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SOME RELATIONSHIPS BETWEEN MEASURES

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Suppose μ and ν are (nonnegative, countably additive) measures on the same sigma-ring. We say that ν is *quasi-dominant* with respect to μ if each measurable set contains a subset with the same ν-measure, where μ is absolutely continuous with respect to ν on that subset. In particular, ν is quasi-dominant with respect to μ if μ is sigma-finite. We say that ν is *strongly recessive* with respect to μ if the zero measure is the only measure that is quasi-dominant with respect to μ and less than or equal to ν. Properties of these relationships are investigated, and applications are given to purely atomic measures, to the Radon-Nikodým theorem and to a decomposition of product measures.

1. Weak singularity and absolute continuity. Let μ and ν be (nonnegative, countably additive) measures on a sigma-ring $\mathscr{S}$. Recall that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $\mu(E) = 0$. If $\nu \ll \mu$ and $\mu \ll \nu$, then μ and ν are said to be equivalent and we write $\mu \sim \nu$. We say that ν is *weakly singular* with respect to μ, denoted $\nu S \mu$, if given E in $\mathscr{S}$, there exists F in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F)$ and $\mu(F) = 0$.

We shall make use of the following form of the Lebesgue Decomposition Theorem [3, Theorem 2.1 or 6, Theorem 1.1]:

THEOREM 1.1. (*Lebesgue Decomposition Theorem*). *Suppose μ and ν are measures on a sigma-ring $\mathscr{S}$. Then there exist measures ν_1 and ν_2 such that* (1) $\nu = \nu_1 + \nu_2$, (2) $\nu_1 \ll \mu$ *and* (3) $\nu_2 S \mu$. *The measure ν_2 is unique. We may arrange to have $\nu_1 S \nu_2$, and under that requirement ν_1 is unique also.*

If ν is a measure on $\mathscr{S}$ and $A \in \mathscr{S}$, let ν_A be the measure given by $\nu_A(E) = \nu(A \cap E)$ for all $E \in \mathscr{S}$.

THEOREM 1.2. *Suppose $M_1(\mathscr{S})$ and $M_2(\mathscr{S})$ are families of measures on $\mathscr{S}$ such that the zero measure is the only measure common to both families and such that ν_A is in one of the families whenever ν is in that family and $A \in \mathscr{S}$. Suppose, moreover, that each measure ν on $\mathscr{S}$ can be written as the sum of measures ν_1 and ν_2 such that $\nu_1 \in M_1(\mathscr{S})$ and $\nu_2 \in M_2(\mathscr{S})$ and $\nu_1 S \nu_2$. Then $\nu \in M_2(\mathscr{S})$ if and only if $\nu(A) = 0$ whenever $\nu_A \in M_1(\mathscr{S})$.*

Proof. Suppose $\nu \in M_2(\mathscr{S})$. Then $\nu_A \in M_2(\mathscr{S})$ for all $A \in \mathscr{S}$. If

$\nu_A \in M_1(\mathscr{S})$, then $\nu_A = 0$ so that $\nu(A) = 0$.

Suppose $\nu(A) = 0$ whenever $\nu_A \in M_1(\mathscr{S})$. In order to show that $\nu \in M_2(\mathscr{S})$, it suffices to show that $\nu_1(E) = 0$ for all E in $\mathscr{S}$. Suppose, then, that $E \in \mathscr{S}$. Since $\nu_1 S \nu_2$, there exists F in $\mathscr{S}$ such that $\nu_1(E) = \nu_1(E \cap F)$ and $\nu_2(F) = 0$. Necessarily, $\nu_F = (\nu_1)_F$. Since $(\nu_1)_F \in M_1(\mathscr{S})$, we have $\nu_F \in M_1(\mathscr{S})$ so that $\nu(F) = 0$ by hypothesis. Then $\nu_1(E) = \nu_1(E \cap F) \leqq \nu(F) = 0$, and we are done.

The following results follow from the definitions or from Theorems 1.1 and 1.2:

(1) If $\nu S \mu$, then $\nu_A S \mu$ for all $A \in \mathscr{S}$.
(2) $\nu_A S \mu$ if and only if $\nu_A S \mu_A$ if and only if $\nu S \mu_A$.
(3) If $\nu \ll \mu$, then $\nu_A \ll \mu$.
(4) $\nu_A \ll \mu$ if and only if $\nu_A \ll \mu_A$.
(5) $\nu S \mu$ if and only if $\nu(A) = 0$ whenever $\nu_A \ll \mu$.
(6) $\nu \ll \mu$ if and only if $\nu(A) = 0$ whenever $\nu_A S \mu$.

The relationships of absolute continuity and weak singularity between measures are determined by the null sets of the measures. That is, suppose $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$. Then $\nu_1 \ll \mu_1$ if and only if $\nu_2 \ll \mu_2$, and $\nu_1 S \mu_1$ if and only if $\nu_2 S \mu_2$. We prove the nontrivial part of these assertions.

THEOREM 1.3. *If $\lambda S \mu$ and $\lambda \sim \nu$, then $\nu S \mu$.*

Proof. Suppose $\nu_A \ll \mu$. It suffices to show that $\nu(A) = 0$. Since $\lambda S \mu$, there exists F in $\mathscr{S}$ such that $\lambda(A) = \lambda(A \cap F)$ and $\mu(F) = 0$. Of course, $\nu_A(F) = 0$. Then since $\nu(A \cap F) = 0$, we have $\lambda(A) = \lambda(A \cap F) = 0$. Hence, $\nu(A) = 0$.

If μ is a measure, then $\infty\mu$ will denote that (necessarily equivalent) measure which is ∞ when μ is positive and 0 when μ is 0. Of course, $\mu_1 \sim \mu_2$ if and only if $\infty\mu_1 = \infty\mu_2$. In view of Theorem 1.3 and the preceding remarks, $\nu \ll \mu$ if and only if $\infty\nu \ll \infty\mu$, while $\nu S \mu$ if and only if $\infty\nu S \infty\mu$.

2. Quasi-dominance and strong recessiveness. We shall say that ν is *quasi-dominant* with respect to μ, denoted $\nu Q \mu$, if given E in $\mathscr{S}$, there exists F in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F)$ and $\mu_F \ll \nu$. It is evident that $\nu Q \mu$ if $\nu S \mu$ or $\mu \ll \nu$.

THEOREM 2.1.

(1) *If $\nu Q \lambda$ and $\mu \ll \lambda$, then $\nu Q \mu$.*
(2) *If $\nu Q \mu$ and $\mu S \nu$, then $\nu S \mu$.*
(3a) *If $\nu_1 Q \mu$ and $\nu_2 Q \mu$, then $(\nu_1 + \nu_2) Q \mu$.*
(3b) *If $\nu Q \mu_1$ and $\nu Q \mu_2$, then $\nu Q(\mu_1 + \mu_2)$.*

(4) *If* $\nu Q\mu$, *then* μ *can be written as the sum of* μ_1 *and* μ_2, *where* $\mu_1 \ll \nu$ *and* $\nu S\mu_2$. *We may arrange to have* $\mu_2 S\nu$ *and* $\mu_1 S\mu_2$, *and under those conditions* μ_1 *and* μ_2 *are unique.*

(5) *If* $\lambda Q\mu$ *and* $\lambda \sim \nu$, *then* $\nu Q\mu$.

(6a) *If* $\nu_1 Q\mu$ *and* $\nu_2 Q\mu$, *then* $(\nu_1 \vee \nu_2)Q\mu$.

(6b) *If* $\nu Q\mu_1$ *and* $\nu Q\mu_2$, *then* $\nu Q(\mu_1 \vee \mu_2)$.

(7) *If* μ *is sigma-finite, then* $\nu Q\mu$ *for any measure* ν *on* $\mathscr{S}$.

(8) *If* $\nu Q\mu$, *then* $\nu_A Q\mu$ *for all* $A \in \mathscr{S}$.

Proof.

(1) Follows from definition of quasi-dominance.

(2) Given $E \in \mathscr{S}$, there exists $F \in \mathscr{S}$ such that $\nu(E) = \nu(E \cap F)$ and $\mu_F \ll \nu$. Since $\mu S\nu$ and $\mu_F \ll \nu$, it follows that $\nu(E \cap F) = 0$. In other words, $\nu S\mu$.

(3a) Suppose $E \in \mathscr{S}$. Then there exist F_1 and F_2 in $\mathscr{S}$ such that $\nu_1(E) = \nu_1(E \cap F_1)$ and $\nu_2(E) = \nu_2(E \cap F_2)$, where $\mu_{F_1} \ll \nu_1$ and $\mu_{F_2} \ll \nu_2$. If $F = F_1 \cup F_2$, then it can be seen that $(\nu_1 + \nu_2)(E) = (\nu_1 + \nu_2)(E \cap F)$ and $\mu_F \ll \nu_1 + \nu_2$.

(3b) Suppose $E \in \mathscr{S}$. Since $\nu Q\mu_1$, there exists F_1 in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F_1)$ and $(\mu_1)_{F_1} \ll \nu$. Since $\nu Q\mu_2$, there exists F_2 in $\mathscr{S}$ such that $\nu(E \cap F_1) = \nu((E \cap F_1) \cap F_2)$ and $(\mu_2)_{F_2} \ll \nu$. If $F = F_1 \cap F_2$, then $\nu(E) = \nu(E \cap F)$ and $(\mu_1 + \mu_2)_F \ll \nu$.

(4) By the Lebesgue Decomposition Theorem, μ can be written as the sum of μ_1 and μ_2, where $\mu_1 \ll \nu$ and $\mu_2 S\nu$ and $\mu_1 S\mu_2$. Since $\nu Q\mu_2$ by (1) and since $\mu_2 S\nu$, we have $\nu S\mu_2$ by (2). Uniqueness under the added conditions amounts to the uniqueness of the Lebesgue Decomposition Theorem for the case $\mu_1 S\mu_2$.

(5) By (4), $\mu = \mu_1 + \mu_2$, where $\mu_1 \ll \lambda$ and $\lambda S\mu_2$. Since $\lambda \sim \nu$, we have $\mu_1 \ll \nu$ and $\nu S\mu_2$. Then $\nu Q\mu$ by (3b).

(6a) Since $(\nu_1 \vee \nu_2) \sim (\nu_1 + \nu_2)$, the result follows from (3a) and (5).

(6b) Since $(\mu_1 \vee \mu_2) \sim (\mu_1 + \mu_2)$, the result follows from (3b) and (1).

(7) By the Lebesgue Decomposition Theorem, $\mu = \mu_1 + \mu_2$, where $\mu_1 \ll \nu$ and $\mu_2 S\nu$. Since μ_2 is sigma-finite, $\nu S\mu_2$ [3, Theorem 3.2]. Then $\nu Q\mu$ by (3b).

(8) Fix $A \in \mathscr{S}$ and suppose $E \in \mathscr{S}$. Since $\nu Q\mu$, there exists $F \in \mathscr{S}$ such that $\nu_A(E) = \nu(A \cap E) = \nu((A \cap E) \cap F)$ and such that $\mu_F \ll \nu$. Necessarily, $\mu_{A \cap F} \ll \nu_A$. Hence, $\nu_A(E) = \nu_A(E \cap (A \cap F))$ and $\mu_{(A \cap F)} \ll \nu_A$, so that $\nu_A Q\mu$.

We say that ν is *strongly recessive* with respect to μ, denoted $\nu <_S \mu$, if λ is the zero measure whenever $\lambda \leqq \nu$ and $\lambda Q\mu$. Clearly, $\nu Q\mu$ and $\nu <_S \mu$ if and only if ν is the zero measure.

THEOREM 2.2. *The following are equivalent*:

(1) *If* $A \in \mathscr{S}$ *and* $\nu_A Q \mu$, *then* $\nu(A) = 0$.

(2) *If* $\lambda \leqq \nu$ *and* $\lambda Q \mu$, *then* $\lambda = 0$.

(3) *If* $\lambda \ll \nu$ *and* $\lambda Q \mu$, *then* $\lambda = 0$.

(4) $\mu S \nu$ *and* $\nu \ll \mu$.

(5) $\mu S \nu$ *and* $\nu \leqq \mu$.

(6) $(\mu + \nu) S \nu$.

(7) $\mu S \nu$ *and* $\nu(A) = 0$ *whenever* $\mu(A) < \infty$.

Proof.

(1) implies (7): Suppose $\mu_A \ll \nu$. Assuming (1), we first show that $\mu S \nu$ by showing that $\mu(A) = 0$. Since $\mu_A \ll \nu_A$, we have $\nu_A Q \mu_A$ so that $\nu_A Q \mu$. Assuming (1), we have $\nu(A) = 0$ so that $\mu(A) = 0$. Hence, $\mu S \nu$. Now suppose $\mu(A) < \infty$. Assuming (1), we show that $\nu(A) = 0$. We already know that $\mu S \nu$, so that $\mu_A S \nu$. Since μ_A is finite, we have $\nu S \mu_A$ [3, Theorem 3.2] so that $\nu_A S \mu$. Hence, $\nu_A Q \mu$ and assuming (1), we have $\nu(A) = 0$ as was to be shown.

(7) implies (6): Since $\nu(A) = 0$ whenever $\mu(A) < \infty$, we have $\mu = \mu + \nu$. Hence, $(\mu + \nu) S \nu$.

(6) implies (5): Clearly, $\nu \leqq \mu + \nu$. It suffices to show that $\mu = \mu + \nu$. Suppose $E \in \mathscr{S}$. Since $(\mu + \nu) S \nu$, there exists F in $\mathscr{S}$ such that $(\mu+\nu)(E) = (\mu+\nu)(E \cap F)$ and $\nu(F) = 0$. Hence, $(\mu+\nu)(E) = (\mu + \nu)(E \cap F) = \mu(E \cap F) \leqq \mu(E)$ so that $(\mu + \nu)(E) = \mu(E)$.

(4) implies (3): Suppose $\mu S \nu$ and $\nu \ll \mu$. Suppose, moreover, that $\lambda \ll \nu$ and $\lambda Q \mu$. It suffices to show that $\lambda = 0$. Since $\mu S \nu$ and $\lambda \ll \nu$, we have $\mu S \lambda$. Since $\mu S \lambda$ and $\lambda Q \mu$, we have $\lambda S \mu$ by (2) of Theorem 2.1. Since $\lambda S \mu$ and since $\lambda \ll \nu$, we have $\lambda = 0$.

Clearly, (5) implies (4), (3) implies (2) and (2) implies (1).

We shall see that the second condition in (7) of Theorem 2.2 is enough to insure that $\nu <_S \mu$ whenever μ enjoys the property of semifiniteness. We say that μ is *semifinite* (or locally finite) if it satisfies any of the following equivalent conditions [1, Exercise 25.9 or 9, Theorem 8.3]: (1) If $E \in \mathscr{S}$, then $\mu(E) = \sup\{\mu(E \cap F): \mu(F) < \infty\}$. (2) Every measurable set of positive measure contains a measurable set of finite positive measure. (3) Every measurable set E contains a measurable set F such that F has sigma-finite μ-measure and $\mu(E) = \mu(F)$. A measure is called *degenerate* if the only values taken on by the measure are 0 and ∞.

Following [5, page 396], we shall say that ν is *totally incompatible* with μ if $\nu(E) > 0$ implies $\mu(E) = \infty$. Equivalently, ν is totally incompatible with μ if $\mu + \nu = \mu$. In view of Theorem 2.2, ν is totally incompatible with μ whenever $\nu <_S \mu$. If ν is totally incompatible with μ, then clearly $\nu \ll \mu$. If $\nu \ll \mu$ and μ is degenerate, then ν is totally incompatible with μ.

THEOREM 2.3. *If ν is totally incompatible with μ and μ is semifinite, then $\nu <_S \mu$.*

Proof. Evidently, $\nu \ll \mu$ and it suffices to show that $\mu S \nu$. Given E in $\mathscr{S}$, by virtue of the semifiniteness of μ there exists F in $\mathscr{S}$ such that $\mu(E) = \mu(E \cap F)$ and such that $\mu(F)$ is sigma-finite. Since ν is totally incompatible with μ and since $\mu(F)$ is sigma-finite, we have $\nu(F) = 0$.

THEOREM 2.4.

(1a) *If $\nu \ll \lambda$ and $\lambda <_S \mu$, then $\nu <_S \mu$.*

(1b) *If $\nu <_S \lambda$ and $\lambda \ll \mu$, then $\nu <_S \mu$.*

(2) *If $\nu Q \mu$ and $\lambda <_S \mu$, then $\nu S \lambda$.*

(3a) *If $\nu <_S (\mu + \lambda)$ and $\nu Q \lambda$, then $\nu <_S \mu$. Hence, if $\nu <_S (\mu + \lambda)$ and $\nu S \lambda$, then $\nu <_S \mu$.*

(3b) *If $(\nu + \lambda) Q \mu$ and $\lambda <_S \mu$, then $\nu Q \mu$.*

(4) *If* (i) *$\nu <_S \mu$,* (ii) *$\nu Q \lambda$ and* (iii) *$\lambda Q \mu$, then $\nu S \lambda$.*

(5) *If $\nu_1 <_S \mu$ and $\nu_2 <_S \mu$, then $\nu_1 + \nu_2 <_S \mu$.*

(6) *If $\nu <_S \mu$, then $\nu_A <_S \mu_A$ for all A in $\mathscr{S}$.*

Proof.

(1a) Since $\lambda <_S \mu$, we have $\mu S \lambda$ and $\lambda \ll \mu$ by Theorem 2.2. Since $\nu \ll \lambda$, we have $\mu S \nu$ and $\nu \quad \mu$. Hence, $\nu <_S \mu$ by Theorem 2.2.

(1b) Suppose $A \in \mathscr{S}$ and $\nu_A Q \mu$. Since $\lambda \ll \mu$, we have $\nu_A Q \lambda$. Since $\nu <_S \lambda$ and since $\nu_A Q \lambda$, we have $\nu(A) = 0$. Therefore, $\nu <_S \mu$ by Theorem 2.2.

(2) Suppose $\nu_A \ll \lambda$. It suffices to show that $\nu(A) = 0$. Since $\lambda <_S \mu$, we have $\nu_A <_S \mu$ by (1a) of this theorem. By (8) of Theorem 2.1, we have $\nu_A Q \mu$. Hence, $\nu_A = 0$ and $\nu(A) = 0$.

(3a) Suppose $A \in \mathscr{S}$ and $\nu_A Q \mu$. It suffices to show that $\nu(A) = 0$. Since $\nu Q \lambda$, we have $\nu_A Q \lambda$ by (8) of Theorem 2.1. Then $\nu_A Q(\mu + \lambda)$, so that $\nu(A) = 0$ by Theorem 2.2.

(3b) $(\nu + \lambda) S \lambda$ by (2) of this theorem. Hence, $\lambda <_S \nu$ and $\nu + \lambda = \nu$ by Theorem 2.2.

(4) Since $\lambda Q \mu$ and $\nu <_S \mu$, we have $\lambda S \nu$ by (2) of this theorem. Since $\nu Q \lambda$ and $\lambda S \nu$, we have $\nu S \lambda$ by (2) of Theorem 2.1.

(5) and (6) follow immediately from Theorem 2.2.

For reference and for comparison, we restate the Lebesgue Decomposition Theorem (Theorem 1.1). In stating this theorem, we may replace the requirement that $\nu_1 S \nu_2$ by $\nu_1 Q \nu_2$ because of (2) in Theorem 2.1 and the fact that $\nu_2 S \nu_1$. We then prove an analogous decomposition theorem involving strong recessiveness and quasi-dominance.

Lebesgue Decomposition Theorem. Suppose μ and ν are measures on a sigma-ring $\mathscr{S}$. Then ν can be written as $\nu_1 + \nu_2$, where $\nu_1 \ll \mu$ and $\nu_2 S \mu$. Necessarily, ν_2 is unique. We may arrange to have $\nu_1 Q \nu_2$ (or $\nu_1 S \nu_2$), and in that case ν_1 is unique also.

THEOREM 2.5. *Suppose μ and ν are measures on a sigma-ring $\mathscr{S}$. Then ν can be written as $\nu_1 + \nu_2$, where $\nu_1 <_S \mu$ and $\nu_2 Q \mu$. The measure ν_2 is unique. We may arrange to have $\nu_1 Q \nu_2$ (or $\nu_1 S \nu_2$), and under that requirement ν_1 is unique also.*

Proof. By the Lebesgue Decomposition Theorem, μ can be written as $\mu_1 + \mu_2$, where $\mu_1 \ll \nu$ and $\mu_2 S \nu$ and $\mu_1 S \mu_2$. Again by the Lebesgue Decomposition Theorem, ν can be written as $\nu_1 + \nu_2$, where $\nu_1 \ll \mu_2$ and $\nu_2 S \mu_2$ and $\nu_1 S \nu_2$. Notice that $\mu_1 S \nu_1$ since $\mu_1 S \mu_2$ and $\nu_1 \ll \mu_2$. We show that ν_1 and ν_2 are the required measures.

Let us show that $\nu_1 <_S \mu$. Of course, $\nu_1 \ll \mu$ since $\nu_1 \ll \mu_2$ and $\mu_2 \leqq \mu$. Since $\mu_2 S \nu$, we have $\mu_2 S \nu_1$. Since $\mu_1 S \nu_1$ and $\mu_2 S \nu_1$, we have $\mu S \nu_1$ so that $\nu_1 <_S \mu$.

Now we show that $\nu_2 Q \mu$. Since $\mu_1 \ll \nu_1 + \nu_2$ and since $\mu_1 S \nu_1$, we have $\mu_1 \ll \nu_2$ [3, page 630]. Since $\mu_1 \ll \nu_2$ and since $\nu_2 S \mu_2$, we have $\nu_2 Q \mu$.

To prove uniqueness of the decomposition, suppose $\nu = \nu_3 + \nu_4$, where $\nu_3 <_S \mu$ and $\nu_4 Q \mu$. Then $\nu_4 S \nu_1$ by (2) of Theorem 2.4. Since $\nu_4 \leqq \nu_1 + \nu_2$ and since $\nu_4 S \nu_1$, we have $\nu_4 \leqq \nu_2$. Similarly, $\nu_2 \leqq \nu_4$, so that ν_2 is unique.

Since $\nu_1 S \nu_2$, we have $\nu_1 Q \nu_2$. Now suppose $\nu = \nu_3 + \nu_2$, where $\nu_3 <_S \mu$ and $\nu_2 Q \mu$ and $\nu_3 Q \nu_2$. Then $\nu_3 S \nu_2$ by (4) of Theorem 2.4. Since $\nu_3 \leqq \nu_1 + \nu_2$ and since $\nu_3 S \nu_2$, we have $\nu_3 \leqq \nu_1$. Similarly, $\nu_1 \leqq \nu_3$, so that ν_1 is unique in this case.

We have already seen that $\nu <_S \mu$ if and only if $\nu(A) = 0$ whenever $\nu_A Q \mu$. We now prove the corresponding result for $\nu Q \mu$.

THEOREM 2.6. *$\nu Q \mu$ if and only if $\nu(A) = 0$ whenever $\nu_A <_S \mu$.*

Proof. Let $M_1(\mathscr{S})$ be the family of measures on $\mathscr{S}$ which are strongly recessive with respect to μ, and let $M_2(\mathscr{S})$ be the family of all measures on $\mathscr{S}$ which are quasi-dominant with respect to μ. The desired result follows from Theorem 1.2 and the decomposition of Theorem 2.5.

As an application of Theorem 2.6 we have the following:

THEOREM 2.7. *If $(\nu + \lambda) Q \mu$ and $\nu Q \lambda$, then $\nu Q \mu$.*

Proof. Suppose $\nu_A <_S \mu$. Then $(\nu + \lambda) S \nu_A$ by (2) of Theorem 2.4. Hence, $(\nu_A + \lambda_A) S \nu_A$, so that $\nu_A <_S \lambda_A$ by Theorem 2.2. Since $\nu_A Q \lambda_A$, and $\nu_A <_S \lambda_A$, we see that ν_A is the zero measure. Hence, $\nu(A) = 0$ so that $\nu Q \mu$.

Suppose $\mathscr{S}$ is a sigma-ring and $\mathscr{T}$ is a sigma-ring containing $\mathscr{S}$. We say that $\mathscr{S}$ is an ideal in $\mathscr{T}$ if $E \cap F \in \mathscr{S}$ whenever $E \in \mathscr{T}$ and $F \in \mathscr{S}$. If $\mathscr{S}$ is a sigma-ring, let $\mathscr{S}_\lambda$ denote the class of locally measurable sets; that is, $\mathscr{S}_\lambda = \{E : E \cap F \in \mathscr{S}$ whenever $F \in \mathscr{S}\}$. The class $\mathscr{S}_\lambda$ is a sigma-algebra since it contains X, and it is the largest sigma-ring having $\mathscr{S}$ as an ideal. If μ is a measure on $\mathscr{S}$ and $\mathscr{S}$ is an ideal in $\mathscr{T}$, define μ_λ on $\mathscr{T}$ by $\mu_\lambda(E) = \sup\{\mu(E \cap F) : F \in \mathscr{S}\}$ for all $E \in \mathscr{T}$. Then μ_λ is an extension of μ to a smallest measure on $\mathscr{T}$ [1, Exercise 17.1].

THEOREM 2.8. *Suppose the sigma-ring $\mathscr{S}$ is an ideal in the sigma-ring $\mathscr{T}$. Suppose, moreover, that μ and ν are measures on $\mathscr{S}$ and that μ_λ and ν_λ are their respective extensions to smallest measures on $\mathscr{T}$. Then:*

(1) *$\nu S \mu$ if and only if $\nu_\lambda S \mu_\lambda$. Indeed, given E in $\mathscr{T}$, there exists F in $\mathscr{S}$ such that $\nu_\lambda(E) = \nu(E \cap F)$ and $\mu(F) = 0$.*

(2) *$\nu \ll \mu$ if and only if $\nu_\lambda \ll \mu_\lambda$.*

(3) *$\nu Q \mu$ if and only if $\nu_\lambda Q \mu_\lambda$. Indeed, given E in $\mathscr{T}$, there exists F in $\mathscr{S}$ such that $\nu_\lambda(E) = \nu(E \cap F)$ and $\mu_F \ll \nu$.*

(4) *$\nu <_S \mu$ if and only if $\nu_\lambda <_S \mu_\lambda$.*

Proof. The relationships on $\mathscr{T}$ clearly imply the same relationships on $\mathscr{S}$. It suffices to prove the results which extend relationships on $\mathscr{S}$ to relationships on $\mathscr{T}$.

(1) Suppose $E \in \mathscr{T}$. Then $\nu_\lambda(E) = \sup\{\nu(E \cap F) : F \in \mathscr{S}\}$. Hence, there exists a sequence $\{E_n\}$ in $\mathscr{S}$ such that $\nu_\lambda(E) = \lim \nu(E \cap E_n)$. For each n, there exists F_n in $\mathscr{S}$ such that $\nu(E \cap E_n) = \nu(E \cap E_n \cap F_n)$ and $\mu(F_n) = 0$. If $F = \cup F_n$, then $\nu(E \cap E_n) = \nu(E \cap E_n \cap F)$ for all n. Hence, $\nu_\lambda(E) = \nu(E \cap F)$ and $\mu(F) = 0$.

(2) Suppose $\nu \ll \mu$ and suppose $\mu_\lambda(E) = 0$. Then $\mu(E \cap F) = 0$ for all F in $\mathscr{S}$. Since $\nu \ll \mu$, we have $\nu(E \cap F) = 0$ for all F in $\mathscr{S}$ so that $\nu_\lambda(E) = 0$.

(3) The proof is similar to that of (1). If $\mu_{F_n} \ll \nu$ for all n and $F = \cup F_n$, then $\mu_F \ll \nu$. Then $(\mu_F)_\lambda \ll \nu_\lambda$ by (2), and we use the fact that $(\mu_F)_\lambda = (\mu_\lambda)_F$.

(4) This result follows from (1) and (2) and the fact that $\mu S \nu$ and $\nu \ll \mu$.

3. Convergence of measures. In this section we examine the extent to which quasi-dominance or strong recessiveness is preserved under convergence of measures. Our notation is as follows: If μ_n and μ are measures such that $\mu_n(A) \to \mu(A)$ for all $A \in \mathscr{S}$, we write $\mu_n \to \mu$. If μ_α and μ are measures such that $\mu_\alpha(A) \to \mu(A)$ for each $A \in \mathscr{S}$, where the α's are members of some directed set, we write $\mu_\alpha \to \mu$. If $\mu_m \leqq \mu_n$ whenever $m \leqq n$ [$\mu_\alpha \leqq \mu_\beta$ whenever $\alpha \leqq \beta$], then we write $\mu_n \uparrow \mu$ [resp., $\mu_\alpha \uparrow \mu$]. An increasingly directed net of measures always converges to a measure, namely its supremum, but we have no need of this fact.

THEOREM 3.1. *Suppose $\nu_n Q \mu$ for all n and $\nu_n \to \nu$ or suppose $\nu_\alpha Q \mu$ for all α and $\nu_\alpha \uparrow \nu$. Then $\nu Q \mu$.*

Proof. Suppose $\nu_A <_S \mu$. Since $\nu_n Q \mu$ for all n [$\nu_\alpha Q \mu$ for all α], we have $\nu_n S \nu_A$ for all n [resp., $\nu_\alpha S \nu_A$ for all α] by (2) of Theorem 2.4. In either case, we have $\nu S \nu_A$ [3, page 630]. Necessarily, $\nu(A) = 0$ so that $\nu Q \mu$.

We cannot weaken the convergence in Theorem 3.1 to ordinary convergence of a generalized sequence. That is, there exist measures ν_α, ν and μ such that $\nu_\alpha Q \mu$ for all α and $\nu_\alpha \to \nu$, but it is false that $\nu Q \mu$. Indeed, we can have $\nu <_S \mu$ even though ν is a finite, nonzero measure and μ is a semifinite measure.

Example 3.2. Let X be the set of ordinals less than or equal to the first uncountable ordinal ω_1. Let $\mathscr{S}_1$ be the set of countable subsets of $X - \{\omega_1\}$ or their complements in X. Let $\rho(E) = 0$ if E is countable and 1 if E is the complement of a countable set. For each $\alpha < \omega_1$, let $\rho_\alpha(E) = 1$ if $\alpha \in E$ and 0 otherwise. It is easy to see that $\rho_\alpha \to \rho$. Let $\mathscr{S}_2$ be the Borel sets of the unit interval Y, let λ be Lebesgue measure on $\mathscr{S}_2$, and let $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$. If $\nu_\alpha = \rho_\alpha \times \lambda$ and $\nu = \rho \times \lambda$, then it is clear that $\nu_\alpha \to \nu$. Now let κ be counting measure on $\mathscr{S}_2$, and let μ be the smallest measure on $\mathscr{S}_1 \times \mathscr{S}_2$ such that $\mu(A \times B) = \rho(A)\kappa(B)$ [1, Theorem 39.1 and Exercise 39.18]. Then $\nu_\alpha S \mu$ for all α and $\nu <_S \mu$. Since ν is nonzero, it is false that $\nu Q \mu$.

THEOREM 3.3. *If* (1) *$\nu Q \mu_n$ for all n,* (2) *$\mu_n \to \mu$ and* (3) *ν is semifinite, then $\nu Q \mu$.*

Proof. If $\nu(A) < \infty$, we show that $\nu_A Q \mu$. Suppose $\nu(A) < \infty$. Since $\nu Q \mu_1$, there exists F_1 in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F_1)$, where $(\mu_1)_{F_1} \ll \nu$. We find, inductively, F_n in $\mathscr{S}$ such that

(i) F_n is contained in F_{n-1},

(ii) $\nu(A) = \nu(A \cap F_n)$, and

(iii) $(\mu_n)_{F_n} \ll \nu$.
Let $F = \cap F_n$. Then $(\mu_n)_F \ll \nu$ for all n, and we have $\mu_F \ll \nu$. Since $\nu(A) < \infty$, we have $\nu(A) = \nu(A \cap F)$ and $\nu(A - F) = 0$. Hence, $\nu_A Q \mu$ if $\nu(A) < \infty$. Since ν is semifinite, $\nu Q \mu$ by Theorem 3.1.

It is possible to have measures μ_α, μ and ν such that $\nu Q \mu_\alpha$ for all α and such that $\mu_\alpha \uparrow \mu$ and yet not have $\nu Q \mu$. Indeed, we can arrange to have ν be finite, μ be semifinite, $\nu S \mu_\alpha$ for each α and have $\nu <_S \mu$ where ν is not the zero measure. Choose nonzero measures ν and μ such that $\nu <_S \mu$, where μ is semifinite (and where ν is finite, if desired). The measures $\{\mu_A: \mu(A) < \infty\}$ are directed in the obvious sense and $\mu_A \uparrow \mu$. If $E \in \mathscr{S}$ and $\mu(A) < \infty$, then $\nu(E) = \nu(E - A)$ and $\mu_A(E - A) = 0$. Hence, $\nu S \mu_A$ for each such A.

We now show that the semifiniteness of ν cannot be dropped in the statement of Theorem 3.3. We shall find a nonzero measure ν and an increasing sequence of measures μ_n such that ν is quasi-dominant with respect to each μ_n and such that ν is not quasi-dominant with respect to the limit of the μ_n's.

Example 3.4. For each positive integer i, let X_i be a copy of the unit interval, let $\mathscr{T}_i$ be the Borel sets of X_i, let κ_i be counting measure on $\mathscr{T}_i$, and let λ_i be Lebesgue measure on $\mathscr{T}_i$. Let $Y = \times X_i$ and let $\mathscr{T} = \times \mathscr{T}_i$. Let ρ_n be the smallest product measure of the form $\kappa_1 \times \cdots \times \kappa_n \times \lambda_{n+1} \times \cdots$. If desired, ρ_n can be thought of as the smallest product of $\kappa_1 \times \cdots \times \kappa_n$ and $\lambda_{n+1} \times \cdots$. Then $\rho_1 <_S \rho_2 <_S \rho_3 <_S \cdots$. If $\rho = \sup \rho_n$, then $\rho S \rho_n$ for all n.

Now let κ and λ be counting measure and Lebesgue measure, respectively, on the Borel sets $\mathscr{S}$ of the unit interval X. Let ν be the smallest measure on $\mathscr{S} \times \mathscr{T}$ such that $\nu(A \times B) = \kappa(A)\rho(B)$ [1, Theorem 39.1 and Exercise 39.18]. Let μ_n be the smallest measure on $\mathscr{S} \times \mathscr{T}$ such that $\mu_n(A \times B) = \lambda(A)\rho_n(B)$, and let $\mu = \sup \mu_n$. It is easy to see that $\nu S \mu_n$ for all n (and hence, $\nu Q \mu_n$ for all n), $\mu_n \uparrow \mu$, and $\nu <_S \mu$. Since $\nu \neq 0$, it is false that $\nu Q \mu$.

THEOREM 3.5. *Suppose* $\nu <_S \mu_\alpha$ *for all* α *and* $\mu_\alpha \to \mu$. *If* μ *is semifinite or if* $\mu_\alpha \uparrow \mu$, *then* $\nu <_S \mu$.

Proof. If $\nu(E) > 0$, then $\mu_\alpha(E) = \infty$ for all α. Hence, $\mu(E) = \infty$ if $\nu(E) > 0$. If μ is semifinite, then $\nu <_S \mu$ by Theorem 2.3. On the other hand, suppose $\mu_\alpha \uparrow \mu$. It suffices to show that $\mu S \nu$, but this is the case since $\mu_\alpha S \nu$ for all α [3, Theorem 3.1].

If $\nu <_S \mu_\alpha$ for all α and $\mu_\alpha \to \mu$, it does not follow that $\nu <_S \mu$.

Example 3.6. As in Example 3.2, let X be the set of ordinals less than or equal to the first uncountable ordinal ω_1. Let $\mathscr{S}_1$ be the class of countable subsets of $X - \{\omega_1\}$ or their complements in X. Let $\nu(E) = 0$ if E is countable and 1 if E is the complement of a countable set. For each $\alpha < \omega_1$, let $\mu_\alpha(E)$ be the number of points in E which are greater than α. Let $\mu = \infty\nu$. It is easy to see that $\nu <_S \mu_\alpha$ for all α and $\mu_\alpha \to \mu$, but it is false that $\nu <_S \mu$. Indeed, $\nu Q \mu$ in this case.

THEOREM 3.7. *If* (1) $\nu_\alpha <_S \mu$ *for all* α, (2) $\nu_\alpha \to \nu$ *and* (3) μ *or* ν *is semifinite, then* $\nu <_S \mu$.

Proof. Since $\nu_\alpha <_S \mu$ for all α and since $\nu_\alpha \to \nu$, it is easy to see that $\mu(E) = \infty$ whenever $\nu(E) > 0$. In other words, ν is totally incompatible with μ. By Theorem 2.3, we have $\nu <_S \mu$ if μ is semifinite. We will be able to use this part of the theorem to show that $\nu S \mu$ in the case that ν is semifinite.

Suppose ν is semifinite and suppose $\mu_A \ll \nu$. Then $(\nu_\alpha)_A <_S \mu_A$ for all α, and we have $(\nu_\alpha)_A <_S \nu$ for all α. Since $(\nu_\alpha)_A \to \nu_A$, we use the first part of this theorem to assert that $\nu_A <_S \nu$. Necessarily, $\nu(A) = 0$ so that $\mu(A) = 0$. Hence, $\mu S \nu$ and we are done.

If $\nu_n <_S \mu$ for all n and $\nu_n \uparrow \nu$, does it follow that $\nu <_S \mu$? The answer is no. Indeed, there exist nonzero measures ρ_n and ρ such that $\rho_n <_S \rho$ for all n and such that $\rho_n \uparrow \rho$. Use the measures ρ_n and ρ given in Example 3.4.

4. Atomic and nonatomic measures.

A measurable set will be called an *atom* for μ if it has positive μ-measure and does not contain two disjoint sets of positive μ-measure. We say that a measure is *purely atomic* if every chunk (measurable set of positive measure) contains an atom. We say that a measure is *nonatomic* if it has no atoms. Using these definitions, it is easy to see that a measure is purely atomic [nonatomic] if an equivalent measure is purely atomic [resp., nonatomic]. In Theorem 4.2 and Corollary 4.3 we consider some ways in which quasi-dominance plays a role in the study of purely atomic measures and nonatomic measures.

THEOREM 4.1.

(1) *If* μ *is purely atomic, then so is* μ_A *for each* A *in* $\mathscr{S}$.

(2) *If* μ *is nonatomic, then so is* μ_A *for each* A *in* $\mathscr{S}$.

(3) μ *is purely atomic if and only if* $\mu(A) = 0$ *whenever* μ_A *is nonatomic.*

(4) μ *is nonatomic if and only if* $\mu(A) = 0$ *whenever* μ_A *is purely atomic.*

Proof.

(1) If $\mu_A(E) > 0$, then $\mu(A \cap E) > 0$. Hence, $A \cap E$ contains a set F which is an atom for μ. It is easy to see that F is an atom for μ_A also.

(2) If E were an atom for μ_A, then $A \cap E$ would be an atom for μ.

(3) and (4). By [4, Theorem 2.1], μ can be written as $\mu_1 + \mu_2$, where μ_1 is purely atomic, μ_2 is nonatomic, $\mu_1 S \mu_2$ and $\mu_2 S \mu_1$. The assertions of (3) and (4) then follow from Theorem 1.2.

THEOREM 4.2. *Suppose $\nu \ll \mu$ and $\nu Q \mu$.*

(1) *If μ is purely atomic, then so is ν.*

(2) *If μ is nonatomic, then so is ν.*

Proof. We first notice that $\nu_A Q \mu$ for all A in $\mathscr{S}$ by (8) of Theorem 2.1. To prove (1), suppose μ_A is nonatomic. Since $\nu_A \ll \mu$ and since μ is purely atomic, we have $\mu S \nu_A$ by [4, Theorem 2.3]. In other words, $\nu_A <_S \mu$. Since $\nu_A Q \mu$, we have $\nu(A) = 0$. Hence, ν is purely atomic by (3) of Theorem 4.1.

To prove (2), suppose ν_A is purely atomic. Since $\nu_A \ll \mu$ and since μ is nonatomic, we have $\mu S \nu_A$ by [4, Theorem 1.6]. In other words, $\nu_A <_S \mu$. Since $\nu_A Q \mu$, we have $\nu(A) = 0$. Hence, ν is nonatomic by (4) of Theorem 4.1.

COROLLARY 4.3. (Cf. [4, Theorem 1.5]). *Suppose $\mu = \nu + \lambda$ and $\nu Q \lambda$.*

(1) *If μ is purely atomic, then so is ν.*

(2) *If μ is nonatomic, then so is ν.*

Proof. Suppose $\mu = \nu + \lambda$ and $\nu Q \lambda$. Of course, $\nu Q \nu$ so that $\nu Q(\nu + \lambda)$. That is, $\nu Q \mu$. Then since $\nu \ll \mu$, the conclusions follow from Theorem 4.2.

5. Quasi-dominance and the Radon-Nikodým theorem.

If f is a real-valued function on X, we say that f is *locally measurable* if the inverse image of each Borel set is a locally measurable set. Equivalently, f is locally measurable if and only if $\{x: f(x) > a\} \cap F$ is in $\mathscr{S}$ for all real numbers a and all F in $\mathscr{S}$.

THEOREM 5.1. *Suppose there exists a nonnegative locally measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all E in $\mathscr{S}$. Then $\nu \ll \mu$ and $\nu Q \mu$.*

Proof. It is evident and well-known that $\nu \ll \mu$. Now let

$F = \{x: f(x) > 0\}$. It is easy to see that $\nu(E - F) = 0$ so that $\nu(E) = \nu(E \cap F)$ for all $E \in \mathscr{S}$. We show that $\mu_F \ll \nu$. If $\nu(G) = 0$, then $0 = \int_G f\, d\mu = \int_{F \cap G} f\, d\mu$. Since $f(x) > 0$ for all x in $F \cap G$, we have $0 = \mu(F \cap G) = \mu_F(G)$. Hence, $\nu Q \mu$.

THEOREM 5.2. *Suppose ν is finite, μ is semifinite, $\nu \ll \mu$ and $\nu Q \mu$. Then there exists a nonnegative measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all E in $\mathscr{S}$.*

Proof. Since ν is finite, we can find a set E_0 in $\mathscr{S}$ such that $\nu(E_0) \geqq \nu(E)$ for all E in $\mathscr{S}$. Since $\nu Q \mu$, there exists F in $\mathscr{S}$ such that $\nu(E_0) = \nu(E_0 \cap F)$ and $\mu_F \ll \nu$. Since ν is finite and μ_F is semifinite, it is easy to see that μ_F is sigma-finite. By the usual Radon-Nikodým theorem, there is a nonnegative measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all measurable sets E contained in F. If we let f be zero on the complement of F, then it is clear that $\nu(E) = \int_E f d\mu$ for all E in $\mathscr{S}$.

If ν has a Radon-Nikodým derivative with respect to μ, then ν enjoys a strong form of quasi-dominance in that the set F does not depend on E and that $\nu(E) = \nu(E \cap F)$ can be replaced by $\nu(E - F) = 0$ for all E in $\mathscr{S}$. It is easy to see that if ν is finite and $\nu Q \mu$, then ν enjoys this strong form of quasi-dominance with respect to μ. We might ask if a Radon-Nikodým derivative exists for semifinite measures in the presence of absolute continuity and strong quasi-dominance, and the answer is no. Indeed, even if μ and ν are equivalent semifinite measures, a standard example shows that it may be impossible to find a nonnegative function f such that $\chi_E f$ is measurable and $\nu(E) = \int_E f d\mu$ whenever $\mu(E) < \infty$. (It can be seen that two equivalent semifinite measures have the same sets of sigma-finite measure. Consequently, the Radon-Nikodým theorem holds for such measures if $\nu(E) = \int_E f d\mu$ whenever $\mu(E) < \infty$ [5, Theorem 3.1].)

Example 5.3. [Cf 2, Exercise 31.9]. Let A and B be uncountable sets such that card $A <$ card B. Let $X = A \times B$. A set $\{(a, b): a = a_0\}$ is a vertical line and $\{(a, b): b = b_0\}$ is a horizontal line. Let $\mathscr{S}$ be the smallest sigma-algebra containing vertical lines, horizontal lines and countable sets. Let $\alpha(E)$ be the number of horizontal lines L such that $L - E$ is countable, and let $\beta(E)$ be the number of vertical lines L such that $L - E$ is countable. Let $\mu = \alpha + \beta$ and $\nu = \alpha + 2\beta$. Then $\nu \ll \mu$ and ν is strongly quasi-dominant over μ since $\mu \ll \nu$. Although μ and ν are semifinite, it can be seen that no function f exists such that $\nu(E) = \int_E f d\mu$ for all E in $\mathscr{S}$ such that $\mu(E) < \infty$.

THEOREM 5.4. *Suppose ν is a degenerate measure such that $\nu \ll \mu$. Suppose, moreover, that there exists a locally measurable set F such that $\nu(E - F) = 0$ for all E in $\mathscr{S}$ and such that $\mu_F \ll \nu$. If $f = \infty \chi_F$, then $\nu(E) = \int_E f d\mu$ for all E in $\mathscr{S}$.*

Proof. Suppose $E \in \mathscr{S}$. We wish to show that $\nu(E) = \int_E f d\mu$. It is easy to see that $\nu(E) = \nu(E \cap F)$ and that $\int_E f d\mu = \infty \mu(E \cap F)$. If $\nu(E) = \infty$, then $\nu(E \cap F) = \infty$ so that $\mu(E \cap F) > 0$ and $\infty \mu(E \cap F) = \infty$. If $\nu(E) = 0$, then $\mu_F(E) = 0$ so that $\mu(E \cap F) = 0$ and $\infty \mu(E \cap F) = 0$.

We now look at the Radon-Nikodým theorem from a slightly different point of view. In keeping with [5, page 395], we say that ν is *compatible* with μ if $0 < \nu(E) < \infty$ implies there exists F in $\mathscr{S}$ such that $\nu(E \cap F) > 0$ and $\mu(F) < \infty$. Let us say that ν is *strongly compatible* with μ if $\nu(E) > 0$ implies there exists F in $\mathscr{S}$ such that $\nu(E \cap F) > 0$ and $\mu(F) < \infty$. For example, ν is strongly compatible with μ whenever $\nu S \mu$. Of course, if ν is strongly compatible with μ, then ν is compatible with μ. If ν is (strongly) compatible with μ, then clearly ν_A is (strongly) compatible with μ or μ_A.

Recall that ν is totally incompatible with μ if $\mu(E) = \infty$ whenever $\nu(E) > 0$. If ν is compatible with μ and if ν is totally incompatible with μ, then it is easy to see that ν is degenerate (i.e., has a subset of $\{0, \infty\}$ for its range). A degenerate measure is clearly compatible with any measure.

THEOREM 5.5. *If ν is strongly compatible with μ, then $\nu Q \mu$.*

Proof. Suppose $\nu_A <_S \mu$. We want to show $\nu(A) = 0$. Suppose, to the contrary, that $\nu(A) > 0$. Then there exists F in $\mathscr{S}$ such that $\nu(A \cap F) > 0$ and $\mu(F) < \infty$. In other words $\nu_A(F) > 0$ and $\mu(F) < \infty$, which is impossible since ν_A is totally incompatible with μ by Theorem 2.2.

THEOREM 5.6. *If ν is semifinite and ν is compatible with μ, then ν is strongly compatible with μ. Hence, $\nu Q \mu$ in this case.*

Proof. The result follows immediately from the definitions.

If $\nu Q \mu$, it does not follow that ν is even compatible with μ.

For example, let ν be Lebesgue measure on the Borel sets of [0, 1] and let μ be $\infty\nu$. However, we have the following result if μ is semifinite:

THEOREM 5.7. *If $\nu Q\mu$ and μ is semifinite, then ν is strongly compatible with μ.*

Proof. Suppose $0 < \nu(E)$. Since $\nu Q\mu$, there exists F in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F)$ and $\mu_F \ll \nu$. If $\mu(E \cap F) < \infty$, we are done. Otherwise, there exists a measurable set G contained in $E \cap F$ such that $0 < \mu(G) < \infty$. Since $\mu_F(G) > 0$, we have $\nu(G) > 0$, so that ν is strongly compatible with μ.

We may combine Theorems 5.5, 5.6 and 5.7 as follows:

COROLLARY 5.8. *Suppose μ and ν are semifinite. Then the following are equivalent:*

(1) *ν is compatible with μ.*
(2) *ν is strongly compatible with μ.*
(3) *ν is quasi-dominant with respect to μ.*

If f is a real-valued function on X, let us say that f is μ-measurable if $\{x: f(x) > a\} \cap F$ is in $\mathscr{S}$ for all real numbers a and all measurable F such that $\mu(F) < \infty$. Let $\mathscr{S}_{\varphi\lambda} = \{E: E \cap F \in \mathscr{S}$ whenever $F \in \mathscr{S}$ and $\mu(F) < \infty\}$. Define $\mu_{\varphi\lambda}$ on $\mathscr{S}_{\varphi\lambda}$ by $\mu_{\varphi\lambda}(E) = \sup \{\mu(F): F \in \mathscr{S}$ and $F \subset E$ and $\mu(F) < \infty\}$ for all E in $\mathscr{S}_{\varphi\lambda}$. If μ is semifinite, it is easy to see that $\mu_{\varphi\lambda}$ is an extension of μ to a smallest measure on $\mathscr{S}_{\varphi\lambda}$ [cf. 1, Exercise 17.1]. We shall use these ideas in our next theorem, which is a variation of Theorem 5.1.

THEOREM 5.9. (Cf. [5, Theorem 2.1].) *Suppose μ is semifinite and suppose there exists a nonnegative μ-measurable function f such that $\nu(E) = \int_E f d\mu_{\varphi\lambda}$ for all E in $\mathscr{S}$. Then $\nu \ll \mu$ and $\nu Q\mu$.*

Proof. It is easy to see that $\nu \ll \mu$. We show that ν is strongly compatible with μ. Suppose $0 < \nu(E)$, and let $A = \{x: f(x) > 0\}$. Since $\nu(E) > 0$, it follows that $\mu_{\varphi\lambda}(A \cap E) > 0$. Hence, there exists F in $\mathscr{S}$ such that F is a subset of $A \cap E$ and $0 < \mu(F) < \infty$. Since f is positive on F and since $\mu(F) > 0$, we have $\nu(F) > 0$. Hence, ν is strongly compatible with respect to μ, and we have $\nu Q\mu$ by Theorem 5.5.

If desired, an alternate proof of Theorem 5.2 is possible. Since

μ is semifinite and since $\nu Q \mu$, we have ν is compatible with μ by Theorem 5.7. Then the existence of the Radon-Nikodým derivative follows from [5, Theorem 2.2].

6. **Largest product measures.** Suppose μ and ν are semifinite measures on sigma-rings $\mathscr{S}$ and $\mathscr{T}$, respectively. We say that a measure ρ on $\mathscr{S} \times \mathscr{T}$ is a product of μ with ν if $\rho(A \times B) = \mu(A)\nu(B)$ whenever $A \in \mathscr{S}$ and $B \in \mathscr{T}$. More than one product of μ with ν may exist. Nevertheless, there is always a largest product of μ with ν given by outer measure extension [7, page 265].

In order to see something of the role quasi-dominance and strong recessiveness can play in the study of largest product measures, we state some results without proof. In Theorem 6.2 we see that things work out well if ν is quasi-dominant or strongly recessive with respect to ν'.

THEOREM 6.1. *Suppose*

(1) μ *and* μ' *are semifinite measures on the sigma-ring* $\mathscr{S}$,

(2) ν *and* ν' *are semifinite measures on the sigma-ring* $\mathscr{T}$ *and* $\nu \ll \nu'$,

(3) ρ *is the largest product of* μ *with* ν, *and*

(4) ρ' *is the largest product of* μ' *with* ν'.

Then ρ *can be written as the sum of measures* ρ_1 *and* ρ_2 *such that* $\rho_1 \ll \rho'$ *and* $\rho_2 S \rho'$, *where* ρ_1 *is a product of some measure* μ_1 *with* ν *and* ρ_2 *is a product of some measure* μ_2 *with* ν.

THEOREM 6.2. *Assume the hypotheses of Theorem* 6.1, *and suppose, in addition, that* ν *is quasi-dominant with respect to* ν' *or that* ν *is strongly recessive with respect to* ν'. *Then the measure* ρ_1 *of Theorem* 6.1 *can be taken to be the* largest *product of some measure* μ_1 *with* ν.

In general, the measure ρ_1 of Theorem 6.1 cannot be expressed as the *largest* product of μ_1 with ν. For example, let $\mathscr{S}$ be the Borel sets of the unit interval and let $\mathscr{T}$ be the Borel sets of the product of the unit interval with the two-point set $\{0, 1\}$. Define μ and μ' on $\mathscr{S}$ by $\mu = \kappa$ and $\mu' = \lambda$, where κ is counting measure and λ is Lebesgue measure. Define ν and ν' on $\mathscr{T}$ by

$$\nu(B) = \lambda(\{y: (y, 0) \in B\}) + \lambda(\{y: (y, 1) \in B\})$$

and

$$\nu'(B) = \lambda(\{y: (y, 0) \in B\}) + \kappa(\{y: (y, 1) \in B\}) .$$

Let ρ be the largest product of μ with ν, and let ρ' be the largest product of μ' with ν'. By Theorem 6.1, we may write ρ as a sum of product measures ρ_1 and ρ_2 such that $\rho_1 \ll \rho'$ and $\rho_2 S \rho'$. It can be seen that

$$\rho_1(\{(x, (y, 0)) \colon x = y\}) = 0$$

and

$$\rho_1(\{(x, (y, 1)) \colon x = y\}) = \infty \,.$$

If ρ_1 could be expressed as the largest product of some measure μ_1 with ν, we would have the impossible conclusion that

$$\mu_1 \times \lambda(\{(x, y) \colon x = y\}) = 0$$

and

$$\mu_1 \times \lambda(\{(x, y) \colon x = y\}) = \infty \,,$$

where $\mu_1 \times \lambda$ is the largest product of μ_1 and λ in each case.

We close by stating a theorem with the same hypotheses as Theorem 6.1 but with a conclusion that uses Theorem 2.5 to decompose ν with respect to ν'.

THEOREM 6.3. *Assume the hypotheses of Theorem* 6.1. *Then* ρ *can be written as the sum of measures* ρ_0, ρ_1 *and* ρ_2 *such that* $\rho_0 + \rho_1 \ll \rho'$ *and* $\rho_2 S \rho'$, *where* $\rho_0[\rho_1]$ *is the largest product of some* μ_0 *with* ν_0 [*resp., some* μ_1 *with* ν_1] *and* ρ_2 *is a product of some* μ_2 *with* ν.

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