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WORD EQUATIONS IN A BAND OF PATHS

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In this paper we introduce a multiplication of paths which yields an idempotent semigroup. We study the properties of this band and solve all word equations in this band.

Multiplying paths in a topological space by concatenation is a classical idea in algebraic topology. However, in $\boldsymbol{R}^n$, identifying up to homotopy trivializes all paths. There are many ways of obtaining associativity with less identification. Looking only at the images of the paths in $\boldsymbol{R}^n$, yields an inverse semigroup (cf. [6]). Lesser identifications lead to semigroups which locally resemble free semigroups [4, 5].

1. Preliminaries.

Throughout this paper, $\boldsymbol{R}_0$, $\boldsymbol{R}$, Z^+ will denote the sets of nonnegative reals, reals and positive integers, respectively. If S is a semigroup, then $S^1 = S$ if S has an identity element; $S^1 = S \cup \{1\}$ with obvious multiplication if S does not have an identity element. For basic notions of semigroups, see [1].

We will let $\mathscr{Y}$ denote the set of all strictly increasing continuous self-maps ϕ of $[0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. Let $n \in Z^+$ remain fixed throughout this paper and let $\mathscr{M}$ denote the set of all rectifiable, continuous functions f from $[0, 1]$ into $\boldsymbol{R}^n$ such that $f(0) = 0$ and f is not constant on any subinterval of $[0, 1]$. If $f \in \mathscr{M}$, then let $l(f)$ denote the length of f. If $f, g \in \mathscr{M}$, then let $f * g \in \mathscr{M}$ be defined by

$$f * g(x) = \begin{cases} f(2x) & , \quad 0 \leqq x \leqq 1/2 \\ f(1) + g(2x - 1) , & 1/2 \leqq x \leqq 1 . \end{cases}$$

For $f, g \in \mathscr{M}$, define $f \equiv g$ if $g = f \circ \phi$ for some $\phi \in \mathscr{Y}$. Note that if $f \equiv g$, then $l(f) = l(g)$. Intuitively, considered as a function of time, we are interested in the way our path is traced but not the speed. $\equiv$ is an equivalence relation on $\mathscr{M}$. Let $\overline{\mathscr{M}} = \mathscr{M}/\equiv$. The operation $*$ defined above remains well defined on $\overline{\mathscr{M}}$ and $(\overline{\mathscr{M}}, *)$ is a cancellative semigroup [4]. (Note that in [4], $*$ was not used to denote this operation.) If $f \in \mathscr{M}$, then let $\bar{f}$ denote the equivalence class of f and define $l(\bar{f}) = l(f)$. Then $l(\bar{f})$ is well defined. Let $f \in \mathscr{M}$. For $\alpha \in (0, 1]$, define $g(\alpha) = l(f_{[0,\alpha]})$ where $f_{[0,\alpha]}$ denotes the path from 0 to α (cf. [4]); $g(0) = 0$. By the usual arguments of analysis, g is continuous. So for any $\beta \in (0, l(f))$, there exists $\alpha \in$

$(0, 1]$ such that $l(f_{[0,\alpha]}) = \beta$. Let $\mathscr{B} = \bar{\mathscr{M}}^1$ and set $l(1) = 0$. It follows from the above that for any $a \in \mathscr{B}$, $\alpha \in [0, l(a)]$, there exist $b, c \in \mathscr{B}$ such that $a = b * c$, $l(b) = \alpha$, $l(c) = l(a) - \alpha$. Also note that for any $a, b \in \mathscr{B}$, $l(a * b) = l(a) + l(b)$. Let $a, b, c, d \in \mathscr{B}$ such that $a * b = c * d$, $l(a) = l(c)$. Then by [4], $a = c$ and $b = d$.

We now define a new operation on $\mathscr{B}$. Let $a, b \in \mathscr{B}$. First assume $l(a) \leqq l(b)$. Let $a = a_1 * a_2$, $b = b_1 * b_2$ where $l(a_1) = l(a_2) = l(b_1)$. Then define $ab = a_1 * b_2$. Next assume $l(b) \leqq l(a)$. Let $a = a_1 * a_2$, $b = b_1 * b_2$ where $l(a_2) = l(b_1) = l(b_2)$. Then define $ab = a_1 * b_2$. From now on when we talk about $\mathscr{B}$, it is to be understood that we are talking about $\mathscr{B}$ with respect to the operation just defined. Visually we can think of the paths colliding and exactly half of the smaller path and an equal part of the larger path destroying each other. Applications, outside of mathematics, of this and similar models will be developed at a later date.

THEOREM 1.1. *$\mathscr{B}$ is a band (idempotent semigroup) with identity* 1. *For any $a, b \in \mathscr{B}$, $l(ab) = \max. \{l(a), l(b)\}$. For any $a \in \mathscr{B}$, $\mathscr{B}_a = \{b \mid b \in \mathscr{B}, l(a) = l(b)\}$ is the rectangular band component of a. $\boldsymbol{R}_0$, with reversed order, is the maximal semilattice image of $\mathscr{B}$ and l is the corresponding homomorphism.*

Proof. First we show that $\mathscr{B}$ is associative. Let $a, b, c \in \mathscr{B}$. We will show $(ab)c = a(bc)$. First assume $l(a) \leqq l(b) \leqq l(c)$. There exist $a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathscr{B}$ such that $a = a_1 * a_2$, $b = b_1 * b_2 * b_3$, $c = c_1 * c_2 * c_3$, $l(a_1) = l(a_2) = l(b_1) = l(c_1)$, $l(b_2) = l(c_2)$ and $l(b_1 * b_2) = l(b_3)$. Then $ab = a_1 * b_2 * b_3$ and $(ab)c = a_1 * b_2 * c_3$. Also $bc = b_1 * b_2 * c_3$ and $a(bc) = a_1 * b_2 * c_3$. So a $(bc) = (ab)c$. Next assume $l(a) \leqq l(c) \leqq l(b)$. There exist $a_1, a_2, b_1, b_2, b_3, c_1, c_2 \in \mathscr{B}$ such that $a = a_1 * a_2$, $b = b_1 * b_2 * b_3$, $c = c_1 * c_2$, $l(a_1) = l(b_1) = l(a_2)$, $l(b_3) = l(c_1) = l(c_2)$. Then $ab = a_1 * b_2 * b_3$ and $(ab)c = a_1 * b_2 * c_2$. Also $bc = b_1 * b_2 * c_2$ and a $(bc) = a_1 * b_2 * c_2$. So $(ab)c = a(bc)$. This takes care of the case when a has smallest length. The case when c has smallest length is dual. So we are left with the case when b has smallest length. By right-left duality, we can assume $l(b) \leqq l(a) \leqq l(c)$. There exist $a_1, a_2, a_3, b_1, b_2, c_1, c_2, c_3 \in \mathscr{B}$ such that $a = a_1 * a_2 * a_3$, $b = b_1 * b_2$, $c = c_1 * c_2 * c_3$, $l(a_1) = l(a_2 * a_3) = l(c_1 * c_2)$ and $l(b_1) = l(b_2) = l(c_1) = l(a_3)$. Then $ab = a_1 * a_2 * b_2$ and $(ab)c = a_1 * c_3$. Also, $bc = b_1 * c_2 * c_3$ and a $(bc) = a_1 * c_3$. So $(ab)c = a(bc)$ and $\mathscr{B}$ is associative. It is clear that $a^2 = a$ for all $a \in \mathscr{B}$ and so $\mathscr{B}$ is a band. It is also clear that $l(ab) = \max\{l(a), l(b)\}$ for all $a, b \in \mathscr{B}$. Let $b, c \in \mathscr{B}_a$. Then $l(b) = l(c)$. There exist $b_1, b_2, c_1, c_2 \in \mathscr{B}$ such that $b = b_1 * b_2$, $c = c_1 * c_2$ and $l(b_1) = l(b_2) = l(c_1) = l(c_2)$. So $bc = b_1 * c_2$ and $bcb = b_1 * b_2 = b$. So each $\mathscr{B}_a$ is a rectangular band.

Conversely if R is any subrectangular band of B containing a

and if $b \in R$, then $bab = b$ so $l(b) = \max\{l(a), l(b)\}$, and hence $l(b) \geqq l(a)$. Similarly $aba = a$ implies $l(a) \geqq l(b)$, so $l(b) = l(a)$ and $R \subseteqq B_a$.

LEMMA 1.2. *Let $a, b, c \in \mathscr{B}$ such that $l(b) \leqq l(a)$ and $l(b) \leqq l(c)$. Then $abc = ac$.*

Proof. By symmetry, assume $l(a) \leqq l(c)$. There exist $a_1, a_2, b_1, b_2, c_1, c_2, c_3 \in \mathscr{B}$ such that $a = a_1 * a_2$, $b = b_1 * b_2$, $c = c_1 * c_2 * c_3$, $l(a_1) = l(c_1 * c_2) = l(a_2)$ and $l(b_1) = l(b_2) = l(c_1)$. So $bc = b_1 * c_2 * c_3$ and a $(ab) = a_1 * c_3$. Also, $ac = a_1 * c_3$. So $a(bc) = ac$.

LEMMA 1.3. *Let $a, b, c \in \mathscr{B}$ such that $l(a) \leqq l(b) \leqq l(c)$ and $ac = bc$. Then $ab = b$ and there exists $a' \in \mathscr{B}$ such that $l(a) = l(a')$ and $c = a'bc$.*

Proof. Let $a = a_1 * a_2$, $b = b_1 * b_2 * b_3$, $c = c_1 * c_2 * c_3$ with $l(a_1) = l(a_2) = l(b_1) = l(c_1)$, $l(b_2) = l(c_2)$ and $l(b_1 * b_2) = l(b_3)$. Then $ac = a_1 * c_2 * c_3$ and $bc = b_1 * b_2 * c_3$. So $a_1 = b_1$ and $b_2 = c_2$. Thus, $ab = a_1 * b_2 * b_3 = b_1 * b_2 * b_3 = b$. Let $a' = c_1 * a_2$. Then $l(a) = l(a')$ and $a'b = c_1 * b_2 * b_3$. So $a'bc = c_1 * b_2 * c_3 = c_1 * c_2 * c_3 = c$.

Following is the right-left dual of Lemma 1.3.

LEMMA 1.4. *Let $a, b, c \in \mathscr{B}$ such that $l(a) \leqq l(b) \leqq l(c)$ and $ca = cb$. Then $ba = b$ and there exists $a' \in \mathscr{B}$ such that $l(a) = l(a')$ and $c = cba'$.*

LEMMA 1.5. *Let $a, b, c \in \mathscr{B}$ such that $abc = b$. Then for $d \in \mathscr{B}$, the following are equivalent.*

(1) *$adc = b$ and $l(d) = l(b)$.*

(2) *$d = a'bc'$ for some $a', c' \in \mathscr{B}$ with $l(a) = l(a')$ and $l(c) = l(c')$.*

Proof. First note that $l(a) \leqq l(b)$ and $l(c) \leqq l(b)$.

(1) $\Rightarrow$ (2). By Lemma 1.2, $ad = adcd = bd$. Similarly $dc = db$. By Lemmas 1.3 and 1.4, there exists $a', c' \in \mathscr{B}$ such that $l(a) = l(a')$, $l(c) = l(c')$ and $d = a'bd = dbc'$. So $d = a'bdbc' = a'bc'$.

(2) $\Rightarrow$ (1). Clearly $l(d) = l(b)$. By Lemma 1.2, $adc = aa'bc'c = abc = b$.

LEMMA 1.6. *Let $a, b, c, d \in \mathscr{B}$ such that $l(b) \geqq l(a)$, $l(c) \geqq l(d)$ and $abcd = bc$. Then $ab = b$ and $cd = c$.*

Proof. Since $l(bc) \geqq l(d)$, $abc = abcdbc = bc$. So $(ab)bc = b(bc)$. Since $l(ab) = l(b) \leqq l(bc)$, Lemma 1.3 implies that $(ab)b = b$. So $ab = b$. The other assertion is proved dually using Lemma 1.4.

LEMMA 1.7. *Let* $a_1, a_2, b_1, b_2, c \in \mathscr{B}$ *such that* $l(c) \geqq l(a_1)$, $l(a_2)$, $l(b_1)$ $l(b_2)$ *and* $a_1ca_2 = b_1cb_2$. *Then*

(i) *If* $l(a_1) \leqq l(b_1)$ *and* $l(b_2) \leqq l(a_2)$, *then* $a_1b_1a_2b_2 = b_1a_2$.

(ii) *If* $l(a_1) \leqq l(b_1)$ *and* $l(a_2) \leqq l(b_2)$, *then* $a_1b_1b_2a_2 = b_1b_2$.

Proof. $a_1ca_2c = b_1cb_2c$ and so by Lemma 1.2, $a_1c = b_1c$. Similarly $ca_2 = cb_2$.

(i) By Lemmas 1.3 and 1.4, $a_1b_1 = b_1$ and $a_2b_2 = a_2$. So $a_1b_1a_2b_2 = b_1a_2$.

(ii) By Lemmas 1.3 and 1.4, $a_1b_1 = b_1$ and $b_2a_2 = b_2$. So $a_1b_1b_2a_2 = b_1b_2$.

LEMMA 1.8. *Let* $a_1, a_2, b_1, b_2 \in \mathscr{B}$ *such that* $l(b_1) \geqq l(a_1)$, $l(a_2) \geqq l(b_2)$ *and* $a_1b_1a_2b_2 = b_1a_2$. *Then for* $c \in \mathscr{B}$, *the following are equivalent.*

(1) $a_1ca_2 = b_1cb_2$, $l(c) \geqq l(b_1)$ *and* $l(c) \geqq l(a_2)$.

(2) $c = a_1'b_1da_2b_2'$ *for some* $a_1', b_2', d \in \mathscr{B}$ *with* $l(a_1) = l(a_1')$, $l(b_2) = l(b_2')$, $l(d) \geqq l(b_1)$, $l(d) \geqq l(a_2)$.

Proof. (1) $\Rightarrow$ (2). $a_1ca_2c = b_1cb_2c$ and so $a_1c = b_1c$. Similarly $ca_2 = cb_2$. By Lemmas 1.3 and 1.4, there exist $a_1', b_2' \in \mathscr{B}$ such that $l(a_1) = l(a_1')$, $l(b_2) = l(b_2')$ and $c = a_1'b_1c = ca_2b_2'$. So $c = a_1'b_1ca_2b_1'$.

(2) $\Rightarrow$ (1). By Lemma 1.6, $a_1b_1 = b_1$ and $a_2b_2 = a_2$. By Lemma 1.2, $a_1ca_2 = a_1a_1'b_1da_2b_2'a_2 = a_1b_1da_2 = b_1da_2$. Also $b_1cb_2 = b_1a_1'b_1da_2b_2'b_2 = b_1da_2b_2 = b_1da_2$. So $a_1ca_2 = b_1cb_2$ Clearly $l(c) \geqq l(b_1)$, $l(c) \geqq l(a_2)$.

LEMMA 1.9. *Let* $a_1, a_2, b_1, b_2 \in \mathscr{B}$ *such that* $l(b_1) \geqq l(a_1)$, $l(b_2) \geqq l(a_2)$ *and* $a_1b_1b_2a_2 = b_1b_2$. *Then for* $c \in \mathscr{B}$, *the following are equivalent.*

(1) $a_1ca_2 = b_1cb_2$ *and* $l(c) \geqq l(b_1)$, $l(c) \geqq l(b_2)$.

(2) $c = a_1'b_1db_2a_2'$ *for some* $d, a_1', a_2' \in \mathscr{B}$ *with* $l(a_1) = l(a_1')$, $l(a_2) = l(a_2')$, $l(d) \geqq l(b_1)$ *and* $l(d) \geqq l(b_2)$.

Proof. (1) $\Rightarrow$ (2). $a_1ca_2c = b_1cb_2c$ and so $a_1c = b_1c$. Similarly $ca_2 = cb_2$. By Lemmas 1.3 and 1.4, there exist a_1', a_2' such that $c = a_1'b_1c = cb_2a_2'$, $l(a_1) = l(a_1')$ and $l(a_2) = l(a_2')$. Then $c = a_1'b_1cb_2a_2'$.

(2) $\Rightarrow$ (1). By Lemma 1.6, $a_1b_1 = b_1$ and $b_2a_2 = b_2$. Then by Lemma 1.2, $a_1ca_2 = a_1a_1'b_1db_2a_2'a_2 = a_1b_1db_2a_2 = b_1db_2$. Also $b_1cb_2 = b_1a_1'b_1db_2a_2'b_2 = b_1db_2$. So $a_1ca_2 = b_1cb_2$. Clearly, $l(c) \geqq l(b_1)$ and $l(c) \geqq l(b_2)$.

2. **Word equations.** If Γ is a nonempty set, then let $\mathscr{F} = \mathscr{F}(\Gamma)$ denote the free semigroup on Γ. If $\Gamma = \{X_1, \cdots, X_m\}$, $w = w(X_1, \cdots, X_m) \in \mathscr{F}^1$ and $a_1, \cdots, a_m \in \mathscr{B}$, then let $w(a_1, \cdots, a_m)$ be the element of $\mathscr{B}$ obtained by replacing $X_1, \cdots, X_m$ in w by $a_1, \cdots, a_m$

respectively; if $w = 1$, then $w(a_1, \cdots, a_m) = 1$. For introduction to word equations in free semigroups, see [2, 3].

DEFINITION. By a word equation in variables $X_1, \cdots, X_m$ we mean $\{w_1, w_2\}$ where $w_1 = w_1(X_1, \cdots, X_m)$, $w_2 = w_2(X_1, \cdots, X_m) \in \mathscr{F}(X_1, \cdots, X_m)^1$. It is not necessary that each X_i appears in $w_1 w_2$. By a solution of $\{w_1, w_2\}$ in $\mathscr{B}$, we mean $(a_1, \cdots, a_m)$ where $a_1, \cdots, a_m \in \mathscr{B}$ and $w_1(a_1, \cdots, a_m) = w_2(a_1, \cdots, a_m)$. A solution $(a_1, \cdots, a_m)$ is an ordered solution if $l(a_1) \leqq l(a_2) \leqq \cdots \leqq l(a_m)$.

REMARK 2.1. In the above situation, note that the solutions of $\{w_1, w_2\}$ are exactly all the ordered solutions obtained by relabeling the X_i's in the $m!$ possible ways. So we will concentrate on obtaining the ordered solutions of word equations.

THEOREM 2.2. *Let $m \in Z^+$, $m \geqq 2$. Let $w_1, w_2 \in \mathscr{F} = \mathscr{F}(X_1, \cdots, X_m)$. Suppose $w_1 = u_1 X_m u_2$ for some $u_1, u_2 \in \mathscr{F}^1$ such that X_m does not occur in u_1, u_2, w_2, and X_{m-1} occurs in $w_1 w_2$. Let $v_1 = u_1 w_2 u_2$. Consider the word quation $\{v_1, v_2\}$ in variables $X_1, \cdots, X_{m-1}$. Let $(a_1, \cdots, a_{m-1})$ be an ordered solution of $\{v_1, w_2\}$ in $\mathscr{B}$. Set $a = u_1(a_1, \cdots, a_{m-1})$, $b = w_2(a_1, \cdots, a_{m-1})$, $c = u_2(a_1, \cdots, a_{m-1})$. Let $a', c' \in \mathscr{B}$ such that $l(a) = l(a')$ and $l(c) = l(c')$. If $a_m = a'bc'$, then $(a_1, \cdots, a_{m-1}, a_m)$ is an ordered solution of $\{w_i, w_2\}$. Moreover, every ordered solution of $\{w_1, w_2\}$ in $\mathscr{B}$ is obtained in this manner.*

Proof. Let $(a_1, \cdots, a_m)$ be an ordered solution of $\{w_1, w_2\}$. Let $a = u_1(a_1, \cdots, a_{m-1})$, $b = w_2(a_1, \cdots, a_{m-1})$, $c = u_2(a_1, \cdots, a_{m-1})$. Then $aa_mc = b$. Clearly then, $abc = aaa_mcc = aa_mc = b$. So $(a_1, \cdots, a_{m-1})$ is an ordered solution of $\{v_1, w_2\}$. New it follows from Lemma 1.5, that a_m has the prescribed form. The converse also follows from Lemma 1.5.

In what follows, if $w \in \mathscr{F}(X_1, \cdots, X_m)^1$, then let $\theta(w) = \max\{i \mid X_i \text{ appears in } w\}$; $\theta(1) = 0$.

THEOREM 2.3. *Let $m \in Z^+$, $m \geqq 2$. Let $w_1, w_2 \in \mathscr{F} = \mathscr{F}(X_1, \cdots, X_m)$. Suppose $w_1 = u_1 X_m u_2$, $w_2 = v_1 X_m v_2$ for some $u_1, u_2, v_1, v_2 \in \mathscr{F}^1$ such that X_m does not appear in $u_1 u_2 v_1 v_2$.*

(i) *Suppose $\theta(u_1) \leqq \theta(v_1)$ and $\theta(v_2) \leqq \theta(u_2)$. Let $f_1 = u_1 v_1 u_2 v_2$ and $f_2 = v_1 u_2$. Consider the word equation $\{f_1, f_2\}$ in variables $X_1, \cdots, X_{m-1}$ and let $(a_1, \cdots, a_{m-1})$ be an ordered solution of $\{f_1, f_2\}$ in $\mathscr{B}$. Set $A_1 = u_1(a_1, \cdots, a_{m-1})$, $A_2 = u_2(a_1, \cdots, a_{m-1})$, $B_1 = v_1(a_1, \cdots, a_{m-1})$ and $B_2 = v_2(a_1, \cdots, a_{m-1})$. Let $A_1', B_2', D \in \mathscr{B}$ such that $l(A_1') = l(A_1)$, $l(B_2') = l(B_2)$, $l(D) \geqq l(a_{m-1})$. Set $a_m = A_1' B_1 D A_2 B_2'$. Then $(a_1, \cdots, a_{m-1}, a_m)$ is an ordered solution of $\{w_1, w_2\}$. Moreover, every ordered solution of*

$\{w_1, w_2\}$ in $\mathscr{B}$ is obtained in this manner.

(ii) *Suppose $\theta(u_1) \leqq \theta(v_1)$ and $\theta(u_2) \leqq \theta(v_2)$. Let $f_1 = u_1v_1v_2u_2$, $f_2 = v_1v_2$. Consider the word equation $\{f_1, f_2\}$ in variables $X_1, \cdots, X_{m-1}$. Let $(a_1, \cdots, a_{m-1})$ be an ordered solution of $\{f_1, f_2\}$ in $\mathscr{B}$. Set $A_1 = u_1(a_1, \cdots, a_{m-1})$, $A_2 = u_2(a_1, \cdots, a_{m-1})$, $B_1 = v_1(a_1, \cdots, a_{m-1})$ and $B_2 = v_2(a_1, \cdots, a_{m-1})$. Let $A_1', A_2', D \in \mathscr{B}$ such that $l(A_1') = l(A_1)$, $l(A_2') = l(A_2)$, $l(D) \geqq l(a_{m-1})$. Set $a_m = A_1'B_1DB_2A_2'$. Then $(a_1, \cdots, a_{m-1}, a_m)$ is an ordered solution of $\{w_1, w_2\}$. Moreover, every ordered solution of $\{w_1, w_2\}$ in $\mathscr{B}$ is obtained in this manner.*

Proof. Suppose $(a_1, \cdots, a_m)$ is an ordered solution of $\{w_1, w_2\}$ in $\mathscr{B}$. Let $A_1 = u_1(a_1, \cdots, a_{m-1})$, $A_2 = u_2(a_1, \cdots, a_{m-1})$, $B_1 = v_1(a_1, \cdots, a_{m-1})$ and $B_2 = v_2(a_1, \cdots, a_{m-1})$. So $A_1a_mA_2 = B_1a_mB_2$.

(i) We have $l(A_1) \leqq l(B_1) \leqq l(a_m)$, $l(B_2) \leqq l(A_2) \leqq l(a_m)$. By Lemma 1.7(i), $A_1B_1A_2B_2 = B_1A_2$. So $(a_1, \cdots, a_{m-1})$ is an ordered solution of $\{f_1, f_2\}$. That a_m has the required form, follows from Lemma 1.8. The converse also follows from Lemma 1.8.

(ii) We have $l(A_1) \leqq l(B_1) \leqq l(a_m)$, $l(A_2) \leqq l(B_2) \leqq l(a_m)$. By Lemma 1.7(ii), $A_1B_1B_2A_2 = B_1B_2$. So $(a_1, \cdots, a_{m-1})$ is an ordered solution of $\{f_1, f_2\}$. That a_m has the required form, follows from Lemma 1.9. The converse also follows from Lemma 1.9.

THEOREM 2.4. *Let $m \in Z^+$, $m \geqq 2$, $w_1, w_2 \in \mathscr{F} = \mathscr{F}(X_1, \cdots, X_m)$.*

(i) *Suppose $w_1 = u_1X_mu_2X_mu_3$ for some $u_1, u_2, u_3 \in \mathscr{F}^1$ such that X_m does not occur in $u_1u_3w_2$. Let $f_1 = u_1X_mu_3$, $f_2 = w_2$. Then the ordered solutions of $\{w_1, w_2\}$ in $\mathscr{B}$ are exactly the same as the ordered solutions of $\{f_1, f_2\}$.*

(ii) *Suppose $w_1 = u_1X_mu_2X_mu_3$, $w_2 = v_1X_mv_2X_mv_3$ for some $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathscr{F}^1$ such that X_m does not occur in $u_1u_3v_1v_3$. Let $f_1 = u_1X_mu_3$, $f_2 = v_1X_mv_3$. Then the ordered solutions of $\{w_1, w_2\}$ in $\mathscr{B}$ are exactly the same as the ordered solutions of $\{f_1, f_2\}$.*

(iii) *Suppose $w_1 = u_1X_mu_2X_mu_3$, $w_2 = v_1X_mv_2$ for some $u_1, u_2, u_3, v_1, v_2 \in \mathscr{F}^1$ such that X_m does not occur in $u_1u_3v_1v_2$. Let $f_1 = u_1X_mu_3$, $f_2 = v_1X_mv_2$. Then the ordered solutions of $\{w_1, w_2\}$ in $\mathscr{B}$ are exactly the same as the ordered solutions of $\{f_1, f_2\}$.*

Proof. Let $a_1, \cdots, a_m \in \mathscr{B}$ such that $l(a_1) \leqq \cdots \leqq l(a_m)$. Then in all cases $f_1(a_1, \cdots, a_m) = w_1(a_1, \cdots, a_m)$ and $f_2(a_1, \cdots, a_m) = w_2(a_1, \cdots, a_m)$.

REMARK 2.5. Let $\{w_1, w_2\}$ be a word equation in variables $X_1, \cdots, X_m$. If w_1 or $w_2 = 1$, the solutions are obvious. So assume $w_1 \neq 1$, $w_2 \neq 1$. If $m = 1$, the solutions are again obvious. So let $m \geqq 2$. We claim that the ordered solutions of $\{w_1, w_2\}$ can be described in terms of ordered solutions of a certain word equation in $m-1$ variables. If some X_j does not appear in w_1w_2, then $\{w_1, w_2\}$ can itself be considered as a word equation in $m-1$ variables and our claim holds trivially. Otherwise either $\{w_1, w_2\}$ or $\{w_2, w_1\}$ satisfies Theorems 2.2, 2.3 or 2.4 and our claim still holds. Thus given any word equation, we can completely describe all its solutions in $\mathscr{B}$.

EXAMPLE 2.6. Consider the word equation $\{BCBA, CABAC\}$ in variables A, B, C. There are six ways of ordering A, B, C. Finding the ordered solutions for all these equations (using the theorems of this section) and simplifying, we see that following is the list of all solutions in $\mathscr{B}$ of the above word equation.

(1)
$$A = a$$
$$B = b$$
$$C = bcba\ ,$$

where $a, b, c \in \mathscr{B}$.

(2)
$$A = c'bac$$
$$B = cb$$
$$C = c$$

where $a, b, c, c' \in \mathscr{B}$, $l(c) = l(c') \leqq l(b) \leqq l(a)$.

(3)
$$A = c'babc$$
$$B = b$$
$$C = bc$$

where $a, b, c, c' \in \mathscr{B}$, $l(b) \leqq l(c) = l(c') \leqq l(a)$.

(4)
$$A = ac$$
$$B = caba'$$
$$C = c$$

where $a, a', b, c \in \mathscr{B}$, $l(c) \leqq l(a) = l(a') \leqq l(b)$.

(5)
$$A = a$$
$$B = cbca'$$
$$C = ca$$

where $a, a', b, c \in \mathscr{B}$, $l(a) = l(a') \leqq l(c) \leqq l(b)$.

3. Concluding remarks. Instead of starting with the semigroup of paths, we can start with semigroup of designs around the unit disc of [5] and analogously define a new idempotent, associative multiplication. Then the results of this paper remain true for that band. More generally, let E be any band with an identity element. Let Ω be its maximal semilattice image and l the corresponding homomorphism. Consider Ω with the order given by $e \leqq f$ if and only if $ef = f$. If Ω is linearly ordered and if Lemmas 1.2, 1.3 and 1.4 are true for $\mathscr{E}$, then all of §2 remains true for $\mathscr{E}$.

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