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METABELIAN REPRESENTATIONS OF KNOT GROUPS

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The question of determining which finite metabelian groups may be the homomorphic image of a given knot group G is considered in this paper. As a starting point, it is shown that a homomorphism of a knot group onto a metabelian group H such that $[H: H']=n$ must factor through $Z_n \otimes A_n$, where A_n is the homology group of the n-fold cyclic covering space.

This is similar to a theorem of Burde [1 Satz 4], and Reyner [5] has also proven a similar result, showing in effect that such a homomorphism must factor through $Z \oslash A_n$. Now, A_n can be given the structure of a module over the ring $Z\langle t\rangle$ of L-polynomials, and the problem of determining the metabelian factor groups of G can be reduced to determining the factor modules of A_n.

In this paper, then, a necessary and sufficient condition is given (in terms of the Alexander matrix) for a knot group to have a representation onto any given metabelian group H such that H' contains no cyclic subgroup of order n^2 for any n. (See Theorem 1.5 and the remarks previous to Theorem 1.3.)

Such metabelian groups have a simple structure which is not shared by arbitrary metabelian groups. We therefore limit the scope of this paper to groups with this property. From Theorem 1.5 we deduce necessary and sufficient conditions for a knot group to have a representation on groups of more restricted classes in terms of the Alexander polynomial (Theorems 1.7 and 1.11).

The results obtained are similar in spirit to previous results of Fox [2] about metacyclic representations, and Riley [6] about A_4 representations, and in fact, their theorems are shown to be special cases of the theorems of this paper (see Examples 1.12 and 1.13).

In a final section it is shown how a table of the homology groups of the cyclic coverings may be used to determine the possible metabelian representations. This is included because of the ease with which one can deduce quite complete information from such a table, which may be calculated by computer. The Alexander matrix, of course, contains more information (in fact in a sense complete information) about metabelian representations, but the information therein is not so readily accessible.

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1. Throughout this paper, K is a knot in S^3, and G, or $G(K)$, is its knot group. $G(K)$ can be written as a semi-direct product $Z \otimes G'$ where Z is generated by an element t, which is a meridian of the knot. Any cyclic group Z or Z_n will be thought of as having a distinguished generator, t. If φ is a homomorphism of G, then $G\varphi$ can be written as $Z\varphi \otimes G'\varphi$. Thus, if H is a metabelian factor group of G, then H can be written as $Z_n \otimes H'$. Note that H is a semi-direct product of a cyclic group and the commutator subgroup. We make the convention that all semi-direct products considered in this paper will be of that form. It is true that if H is any group with H' a finitely generated abelian group, and $H/H' \cong C$ is cyclic, then H can be written as $C \otimes H'$ and C acts without fixed points on H. Now, since H' is abelian, φ must factor through $Z \oslash G'/G''$.

Given a semi-direct product $Z_n \otimes A$ with A abelian, A may be given the structure of a $Z\langle t\rangle$ module ($Z\langle t\rangle$ is the ring of integral L-polynomials) by writing $a^t = t^{-1}at$, where module multiplication is indicated by writing the multiplicand as a superscript. Thus, module multiplication by t is an automorphism, denoted throughout this paper by θ, of order n. Conversely, given a $Z\langle t\rangle$ module A in which θ (i.e., multiplication by t) is of order n, one can define a semi-direct product $Z_n \otimes A$. The following lemma is easily proven.

LEMMA 1.1. *Given groups $H = Z_n \otimes A$, and $H^* = Z_m \otimes A^*$ where m divides n, and both A and A^* are abelian, then there is a homomorphism of H onto H^* which takes the distinguished generator t of Z_n to the distinguished generator t^* of Z_m if and only if A^* is a factor module of A.*

Note. By our convention, A and A^* are the commutator subgroups of H and H^*. Thus a homomorphism of H onto H^* must take A to A^*. The rest follows easily.

Now, as a $Z\langle t\rangle$ module, G'/G'' is simply the knot module $\mathscr{M}$. Thus, there is a homomorphism of G onto the metabelian group $Z_n \otimes A$ taking a meridian of the knot to t, if and only if A is a factor module of the knot module. Since, for any a in A, $a^{t^n} = a$, it follows that A must be a factor module of $\mathscr{M}/\mathscr{M}^{t^n-1}$, which latter module will be denoted by A_n.

THEOREM 1.2. *There is a homomorphism of the knot group G onto the metabelian group $Z_n \otimes A$ taking a meridian of G to the distinguished generator of Z_n if and only if A is a factor module of A_n.*

The underlying group of $\mathscr{M}$ is the homology group of the infinite cyclic covering space of the knot, and θ is induced by a covering translation. Factoring out multiples of $t^n - 1$ is thus equivalent to identifying every nth leaf of the covering. It follows that the underlying group of the module A_n is the homology group of the n-fold branched cyclic covering space of the knot. Theorem 1.2 shows that $Z_n \otimes A$ is a factor group of $Z_n \otimes A_n$, which result is very similar to Burde [1, Satz 4]. In fact, since Z_n acts without fixed points on A_n, it follows easily that Z_n acts without fixed points on A. Thus, any metabelian factor group of a knot group is a semi-direct product of the type considered in his theorem.

Note. The condition that a meridian is mapped to the distinguished generator, t, results in no loss of generality of our results. For, if G maps onto some group $H = Z_n \otimes H'$, then a given meridian is mapped to a generator t^* of some cyclic subgroup Z_n^* such that $H = Z_n^* \otimes H'$.

Convention A. When representations of a knot group G onto a semi-direct product $Z_n \otimes A$ are mentioned in future, it will always be supposed implicitly that a meridian is mapped to the generator t of Z_n.

It has now been shown that the search for various metacyclic factor groups of a knot group reduces to the enumeration of the factor modules of A_n.

The complicated structure of A_n makes a complete analysis difficult. Therefore, we limit ourselves to considering factor modules A of A_n, and hence semi-direct products $Z_n \otimes A$, where the group structure of A is just that of an elementary abelian p-group (p a prime); $A = Z_p \oplus \cdots \oplus Z_p$, as a group direct sum.

In this case, A must be a factor module of $A_n/A_n p$ (where $A_n p$ consists of all multiples of p in A_n). This module will be denoted by $A_{n,p}$, and it may be considered as a $Z_p\langle t\rangle$ module, because every element is of order p or 1. Since $Z_p\langle t\rangle$ is a principal ideal domain (PID), this simplifies the problem greatly.

We make two remarks before we begin to examine the structure of $A_{n,p}$. Firstly, suppose that a knot group G maps onto groups $Z_n \otimes A_i; i=1, \cdots, r$ where A_i is an elementary p_i group, the p_i, $i=1, \cdots, r$ being distinct primes. Then G maps onto $Z_n \otimes (A_1 \oplus A_2 \oplus \cdots \oplus A_r)$. To prove this, one simply observes that $Z_n \otimes (A_1 \oplus \cdots \oplus A_k)$ is the pull-back of the diagram $Z_n \otimes (A_1 \oplus \cdots \oplus A_{k-1}) \twoheadrightarrow Z_n \twoheadleftarrow Z_n \otimes A_k$. One proceeds by induction, using the results of § 1 of [4].

Secondly, suppose that G maps onto $Z_n \otimes A$ where A is any group. Then G maps onto $Z_m \otimes A$ for any multiple m of n, for $Z_m \otimes A$ is a pull-back of the diagram $Z_m \twoheadrightarrow Z_n \twoheadleftarrow Z_n \otimes A$. Furthermore, any homomorphism of G onto a group $Z_m \otimes A$, where θ is of order $n < m$, may be obtained in this way by lifting from $Z_n \otimes A$. For this reason, we will call a semi-direct product $Z_n \otimes A$ *nondegenerate* if the order of θ is n.

The condition that θ be of order n has an important practical consequence. In this case, Z_n is a subgroup of $Z_n \otimes A$ which has no nontrivial subgroups which are normal in $Z_n \otimes A$. Therefore, corresponding to Z_n there is a faithful permutation representation of $Z_n \otimes A$ as a group of permutations of the right cosets of Z_n in $Z_n \otimes A$. Finding a faithful permutation representation is the first step in obtaining so called metabelian invariants of the knot.

The structure of a module over a PID is well known. Let f be an element of $Z_p\langle t\rangle$ and $I(f)$ be the ideal it generates. Denote by $X_p(f)$ the $Z_p\langle t\rangle$ module $Z_p\langle t\rangle/I(f)$.

THEOREM 1.3. *If A is a $Z_p\langle t\rangle$-module then A is a direct sum of submodules $X_p(f_1) \oplus X_p(f_2) \oplus \cdots \oplus X_p(f_k)$ where f_i divides f_{i+1}. Any factor module of A is isomorphic to*

$$X_p(g_1) \oplus X_p(g_2) \oplus \cdots \oplus X_p(g_k) \text{ where } g_i \text{ divides } f_i .$$

Theorem 1.3 shows that to determine the factor modules of $A_{n,p}$, we need only find its direct sum decomposition.

REMARK. The module $X_p(f)$ is most easily realised as follows. Let V be a vector space of dimension $\deg(f)$ over Z_p. Let θ be the automorphism of V which is represented by a companion matrix for f (see "Rational Canonical Form" in any text book on linear algebra). Then V becomes a $Z_p\langle t\rangle$-module by defining $x^t = x\theta$, and V is isomorphic to $X_p(f)$.

Suppose R is a commutative ring with identity, and $F_m(R)$ is a free module with basis $\{e_i\}_{i=1,\cdots,m}$. Let I be an ideal of $F_m(R)$ generated by k elements, $r_j = \sum_{i=1}^m a_{ji}e_i$, and $A = F_m(R)/I$, then the matrix $||a_{ji}||$ is called a relation matrix for A over R. A relation matrix for the knot module is the Alexander matrix, M, of the knot (according to the terminology of Rolfsen [7]). Thus, a relation matrix for A_n is of the form $\left(\frac{M}{W}\right)$ where W is a diagonal matrix with diagonal entries $t^n - 1$. Let $\mathrm{mod}_p\colon Z \to Z_p$ be the natural projection. Then mod_p can be extended to a homomorphism, also called mod_p from $Z\langle t\rangle$ to $Z_p\langle t\rangle$, and further, mod_p can be applied entry by entry to a matrix over $Z\langle t\rangle$. Then it is easily seen that the matrix $\binom{M}{W}$ mod_p is a relation matrix for $A_{n,p}$ over $Z_p\langle t\rangle$. Consider the matrix M mod_p. Since $Z_p\langle t\rangle$ is a PID, this may be put in the form of a diagonal matrix, V, with entries $\delta_1(t), \delta_2(t), \cdots, \delta_k(t)$ where $\delta_i(t)$ divides $\delta_{i+1}(t)$. These δ_i will be called the *mod p elementary divisors* of M. They are elements of $Z_p\langle t\rangle$ whose product is equal to $\Delta_p(t)$, the mod p reduction of the Alexander polynomial, $\Delta(t)$. They are, of course, knot invariants.

The relation matrix for $A_{n,p}$ over $Z_p\langle t\rangle$ is thus of the form $\binom{V}{W}$, and this is equivalent to the diagonal matrix with diagonal entries g.c.d. $(\delta_i(t), t^n - 1)$. We have proven:

THEOREM 1.4. *The $Z_p\langle t\rangle$-module $A_{n,p}$ is a direct sum of modules $A_{n,p} = X_p(\gamma_1) \oplus X_p(\gamma_2) \oplus \cdots \oplus X_p(\gamma_k)$ where $\gamma_i(t) =$ g.c.d $(\delta_i(t), t^n - 1)$, and the δ_i are the* mod p *elementary divisors of M.*

Note that $t - 1$ does not divide δ_i for any i, for otherwise $t-1$ divides $\Delta_p(t)$, the mod p Alexander polynomial. This means that $\Delta(1) \equiv 0 \pmod p$ which is of course impossible, since $\Delta(1) = \pm 1$. This provides a proof that θ acts without fixed points on $A_{n,p}$.

THEOREM 1.5. *Let H be an elementary abelian p-group. Then a knot group G maps onto $Z_n \otimes H$ if and only if as a $Z_p\langle t\rangle$-module, $H = X_p(g_1) \oplus \cdots \oplus X_p(g_k)$ where g_i divides g_{i+1} for all i, and g_i divides γ_i for all i.*

Note. Convention A applies. Of course, g_i may equal 1 for some i, in which case $X_p(g_i)$ is just the zero module.

We deduce a simple consequence of the symmetry property of the Alexander matrix. It was shown by Torres and Fox [8] that the Alexander matrix, M, is equivalent to $\bar{M}^T$ where M^T means the transpose of M, and a bar denotes conjugation. Conjugation is the map from $Z\langle t\rangle$ to $Z\langle t\rangle$ taking t to t^{-1}. As a consequence, the mod p elementary divisors of M are symmetric. That is, $\delta_i(t) = u \cdot \delta_i(t^{-1})$ where u is a unit of $Z_p\langle t\rangle$. We write $\delta_i \sim \bar{\delta}_i$. It follows that $\gamma_i \sim \bar{\gamma}_i$. If we factorise γ_i in $Z_p\langle t\rangle$, we see that if f is a factor of γ_i, then so is $\bar{f}$. Thus we have:

PROPOSITION 1.6. *Let $H = X_p(g_1) \oplus \cdots \oplus X_p(g_k)$ and $\bar{H} = X_p(\bar{g}_1) \oplus \cdots \oplus X_p(\bar{g}_k)$. Then a knot group maps onto $Z_n \otimes H$ if and only if it maps onto $Z_n \otimes \bar{H}$.*

It is easily seen that if f is of odd degree then either $t + 1$ divides f or $f \not\sim \bar{f}$. Now, if n is odd, then $t + 1$ does not divide

γ_i, since if $p = 2$, then $t + 1$ does not divide δ_i, and if $p > 2$, then $t + 1$ does not divide $t^n - 1$. Therefore if f is a divisor of γ_i of odd degree then $\bar{f}$ is also a divisor and $f \not\sim \bar{f}$, so the representations of G onto $Z_n \otimes X_p(f)$ and onto $Z_n \otimes X_p(\bar{f})$ are distinct.

Note. If n is odd, it follows from the above that each γ_i is of even degree. This is to be expected in view of the result of Plans [4] that when n is odd, A_n is a direct double, that is, a direct sum of two identical abelian groups.

Theorem 1.5 allows us to find explicitly all possible metabelian groups with commutator subgroup an elementary abelian p-group which are factor groups of a given knot group by calculating the Alexander matrix and the mod p elementary divisors. This may be a little tedious however, so the following theorem allows us to find certain factor modules of $A_{n,p}$ directly from the Alexander polynomial. The proof is easy, so it is omitted.

THEOREM 1.7. *Suppose $f_p \in Z_p\langle t\rangle$ is a factor of $t^n - 1$ which is not divisible by the square of any polynomial in $Z_p\langle t\rangle$. Then G maps onto the group $Z_n \otimes X_p(f_p)$ if and only if f_p divides $\Delta_p(t)$. Convention A applies.*

Note. (i) If p does not divide n, then the condition that f_p is not divisible by a square is automatically fulfilled, since then $t^n - 1$ is a product of distinct irreducible factors in $Z_p\langle t\rangle$. (See below)

(ii) $Z_n \otimes X_p(f_p)$ is nondegenerate if and only if f_p does not divide $t^k - 1$ for any $k < n$.

We now turn aside to consider the factorization of the polynomial $t^n - 1$ in $Z_p\langle t\rangle$.

Firstly, if p divides n and $n = mp^\alpha$ with g.c.d. $(m, p) = 1$, then $t^n - 1 = (t^m - 1)^{p^\alpha}$, thus we are reduced to the case where p does not divide m. Now $t^m - 1$ is a product of cyclotomic polynomials. $t^m - 1 = \prod_{i|m} \sigma_i$ where σ_i is the ith cyclotomic polynomial, that is, the polynomial whose roots are the primitive ith roots of unity. An explicit formula for $\sigma_i(t)$ is

$$\sigma_i(t) = \prod_{d|i} (t^{i/d} - 1)^{\mu(d)}$$

where $\mu(d) = 0$ if d is divisible by an integral square, and otherwise $\mu(d)$ equals $+1$ or -1 depending on whether d is a product of an even or odd number of distinct primes. μ is called the Mobius function. See [9] p. 139.

As for the factoring of the cyclotomic polynomials we have:

LEMMA 1.8. *If p does not divide i then σ_i factors in $Z_p\langle t\rangle$ into $\Phi(i)/s$ distinct factors of length s, where s is the least positive integer such that $p^s = 1 \pmod i$, and Φ is the Euler function.*

See Theorem III 12 E in [9].

Now we examine more closely the conditions that f_p divides $\Delta_p(t)$ in $Z_p\langle t\rangle$.

LEMMA 1.9. *Let R be an integral domain and let $g(t)$ be a polynomial in $R[t]$. If S is a matrix over R with characteristic polynomial f, then*

$$\prod_{i=1}^{r} g(\xi_i) = \det g(S) \text{ where } \xi_1, \cdots, \xi_r$$

are the roots of f in some extension of R.

Proof. In a suitable extension of R, the matrix may be put in Jordan form, in which case, the diagonal elements of S are simply the ξ_i. Then the diagonal elements of $g(S)$ are $g(\xi_i)$, $i = 1, \cdots, r$, and $g(S)$ is upper triangular. The result follows.

LEMMA 1.10. *Let f be a monic polynomial in $Z[t]$ and $\Delta(t) \in Z[t]$ be the Alexander polynomial of a knot K. Let f_p and $\Delta_p(t)$ be the corresponding polynomials in $Z_p[t]$. Then some factor of f_p divides Δ_p if and only if p divides $\prod_{i=1}^{r} \Delta(\xi_i)$ where $\xi_1, \cdots, \xi_r$ are the roots of f in the complex number field.*

Proof. Let S be an integral matrix with characteristic polynomial equal to f. Then if mod_p is the natural projection of Z onto Z_p, and $S_p = S \,\mathrm{mod}_p$, then S_p has characteristic polynomial f_p, and $(\det \Delta(S)) \,\mathrm{mod}_p = \det \Delta_p(S_p)$. Now some divisor of f_p divides Δ_p if and only if $\prod_{i=1}^{r} \Delta_p(\eta_i) = 0$ where the η_i are the roots of f_p. That is, iff $\det \Delta_p(S_p) = 0$, that is, iff p divides $\det \Delta(S) = \prod_{i=1}^{r} \Delta(\xi_i)$.

Of course, the fact that $\Delta(t)$ is the Alexander polynomial played no part in the above proof. Combining this theorem with Theorem 1.7 we obtain

THEOREM 1.11. *If $\Delta(t)$ is the Alexander polynomial of a knot K, then $G(K)$ maps onto $Z_n \otimes X_p(f_p)$ for some irreducible factor f_p of σ_n in $Z_p\langle t\rangle$, if and only if p divides $\prod_{i=1}^{r} \Delta(\xi_i)$ where $\xi_1, \cdots, \xi_r$*

are the primitive nth roots of unity. $Z_n \otimes X_p(f_p)$ *is nondegenerate if and only if p does not divide n.*

EXAMPLE 1.12. Let $n = p-1$. Then t^n-1 factors as $(t-1)(t-2) \cdots (t-(p-1))$ over Z_p. Therefore G maps onto $Z_{p-1} \otimes X_p(t-a) = \langle s, t; t^{p-1}, s^p, t^{-1}st = s^a \rangle$ with a meridian mapping to t, if and only if $t-a$ divides $\Delta_p(t)$. That is, if and only if $p \mid \Delta(a)$. This is a theorem of Fox [2].

EXAMPLE 1.13. If we take $n = 3$ and $p = 2$, then we see that $G(K)$ maps onto $Z_3 \otimes (Z_2 \oplus Z_2) = A_4$ if and only if 2 divides $\Delta(\omega) \cdot \Delta(\omega^2)$ where ω is a primitive cube root of unity. This was previously stated but not proved by Riley [6].

In Theorem 1.11, there is nothing to tell us which of the groups $Z_n \otimes X_p(f)$ the knot group G maps onto, in other words, which irreducible factor f occurs. However as we will show right now, this is not critical, for if f_1 and f_2 are different irreducible factors of σ_n, then $Z_n \otimes X_p(f_1)$ and $Z_n \otimes X_p(f_2)$ are isomorphic. The isomorphism does not send the distinguished generator of Z_n to distinguished generator and so we lose track of where a meridian is mapped. First we prove a preliminary result.

LEMMA 1.14. *Let m be an integer not divisible by p and let* f_1 *be one of the irreducible factors of* σ_m *over* Z_p. *Let V be a vector space over* Z_p *and let* θ *be an automorphism of V with minimal polynomial* f_1^r. *If* f_2 *is another irreducible factor of* σ_m, *then there exists an integer s coprime with m such that* f_2^r *is the minimal polynomial of* θ^s.

Proof. Since an irreducible factor of σ_m has no repeated roots, we may split f_1^r in some splitting field to obtain $f_1^r = [(t-\xi_1) \cdots (t-\xi_k)]^r$. Then if $s \neq 0$ the minimal polynomial of θ^s is $\prod_{j=1}^k (t-\xi_j^s)^r$. Now, since $f_1 = \prod_{j=1}^k (t-\xi_j)$ has coefficients in Z_p, so does $\prod_{j=1}^k (t-\xi_j^s)$. Also, if g.c.d. $(s, m) = 1$, then ξ_j^s is a primitive mth root of unity, since ξ_j is a primitive mth root of unity. Thus $\prod_{j=1}^k (t-\xi_j^s)$ must be one of the irreducible factors, f_i of σ_m. Clearly, for different choices of s, one obtains all the possible f_i.

As an immediate consequence we obtain

THEOREM 1.15. *If* σ_m^r *divides* $t^n - 1$, *and* f_1 *and* f_2 *are two different irreducible factors of* σ_m *over* Z_p, *then* $Z_n \otimes X_p(f_1^r)$ *is isomorphic to* $Z_n \otimes X_p(f_2^r)$.

2. Calculation from a table of homology groups. Theorems 1.7 and 1.11 togther with the remarks about nondegenerate semi-direct products allow a systematic enumeration of all the metabelian factor groups of G of the form $Z_n \otimes X_p(f)$ where f is irreducible. By pull-back techniques, one can furthermore show that if G maps onto $Z_n \otimes X_p(f_1)$ and $Z_n \otimes X_p(f_2)$, where f_1 and f_2 have no common factor over Z_p, then it maps onto $Z_n \otimes X_p(f_1 f_2)$. Thus, we can determine, from the Alexander polynomial alone, all metabelian factor groups of the form $Z_n \otimes X_p(f)$ where f does not contain a repeated factor. (This condition is automatically fulfilled if $p \mid n$.)

If more general factor groups are desired, one must calculate the structure of $A_{n,p}$. This can be done by calculating the mod p elementary divisors (a separate calculation for each p). An alternative method is to use a table of homology groups of the cyclic covering spaces.

If h_1 and h_2 are elements of $Z_p\langle t\rangle$ and g.c.d. $(h_1, h_2) = 1$, then $X_p(h_1 h_2) = X_p(h_1) \oplus X_p(h_2)$. It follows that a module over $Z_p\langle t\rangle$ can be broken up into a direct sum of submodules of the form $X_p(f^\alpha)$ where f is irreducible.

DEFINITION 2.1. Let $A_{n,p}$ be written as a direct sum $\bigoplus X_p(f_i^{\alpha_i})$ where the f_i are irreducible. Let f be in $Z_p\langle t\rangle$. Denote by $H_{n,p}(f)$ the direct sum of just those $X_p(f_i^{\alpha_i})$ such that f_i divides f.

Let $n = qm$ where $q = p^\alpha$ and g.c.d. $(m, p) = 1$. Then, any irreducible factor of $t^n - 1$ in $Z_p\langle t\rangle$ must divide exactly one σ_k where $k \mid m$. It follows that $A_{n,p} = \bigoplus_{k \mid m} H_{n,p}(\sigma_k)$. However it can be deduced from Theorem 1.4 that $H_{n,p}(\sigma_k) = H_{kq,p}(\sigma_k)$. Thus:

$$A_{n,p} = \bigoplus_{k \mid m} H_{kq,p}(\sigma_k) \ . \tag{2.2}$$

We consider two cases.

Case 1. p does not divide n .

In this case, (2.2) becomes

$$A_{n,p} = \bigoplus_{k \mid n} H_{k,p}(\sigma_k) \ . \tag{2.3}$$

Thus, our goal is to determine $H_{k,p}(\sigma_k)$ for each k dividing n. Since Z_p is a field, a module A over $Z_p\langle t\rangle$ can be considered simply as a vector space over Z_p. Define the dimension of A, dim (A) to be its dimension as a vector space. As a group, then, A is a direct sum of dim (A) copies of Z_p. It follows from 2.3 by an easy induction that for any n not divisible by p,

$$\dim H_{n,p}(\sigma_n) = \sum_{i|n} \dim (A_{i,p}) \cdot \mu(n/i) \tag{2.4}$$

where μ is the Mobius function defined previously.

$\dim (A_{i,p})$ is easily read from a table of cyclic homology groups, being the largest integer b such that there is a group homomorphism of A_i onto a direct sum of b copies of Z_p. If σ_n is irreducible over Z_p, then $H_{n,p}(\sigma_n) = X_p(\sigma_n) \oplus \cdots \oplus X_p(\sigma_n)$ is completely determined by its dimension. Otherwise σ_n factors into $\Phi(n)/s = r$ factors $f_1, \cdots, f_r$ of degree s. Then

$$H_{n,p}(\sigma_n) = X_p(f_{i_1}) \oplus X_p(f_{i_2}) \oplus \cdots \oplus X_p(f_{i_N}) .$$

The following considerations are often enough to determine the structure exactly

(i) $Ns = \dim (H_{n,p}(\sigma_n))$.

(ii) $X_p(f_j)$ occurs at least once in this direct sum if and only if f_j divides $\Delta_p(t)$ (over Z_p).

(iii) If $X_p(f_j)$ occurs c times in this direct sum, then f_j^c divides $\Delta_p(t)$, and the Alexander matrix has dimension at least $c \times c$.

Case 2. p divides m .

In this case, using (2.2) it can be shown by induction that

$$\dim H_{kq,p}(\sigma_k) = \sum_{j|k} \dim (A_{jq,p}) \cdot \mu(k/j) . \tag{2.5}$$

If σ_k is irreducible over Z_p, then

$$H_{kq,p}(\sigma_k) = X_p(\sigma_k^{\alpha_1}) \oplus X_p(\sigma_k^{\alpha_2}) \oplus \cdots \oplus X_p(\sigma_k^{\alpha_N}) ,$$

with $\alpha_i \leqq \alpha_{i+1}$ for all i, and $\alpha_N \leqq q$. The following conditions are often sufficient to determine the α_i.

(i) $\dim H_{kq,p}(\sigma_k) = \deg (\sigma_k) \cdot \sum_{i=1}^{N} \alpha_i$

(ii) $N \cdot \deg (\alpha_k) = \dim H_{k,p}(\sigma_k)$.

The second condition follows from the fact that if $H_{kq,p}(\sigma_k)$ is of the form given above, the $H_{k,p}(\sigma_k)$ is a direct sum of N copies of $X_p(\sigma_k)$. Thus, using (2.4) we may determine N. The case where σ_m is not reducible may be considered using a combination of the above methods. Details are omitted.

As an example we consider the knot 9_{40}, which is a 3-bridged knot with Alexander polynomial $1 - 7t + 18t^2 - 23t^3 + 18t^4 - 7t^5 + t^6$ and whose cyclic covering spaces are as follows

Degree	*Torsion*					
2	5	15				
3	4	4	8	8		
4	3	3	3	15	15	
5	11	11	11	11		
6	0	0	4	4	40	40

In explanation, the last line means that the 6 fold cyclic homology group is $Z \oplus Z \oplus Z_4 \oplus Z_4 \oplus Z_{40} \oplus Z_{40}$.

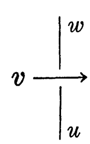
The following is a list of modules $H_{kq,p}(\sigma_k)$

$$H_{2,5}(\sigma_2) = X_5(\sigma_2) \oplus X_5(\sigma_2);\ H_{2,3}(\sigma_2) = X_3(\sigma_2)$$
$$H_{3,2}(\sigma_3) = X_2(\sigma_3) \oplus X_2(\sigma_3)$$
$$H_{4,3}(\sigma_4) = X_3(\sigma_4) \oplus X_3(\sigma_4);\ H_{4,5}(\sigma_4) = 0$$
$$H_{5,11}(\sigma_5) = X_{11}(t-5) \oplus X_{11}(t-5) \oplus X_{11}(t-9) \oplus X_{11}(t-9)$$
$$H_{6,2}(\sigma_3) = X_2(\sigma_3) \oplus X_2(\sigma_3^2);\ H_{6,3}(\sigma_2) = X_3(\sigma_3^2)$$
$$H_{6,5}(\sigma_6) = X_5(\sigma_6)\ .$$

For $p \geqq 7$, $H_{6,p}(\sigma_6) = X_p(\sigma_6)$ and for $p \equiv 1 \bmod 6$, $X_p(\sigma_6) = X_p(t-\alpha) \oplus X_p(t-\beta)$ where α and β are the two roots of σ_6 in Z_p.

EXPLANATION. In the case of $H_{5,11}(\sigma_5)$ we see that σ_5 factors over Z_{11} into $(t-4)(t-5)(t-9)(t-3)$, of which only $t-5$ and $t-9$ divide $\Delta_p(t)$. This explains the structure of $H_{5,11}(\sigma_5)$. As for $H_{6,2}(\sigma_3)$, we see that $\dim(H_{6,2}(\sigma_3)) = 6$ whereas $\dim(H_{3,2}(\sigma_3)) = 4$, so $H_{6,2}(\sigma_3)$ must be $X_2(\sigma_3) \oplus X_2(\sigma_3^2)$. $H_{6,3}(\sigma_2)$ is treated similarly. All the others follow directly from evaluating the dimension.

Finally it seems appropriate to indicate how a given metabelian representation of a knot group is to be found. Suppose that G has a representation on $Z_n \otimes A$, where A is a $Z_p\langle t\rangle$-module of linear dimension d. Elements of A can be written as d-tuples of elements of Z_p, and θ can be represented by a $d \times d$ matrix S over Z_p. In particular, if $A = X_p(f)$ then S is a companion matrix for f. One seeks to label the overcrossings (corresponding to Wirtinger generators) of a knot diagram with elements of Z_p^d(i.e., d-tuples) in such way that at a crossing point with labels as shown,



the labels obey $(u-v)S + v - w = 0$. Then one verifies that if a

Wirtinger generator corresponding to an overcrossing marked u is mapped to the element $t\cdot u$ of $Z_n \otimes A$, then this is a homomorphism of the knot group.

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