binom{2n}{k}a_k t^k + t^{2n}$ *is in* P_{2n}, *then there are constants so that* $m(k, 2n) \leqq a_k \leqq M(k, 2n)$.

Proof. Since $p^{1/2n}(t)$ is convex, by the triangle inequality on the space induced by p, $(1 - |t|)^{2n} \leqq p(t) \leqq (1 + |t|)^{2n}$, so for $t \geqq 0$, $(t-1)^{2n} \leqq p(t) \leqq (t+1)^{2n}$. The set of $2n-1$ equations $\sum_{k=1}^{2n-1}\binom{2n}{k}a_k j^k = p(j) - 1 - j^{2n}$, $1 \leqq j \leqq 2n-1$, has a Vandermonde determinant, hence $\binom{2n}{k}a_k$ may be expressed in terms of $p(j) - 1 - j^{2n}$. Since $p(j)$ is bounded one obtains bounds on a_k which are, in general, wildly generous.

Alternatively, a sequence of polynomials with unbounded a_k's has a subsequence from which can be deduced the existence of $\bar{p}$ in P_{2n}, $\bar{p}(t) = \sum_{k=1}^{2n-1} \binom{2n}{k} \bar{a}_k t^k$, not all $\bar{a}_k$'s equal to zero. This yields a contradiction.

It follows that the set of all points $(a_1, \cdots, a_{2n-1})$, A, in $\boldsymbol{R}^{2n-1}$ so that $1 + \sum_{k=1}^{2n-1} \binom{2n}{k} a_k t^k + t^{2n}$ is in P_{2n} forms a closed (Theorem 2) and bounded (Theorem 9) set. Thus functionals, such as $p(1)$, achieve maxima and minima on A.

The actual values of $m(k, 2n)$ and $M(k, 2n)$ can be found in a few instances. Since $p(t)$ in P_{2n} implies $p(-t)$ and $t^{2n}p(1/t)$ are in P_{2n}, $m(2j + 1, 2n) = -M(2j + 1, 2n)$, $m(2n - k, 2n) = m(k, 2n)$ and $M(2n-k, 2n) = M(k, 2n)$. As L_{2n} spaces are in $\mathscr{P}_{2n}$, $M(k, 2n) \geqq 1$ and $m(k, 2n) \leqq r(k)$. These coefficients are a two-dimensional property; consequently $m(k, 2n)$ and $M(k, 2n)$ are already determined for $2n = 2$ or 4.

In any case, $a_1 = \lim_{t\to\infty} t^{-1}(||x + ty|| - ||x||)$, so $|a_1| \leqq 1$ and $M(1, 2n) = -m(1, 2n) = 1$. Further, $C(p(0)) = (2n)^2(2n - 1)(a_0 a_2 - a_1^2)$ so $a_2 \geqq 0$ and $m(2, 2n) = 0$. The condition in Theorem 9 is, for general p in P_{2n}, $a_k \leqq M(k, 2n)a_0^{1-\alpha}a_{2n}^{\alpha}$, where $\alpha = k/2n$. From the convexity of x^α, extreme values are attained on extreme elements in P_{2n}. In this way, considering Proposition 2, one can show that $M(3, 6) = -m(3, 6) = 1$ and $M(2, 6) = 5^{-5/3}(1565 + 496\sqrt{10})^{1/3} \cong 1.000905$. The general problem remains open.

THEOREM 10. *If X is in $\mathscr{P}_{2n}$ then X is uniformly convex, uniformly smooth and so is reflexive.*

Proof. The uniform convexity follows from Koehler, or by noting that $||x|| = ||y|| = 1$, $||x + y|| = 2$ implies $||x + ty|| = 1 + t$ for $t \geqq 0$ so $p(t) = (1 + t)^{2n}$ and $||x - y|| = 0$. Since the set of coefficients A_ε for which $||x|| = ||y|| = 1$, $||x - y|| \geqq \varepsilon$ is compact, $||x + y||$ achieves a maximum, which is strictly less than 2.

For uniform smoothness, let $||x|| = ||y|| = 1$. For $t \leqq \tau$, by Taylor's theorem, $||x + ty|| + ||x - ty|| = 2 + (2n - 1)(a_2 - a_1^2)t^2 + o(t^2)$. Thus $1/2(||x + ty|| + ||x - ty||) - 1 \leqq c\tau^2 + o(\tau^2)$ so X is uniformly smooth.

If X is any Banach space, suppose $t = ||y|| \geqq ||x|| = 1$ and $u = ||x + y|| \geqq ||x - y|| = v$. Then $u + v \geqq 2t$ so $u^p + v^p \geqq u^p + (2t - u)^p \geqq 2t^p \geqq t^p + 1$. That is, $||x + y||^p + ||x-y||^p \geqq ||x||^p + ||y||^p$ with equality if and only if $||x|| = ||y|| = ||x + y|| = ||x - y|| = 1$. In this case, by the triangle inequality, $\|x + ry\| \equiv 1$ for $|r| \leqq 1$ so X cannot be in $\mathscr{P}_{2n}$. Thus, by the compactness of A, $\|x + y\|^{2n} + \|x - y\|^{2n} \geqq c(n)(\|x\|^{2n} + \|y\|^{2n})$ for x and y in X in $\mathscr{P}_{2n}$. Taking $x = 0$, $c(n) \leqq 2$.

THEOREM 11. *If X is in $\mathscr{P}_{2n}$ for $n \leqq 3$ then $\|x + y\|^{2n} + \|x - y\|^{2n} \geqq 2(\|x\|^{2n} + \|y\|^{2n})$, but this is not necessary true for $n \geqq 4$.*

Proof. For $n \leqq 2$, X is embeddable in L_{2n}. For $n = 3$, let $\|x + ty\|^6 = \sum_{k=0}^{6} \binom{6}{k} a_k t^k$ then $\|x + y\|^6 + \|x - y\|^6 - 2\|x\|^6 - 2\|y\|^6 = 30(a_2 + a_4) \geqq 0$ since $m(2, 6) = m(4, 6) = 0$.

Fix $n \geqq 4$ and set $p_\varepsilon(t) = 1 + \varepsilon(t^2 - 3t^4 + t^6) + t^{2n}$ and $\|x + ty\|^{2n} = p_\varepsilon(t)$, then $\|x + y\|^{2n} + \|x - y\|^{2n} - 2(\|x\|^{2n} + \|y\|^{2n}) = -2\varepsilon > 0$ for $\varepsilon > 0$. A computation shows that $C_{2n}(p_\varepsilon(t)) = 4n^2(2n - 1)t^{2n-2} + \varepsilon(g(t) + \varepsilon h(t))$, where $g(t) = 4n^2(2n - 1)t^{2n-2}(t^2 - 3t^4 + t^6) + 2n(1 + t^{2n})(2 - 36t^2 + 30t^4) - 4n(2n - 1)t^{2n-1}(2t - 12t^3 + 6t^5)$ and $h(t) = 2n(t^6 - 3t^4 + t^2)(30t^4 - 36t^2 + 2) - (2n - 1)(6t^5 - 12t^3 + 2t)^2$.

As $n \geqq 4$, the highest order term of $g + \varepsilon h$ is

$$2n(4n^2 - 26n + 42)t^{2n+4} ,$$

there exist ε_0 and R so that for $0 \leqq \varepsilon \leqq \varepsilon_0$ and $|t| > R$, $(g + \varepsilon h)(t) \geqq 0$ and thus $C_{2n}(p_\varepsilon(t)) > 0$. As $(g + \varepsilon h)(0) = 4n$, for $0 \leqq \varepsilon \leqq \varepsilon_0$ and $|t| < \delta$ or $|t| > R$, $C_{2n}(p_\varepsilon(t)) > 0$. On the remaining (compact) set, t^{2n-2} is positive and $|g| + \varepsilon_0|h|$ is bounded, so for some further reduced range of ε, $C_{2n}(p_\varepsilon) > 0$ and p_ε is in P_{2n}.

For $n = 4$ take $\varepsilon = .04$, then $p_\varepsilon(t) = t^8 + .04t^6 - .12t^4 + .04t^2 + 1$. A direct computation shows that $C_8(p_\varepsilon(t)) = 64(t^{12} + 1) + 11.5392(t^{10} + t^2) + 9.68(t^8 + t^4) + 447.9104t^6$. If we factor out $.64t^6$ and let $u = t^2 + t^{-2}$, then we obtain $u^3 - 18.03u^2 + 12u + 735.92 = q(u)$. (The range for $t^2 + t^{-2}$ is $u \geqq 2$.) Clearly $q(2) > 0$, and q achieves its minimum when $u = u_0 = 6.01 + \sqrt{32.1201} \cong 11.67$. Since $q(u_0) \cong 9.79 > 0$, $C(4) \leqq 1.96$. This bound is not sharp. This example also shows that $m(4, 8) < 0$.

The question of describing spaces dual to spaces in $\mathscr{P}_{2n}$ also remains open. Indeed it is false, in general, that the dual space to a subspace of $L_p(Y, \mu)$ is necessarily embeddable in L_q, $p^{-1} + q^{-1} = 2$. For example, if $p = 2n/(2n - 1)$, $x = (1, 1, 0)$, $y = (1, 0, 1)$ and X is the subspace of ℓ_p^3 generated by x and y, then X^* is not even in $\mathscr{P}_{2n}$, let alone L_{2n}. We omit the proof.

7. **Krivine inequalities.** Krivine [9] has described necessary and sufficient conditions for a space to be embeddable in L_p provided p is not an even integer. Krivine's proof does not apply when $p = 2n$ because it involves the Taylor series remainder of $\cos x$. Theorem 12 discusses this case and provides an underlying reason for this

failure when viewed in conjunction with Corollary 6.

THEOREM (Krivine). *If* $2r - 2 < p < 2r \leqq 4k$ *then a necessary and sufficient condition for* X *to be embeddable in* L_p *is that* (1) *holds for all elements* x_i *and all choices of real scalars* r_i *with* $\Sigma r_i = 0$. *The sum is taken as the* i_j*'s range independently from* 1 *to* m *and as the* ε_j*'s range over all choices of sign* ± 1. *The sum has* $m^{2k}2^{2k-1}$ *terms.*

$$(1)\quad (-1)^r \sum_{i_1=1}^{m} \cdots \sum_{i_{2k}=1}^{m} r_{i_1} \cdots r_{i_{2k}} \sum_{\varepsilon_j} \|x_{i_1} + \varepsilon_2 x_{i_2} + \cdots + \varepsilon_{2k} x_{i_{2k}}\|^p \geqq 0 .$$

THEOREM 12. *If* $4k > 2n$ *and* X *is in* $\mathscr{P}_{2n}$, *then the sum in* (1), *taken with* $p = 2n$, *is identically zero.*

Proof. By Theorem 11 in [11], it suffices to verify any linear identity on one space in $\mathscr{P}_{2n}$, say $\boldsymbol{C}$. Since in (1) all elements are combined with real coefficients, by Theorem 7, we may embed $\boldsymbol{C}$ isometrically in $\boldsymbol{R}$. It therefore suffices to check that (2) holds in $\boldsymbol{R}$.

$$(2)\qquad \sum_{i_1=1}^{m} \cdots \sum_{i_{2k}=1}^{m} r_{i_1} \cdots r_{i_{2k}} \sum_{\pm} (t_{i_1} \pm t_{i_2} \pm \cdots \pm t_{i_{2k}})^{2n} = 0 .$$

Because of the signs in the inner sum, we may rewrite this in the form $\sum_j d_j t_{i_1}^{\pi_j(1)} \cdots t_{i_{2k}}^{\pi_j(2k)}$, where j indexes all partitions of $2n$ into $2k$ even integers and d_j is the positive multinomial coefficient. If we now exchange the order of summation, then (2) becomes (3).

$$(3)\qquad \sum_j d_j \prod_{s=1}^{2k} \left(\sum_{i_s=1}^{m} r_{i_s} t_{i_s}^{\pi_j(s)} \right) = 0 .$$

Fix j; since $4k > 2n$, at least one of the $\pi_j(s)$'s is zero. Thus, one term in the product is $\Sigma r_j = 0$, each term in the sum vanishes and (3) is verified.

For $2n \geqq 4$, there are spaces in $\mathscr{P}_{2n}$ which are not embeddable in L_4, so that Krivine's inequalities do not extend. For $4k = 2n$ and $X = L_{2n}(Y, \mu)$, it is not hard to show that the left hand side of (1) becomes $\left(\int \Sigma r_i x_i^2 d\mu\right)^{2k}$ which is nonnegative. If, on the other hand, X is the space in Theorem 8, $x_1 = x$, $x_2 = y$, $x_3 = z$, $r_1 = -2$, $r_2 = r_3 = 1$, then $\sum_1^3 \sum_1^3 r_i r_j \Sigma \|x_i \pm x_j\|^4 = -16$. It is possible that a careful study of Krivine's inequality for such borderline cases could lead to an embedding theorem for L_p, $p = 2n$.

ACKNOWLEDGMENTS. This work grew out of the author's Ph.D. thesis, produced at Stanford University under the direction of Per Enflo. The author expresses his gratitude both to Professor Enflo and to Professor J. H. B. Kemperman, who kindly answered a written

inquiry in great detail.

REFERENCES

1. Naum Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, London, 1965.
2. Leonard Dor, *Potentials and isometric embeddings in L_1*, Israel J. Math., **24** (1976), 260-268.
3. W. J. Ellison, *Waring's problem*, Amer. Math. Monthly, **78** (1971), 10-36.
4. J. H. B. Kemperman, *On the sharpness of Tchebycheff type inequalities*, Indag. Math., **27** (1965), 554-601.
5. ———, *On a class of moment problems*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. II, pp. 101-126, University of California Press, 1972.
6. ———, personal correspondence, Sept. 2, 1977.
7. Donald Koehler, *G_{2n} spaces*, Trans. Amer. Math. Soc., **150** (1970), 507-518.
8. ———, *The analytic properties of G_{2n} spaces*, Proc. Amer. Math. Soc., **35** (1972), 201-206.
9. Jean-Louis Krivine, *Plongement des espaces normés dans L^p*, C. R. Acad. Sci. (Paris), **261** (1965), 4307-4310.
10. Joram Lindenstrauss, and Lior Tzafriri, *Classical Banach Spaces*, Lecture Notes in Mathematics No. 338, Springer-Verlag, New York, 1973.
11. Bruce Reznick, *Banach spaces which satisfy linear identities*, Pacific J. Math., **74** (1978), 221-233.

Received October 14, 1977.

DUKE UNIVERSITY
DURHAM, NC 27706