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IN SEARCH OF NONSOLVABLE GROUPS OF CENTRAL TYPE

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In 1963 Iwahori and Matsumoto conjectured that a finite group possessing a central simple projective group algebra must be solvable. We verify this conjecture in case all composition factors are known simple groups.

1. Introduction. A natural question in the theory of projective group representations is which finite groups $\bar{G}$ possess a projective group algebra A that has the simplest possible structure. Iwahori and Matsumoto [10] conjectured that $\bar{G}$ must be solvable if A is central simple. DeMeyer and Janusz [2, Theorem 1] showed that such a group possesses a central extension G (of *central type*) such that there is a complex irreducible character χ of G such that $\chi(1)^2 = [G: Z(G)]$. DeMeyer and Janusz also provided the first support for the solvability conjecture.

In this paper we continue the work of these authors and Isaacs [6], Gagola [4] and Yellen [14] and show

MAIN THEOREM. *A nonsolvable group of central type must possess a new simple group as a composition factor.*

We consider the following hypotheses on an arbitrary finite group S:

(1.1) *Hypothesis.* There is a prime p such that S has a nontrivial abelian Sylow p-subgroup and $p \nmid |\mathrm{Out}\, S|$.

(1.2) *Hypothesis.* If there is a proper subgroup I of p-power index, then I is nonsolvable and all composition factors of I satisfy hypothesis (1.1).

Hypothesis (1.1) is satisfied by all known simple groups (3.1) and (1.2) is also satisfied by all known nonabelian simple groups except for (certain) PSL $(2, q)$ (3.2). Theorem 2.6 shows a group of central type having minimal order among those that are nonsolvable and have no composition factor failing (1.1) must have a composition factor S that fails (1.2). Theorem 2.7 shows further that S cannot be a PSL $(2, q)$ and the main theorem follows.

Our notation is standard and follows Gagen [3], Huppert [5] and Isaacs [7] when appropriate.

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2. The structure of a group of central type.

THEOREM 2.1. *Let G be a group of central type. Then no component of G satisfies hypothesis* (1.1).

Proof. Suppose G has a component E_1 satisfying (1.1) and let $Z = Z(G)$. Label the G-conjugates of E_1 as $E_1 \cdots E_m$ and set $K = \langle E_1 \cdots E_m \rangle Z$.

Let R be a Sylow p-subgroup of G containing P, a Sylow p-subgroup of K and set $N = \bigcap N_R(E_i)$. Then $[R: N] \mid m!$, since R acts as a permutation group on $\{E_1 \cdots E_m\}$ and N is the kernel of this action.

Take $x \in N$. By hypothesis (1.1), $\langle x, P \cap E_i \rangle$ induces the same group of automorphisms of $S = E_i/Z(E_i)$ as does $P \cap E_i$. Hence there is $x_i \in E_i \cap P$ such that $xx_i \in C_G(E_i/Z(E_i))$. However $C_G(E_i/Z(E_i)) = C_G(E_i)$ by [3, 10.3a], and $[xx_i, P \cap E_i] = 1$, so $xx_i \in C_R(P \cap E_i)$. Similarly $[x_i, x_j] = 1$ for $i \neq j$, by [3, 10.2a]. It follows that $xx_1 \cdots x_m \in C_R(P)$. This shows $N = C_R(P)$, since P itself is abelian [3, 10.2a].

Now a theorem of DeMeyer and Janusz [2, Theorem 2] implies that R is a group of central type with center $Z \cap R$. Take $\chi \in \mathrm{char}\, R$ and $\zeta \in \mathrm{char}\, Z \cap R$ to be the associated characters, so $\chi|_{Z \cap R} = [R: Z \cap R]^{1/2}\zeta$. Let τ be an irreducible constitutent of the induced character ζ^P. By Clifford's theorem [8, 17.3] $[R: \mathscr{I}(\tau)] = [P: Z \cap R]$, as $P' = 1$. But $\mathscr{I}_R(\tau) \geqq C_R(P) = N$, so we have

$$p^m \mid [P: Z \cap R] \mid [R: N] \mid m!$$

which is absurd. This proves (2.1).

In order to minimize repitition we fix the following notation for the rest of this section. Let G be a nonsolvable group of central type having minimal order among those possessing only composition factors satisfying (1.1). The characters $\chi \in \mathrm{char}\, G$ and $\zeta \in \mathrm{char}\, Z$, $Z = Z(G)$ are supposed to satisfy $\chi|_Z = [G: Z]^{1/2}\zeta$. Take K to be a minimal normal subgroup of G among those properly containing Z and take $\tau \in \mathrm{char}\, K$ to be an irreducible constituent of the induced character ζ^K.

LEMMA 2.2. *K is abelian.*

Proof. By Theorem 2.1 K/Z is an elementary abelian p-group for some prime p. Consider the "bilinear" function $\langle\!\langle\ ,\ \rangle\!\rangle$: $K \times K \to C^*$ defined by $\langle\!\langle x, y\rangle\!\rangle = \zeta([x, y])$ as in Isaacs [7]. By choice of G, ζ is faithful and so $K^{\perp} = Z(K)$ and $Z(K) = Z(G)$ by choice of K. Thus $\langle\!\langle\ ,\ \rangle\!\rangle$ is nondegenerate on K/Z. This implies, [7], that (K, Z, τ) is a fully ramified triple and it follows that $G = \mathscr{I}_G(\tau)$. But now (G, K, τ) is fully ramified by Gagola [4, 2.2a] and Isaacs [7, 8.2] implies a central extension of G/K is of central type and has the same nonabelian composition factors as G, contrary to the choice of G.

LEMMA 2.3. $C_G(K) = C_G(K/Z)$.

Proof. Let $A = C_G(K/Z)/C_G(K)$. Then A stabilizes the normal series $K > Z > 1$ and so is a p-group. Let $\{a_1 \cdots a_s\}$ be a minimal set of generators for A. Consider the commutator map $[a_i, -]$: $K \to Z$. Since Z is cyclic (ζ is faithful by choice of G), $K/C_K(a_i)$ is cyclic also. But $C_K(a_i) \geqq Z$ and K/Z is elementary abelian, so $K/C_K(a_i)$ order 1 or p. It follows that $[K: C_K(A)] = [K: \cap C_K(a_i)] \leqq p^s$, and by Burnside's basis theorem [5, III. 3.2], $[K: C_K(A)] \leqq p^s \leqq [A: \Phi(A)] \leqq |A|$.

Since $C_K(A)$ is G-invariant $C_K(A) = K$ or Z. If $C_K(A) = K$ the lemma holds, so suppose $C_K(A) = Z$. Let K^* be the dual group of K and let $V \leqq K^*$ be the set of characters vanishing on Z. If a power $\tau^l = \tau \otimes \cdots \otimes \tau$ of τ is in V, then $\zeta^l = (\tau|_Z)^l = 1$ and $|Z| \mid l$, as ζ is faithful. Thus $K^* = \langle \tau, V\rangle$ and $C_G(V) \cap \mathscr{I}_G(\tau) = C_G(K^*)$. However, $C_G(K) = C_G(K^*)$ and $C_G(V) = C_G(K/Z)$, so $C_G(K/Z) \cap \mathscr{I}_G(\tau) = C_G(K)$. Now $|A| \geqq [K: C_K(A)] = [K: Z]$ implies $G = \mathscr{I}_G(\tau)C_G(K/Z)$. Since $A_G = C_G(K/Z)/(C_G(K/Z) \cap \mathscr{I}_G(\tau))$ is a p-group, the nonabelian composition factors of G are also composition factors of $\mathscr{I}_G(\tau)$. Once again $(\mathscr{I}_G(\tau), K, \tau)$ is a fully ramified triple [4, 2.2a] and [7, 8.2] implies a central extension of $\mathscr{I}_G(\tau)/K$ is of central type. Since $G = \mathscr{I}_G(\tau)C_G(K/Z)$, $\mathscr{I}_G(\tau)$ has the same nonabelian composition factors as G, contrary to the choice of G.

LEMMA 2.4. *Let K^* denote the dual group of K and $V \leqq K^*$ denote the characters vanishing on Z. If H is a nonsolvable subnormal subgroup of G containing $C_G(K)$ then $\bar{H} = H/C_G(K)$ violates hypothesis* (1.2) *and $H^1(\bar{H}, V) \neq 0$.*

Proof. By Lemma 2.3 $\bar{H}$ acts faithfully on V. The constituents of the induced character ζ^K are τV, by [8, 6.17]. Therefore $\tau^{h-1} = \tau^h\tau^{-1} \in V$ for each $h \in \bar{H}$.

Consider the split extension $\bar{H} \ltimes V$ (with multiplication

$(h_1v_1)(h_2v_2) = (h_1h_2, v_1^{h_2}v_2))$. Then $\phi: \bar{H} \to \bar{H} \ltimes V$ defined by $\phi(h) = (h, \tau^{h^{-1}})$ is an isomorphism. If $\phi(\bar{H})$ is conjugate to $\bar{H} = \{(h, 1) | h \in \bar{H}\}$, say $\phi(\bar{H})^{(g,x)} = \bar{H}$, then $\tau x^{-g^{-1}} \in \tau V$ is H-invariant. Thus $H \leqq \mathscr{I}_G(\tau x^{-g^{-1}})$, and all composition factors of H are composition factors of G. Again $(\mathscr{I}_G(\tau x^{-g^{-1}}), K, \tau x^{g^{-1}})$ is fully ramified, contrary to the choice of G.

This shows $\bar{H} \ltimes V$ contains at least two conjugacy classes of complements to V and so $H^1(\bar{H}, V) \neq 0$.

By Clifford's theorem [8, 17.3] G acts transitively on τV and this set has p-power order. It follows [5, II. 1.5] that every H-orbit on τV has p-power order. Suppose $I \leqq H$ is the stabilizer of $\sigma \in \tau V$. Then $I = \mathscr{I}_H(\sigma) \neq H$ by choice of G. However, $I = H \cap \mathscr{I}_G(\sigma)$ is subnormal in $\mathscr{I}_G(\sigma)$ and the choice of G forces the nonabelian composition factors of I to fail (1.1). This shows $\bar{H}$ fails hypothesis (1.2).

LEMMA 2.5. *Let T be the maximal solvable normal subgroup of G. Then $T = C_G(K)$.*

Proof. Since $C_G(K) \lhd G$ and $C_G(K) \leqq \mathscr{I}_G(\tau)$, the minimality of G implies $C_G(K)$ is solvable and so $T \geqq C_G(K)$.

Suppose $T \neq C_G(K)$ and let R be a minimal normal subgroup of G among those containing $C_G(K)$ and contained in T. Then R acts completely reducibly on V (as in Lemma 2.4) by Clifford's theorem [8, 17.3]. By Lemma 2.3 $C_V(R) = 1$ and $R/C_G(K)$ is a p'-subgroup. It follows that $N_{V\cdot\bar{G}}(R) = \bar{G}$ where $\bar{G} = G/C_G(K)$. Now the Schur-Zassenhaus theorem implies $H^1(R, V) = 0$ and so $0 = H^1(N_{V \in \bar{G}}(\bar{R}), V) = H^1(G, V)$ contrary to Lemma 2.4.

THEOREM 2.6. *Let G be a nonsolvable group of central type of minimal order among those possessing only composition factors that satisfy* (1.1). *Let $Z = Z(G)$ and let K be a minimal subgroup of G among those properly containing Z. Then $F(G/C_G(K)) = 1$ and no component of $G/C_G(K)$ satisfies* (1.2) *for the prime p dividing K/Z.*

Proof. By Theorem 2.1 and Lemma 2.2, $K \leqq C_G(K)$. By Lemma 2.3 $\bar{G} = G/C_G(K)$ acts faithfully on V where V is the subgroup of the dual group of K consisting of characters with kernel containing Z and the fitting group $F(\bar{G})$ is trivial by Lemma 2.5.

If S is a component of $\bar{G}$, then there is a subnormal subgroup $H \geqq C_G(K)$ such that $\bar{H} \cong S$, and Lemma 2.4 applies.

THEOREM 2.7. *Let G and K be as in* 2.6. *Then* PSL $(2, q)$ *is not a component of $G/C_G(K)$.*

Proof. Suppose $H \geqq C_G(K)$ and $H/C_G(K) \cong \mathrm{PSL}(2, q)$ is a component of $\bar{G} = G/C_G(K)$. Observe that H satisfies the hypotheses of Lemma 2.4 and recall the proof of this lemma. It was shown that each H-orbit on τV has nontrivial p-power order.

However, the subgroups of PSL (2, q) are all known [5, II. 8.27] and only for certain q (see 3.2) does there exist a proper subgroup of p-power index and in each of these cases (except PSL (2, 7), $p = 7$), the subgroup is unique up to conjugacy.

Assume at least one of p and q is not 7 and take $H \geqq I \geqq C_G(K)$ so $[H: I]$ is a nontrivial power of p. Then I fixes an element of *each* H-orbit on τV. Therefore $|\tau V| = |C_{\tau V}(I)|[H: I]$ and so $[K^*: C_{K^*}(I)] \leqq [H: I]$. Now $H = \langle I, I^h \rangle$ for h in H but not in I, so

$$[K^*: C_{K^*}(H)] = [K^*: C_{K^*}(I) \cap C_{K^*}(I^h)] \leqq [H: I]^2 .$$

We have shown $\bar{H} = H/C_G(K)$ acts faithfully on $W = K^*/C_{K^*}(H)$ and $|W| \leqq [H: I]^2$.

In case $[H: I] = 2^m$, q is a Mersene prime and I is the normalizer of a Sylow q-group Q. Consequently, an element A of order $(q-1)/2$ normalizing Q acts faithfully on $[W, Q]$. Since Q acts on $[W, Q]$ as the full multiplicative group of $GF(2^m)$, this implies $(q-1)/2 \mid m$, and so $m = 3$. Since A normalizes a second Sylow q-subgroup Q_1 and centralizes $C_W(Q_1)$, it follows that $|W| = 16$ and W has an irreducible submodule of order 8. This situation cannot occur in G since S is subnormal in G and hence acts completely reducibly on V by Clifford's theorem.

In case $[H: I] = 9$, $q = 8$ and we have a homeomorphism of PSL (2, 8) into SL (4, 3) contrary to the fact that 7 does not divide |SL (4, 3)|.

In case $[H: I] = p$, q is a power of 2 and we have SL (2, q) as a subgroup of SL (2, p). This is impossible. (Even in case $p = 5$, $q = 4$, SL (2, 5) has no subgroup of order 60.)

We are left with the case $p = q = 7$. Here there are exactly two possible conjugacy classes for I and so we may choose I so $|\tau V| \leqq 2|C_{\tau V}(I)|[H: I]$ and so $[K^*: C_{K^*}(I)] \leqq 2[H: I]$. However, both $[H: I]$ and $[K^*: C_{K^*}(I)]$ are powers of 7, so we have $[K^*: C_{K^*}(I)]$ just as above. It follows that PSL (2, 7) is a subgroup of SL (2, 7), contrary to the fact that SL (2, 7) is perfect.

3. Hypothesis (1.1) and (1.2) and the known simple groups.

THEOREM 3.1. *If S is a simple alternating group, group of Lie type or one of the first* 26 *sporadic simple groups, S satisfies hypothesis* (1.1).

Proof. In case S is an alternating group, this follows from

Bertram's postulate [11, 8.6] and the fact that $|\text{Out}\, S|\,|\,4$.

Suppose S is of Lie type having characteristic p. The p' part of $|S|$ is a product of terms of the form $(p^i - 1)/k_i$ where $k_i = 1$ or $p^j - 1$ for some $j < i$. Let m be the maximal value of i for which there is such a factor.

Assume there is a p-primitive prime divisor r of $(p^m - 1)$. Then a Sylow r-subgroup of S is cyclic and Out S is generated by diagonal, graph and field automorphisms, Steinberg [13]. The group of diagonal automorphisms D has order dividing the order of the multiplicative group of the underlying field and so $r \nmid |D|$. Fermat's theorem and $p^m \equiv 1 \pmod r$ imply $(r-1)|m$ and so r does not divide the order of the group of field automorphisms. Hypothesis (1.1) now follows unless perhaps $r = 3$ and the diagram of S has 3-fold symmetry, i,e., if $S = D_4(p^t)$. Thus we reduce to the case $S = D_4(p^t)$, $m = 4t$ and 3 is the only p^t-primitive prime divisor of $p^m - 1$. Then $p^{2t} + 1$ has the form $2^a \cdot 3^b$ and consideration of the squares modulo 12 leads to a contradiction. This shows a group of Lie type satisfies hypothesis (1.1) unless perhaps $p^m - 1 = 63$ or p is a Mersene prime and $m = 2$, [15].

In the first case, inspection of the group order formulae leads to the possibilities: PSL (2, 8), PSL (3, 4), PSL (6, 2), PSP (6, 2), $P\Omega\,(5, 2)$, $P\Omega^+\,(8, 2)$ and the solvable group PSU (3, 2). For these groups, the primes $r = 7$, 5, 31, 5, 5 and 7 respectively satisfy hypothesis (1.1). In the second case $S = \text{PSL}\,(2, p)$ is the only possibility and it has a cyclic Sylow p-subgroup.

Suppose finally that S is one of the first 26 sporadic groups. Then inspection of the list of orders of S, Rudvalis and Hurley [6] reveals that $|S|$ is divisible by a prime $r > 7$ to the first power, and inspection of the list of $|\text{Out}\, S|$, Aschbacher and Seitz [1, Table 1] shows $|\text{Out}\, S|\,|\,2$.

THEOREM 3.2. *Let S be as in* 3.1. *If S fails hypothesis* 1.2 *then* $S = \text{PSL}\,(2, 2^m - 1)$ *where* $2^m - 1$ *is prime or* $\text{PSL}\,(2, 2^m)$ *where* $2^m + 1$ *is prime or* 9.

Proof. Suppose S is an alternating group. The group $A_5 \cong \text{PSL}\,(2, 4)$ is exceptional. Observe that A_6 possesses no subgroups of prime power index. Fix a prime p and choose $n > 6$ minimal so that A_n has a subgroup I of p-power index. If I acts transitively A_n/A_{n-1} then $A_{n-1} \cap I$ is of p-power index in A_{n-1}, contrary to choice of n. Therefore, I has at least two orbits on A_n/A_{n-1} and so $|I|\,|\,k!(n-k)!/2$ for some $1 \leqq k \leqq n$. (No element of I can induce an even permutation on one I-orbit and an odd permutation on the complement of this orbit.) Therefore, the binomial coefficient $\binom{n}{k}$

divides $[A_n : I]$ and is a p-power. This forces $k = 1$ and n itself to be a prime power. This shows the only subgroups of p-power index in A_n are A_{n-1} in case n is a p-power and so A_n satisfies hypothesis (1.2) for $n > 6$ since A_{n-1} is a simple group satisfying hypothesis (1.1) by Theorem 3.1.

Next suppose S is a group of Lie type and characteristic r. Suppose $I \leqq S$ is of p-power index. In case $r = p$, Sylow p-subgroup of G acts transitively on G/K and so K acts transitively on the set of Sylow p-subgroups of S. Thus Theorem A of Seitz [12] applies. None of the possible groups I in his list has a composition factor violating (1.1) and the only cases where I is solvable appear in our list. Next assume $r \neq p$. Then a lemma of Tits [12, 1.6] implies that a maximal subgroup K containing I is parabolic. Just as in the proof of 3.1, let m be the maximum value of i for which $|S|$ has a factor of the form $(r^i - 1)/(r^j - 1)$, $i > j$.

In case $(r^m - 1)$ has an r-primitive prime divisor s then s divides the index of every parabolic subgroup of S and so $r = s$ and $K = I$ corresponds to an extremal node in the associated diagram. Now the nonsolvable composition factors of I are groups of Lie type and so they satisfy hypothesis (1.1). The only way I can be solvable is if S has (B, N) rank $\leqq 2$ and the possibilities appear in our list.

In case $p^m - 1$ has no r-primitive prime divisors then either $S = \mathrm{PSL}(2, 2^m - 1)$ (which appears in our list) or $p^m - 1 = 63$, Zsigmondy [15], and S is one of seven explicit groups. Of these only $\mathrm{PSL}(2, 8)$ has a subgroup of prime power index.

A great deal is known about the 26 known sporadic groups and none of them has a solvable subgroup of prime power index, see Aschbacher and Seitz [1].

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