PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

SPACE COVERINGS BY TRANSLATES OF CONVEX SETS

H. GROEMER

Let (C_i) be a sequence of compact convex subsets of euclidean n-dimensional space E^n. Some necessary and sufficient conditions in order that almost all points of E^n can be covered by translates of the sets C_i are established. It is shown that such a covering is possible if and only if all points of E^n can be covered by congruent copies of the sets C_i.

1. **Introduction.** Let $(C_i) = (C_1, C_2, \cdots)$ be a sequence of convex subsets of n-dimensional euclidean space E^n. We say that (C_i) *permits an isometric covering* of E^n if there are proper isometries (rigid motions) $\sigma_1, \sigma_2, \cdots$ so that $E^n \subset \bigcup \sigma_i C_i$. In [1], [2], [3], [4], [5], and [6] it has been shown that in many cases those sequences (C_i) that permit isometric coverings of E^n can be characterized in terms of convergence properties of sequences involving the volumes, diameters or other functionals associated with the sets C_i. Recently S. K. Stein has pointed out that analogous problems can be considered if instead of isometries only translations are permitted. In fact, already Hlawka [9] has proved some results involving this idea. A covering problem regarding translates of strips has been considered in [7]. Generally speaking, translative problems of this kind appear to be more difficult than the corresponding isometric ones. However, in the present paper it will be shown that the situation is a rather different one if, instead of translative coverings of all E^n, one considers translative coverings of almost all points of E^n (with respect to Lebesgue measure). A sequence (C_i) is said to *permit a translative covering of almost all points of* E^n if there are translates $C'_1, C'_2, \cdots$ of $C_1, C_2, \cdots$ so that $E^n \backslash \bigcup C'_i$ is a nullset. First, we establish some necessary and sufficient conditions in order that a given sequence (C_i) permits such a covering of E^n (Theorem 1 and corollaries). Then we are going to prove the rather unexpeced result that a sequence of compact convex subsets of E^n permits a translative covering of almost all points of E^n if and only if it permits an isometric covering of all E^n (Theorem 2). Although these two covering properties are equivalent there exists apparently no direct procedure for obtaining the one kind of covering from the other.

In the following section two lemmas are proved. Our theorems together with proofs and corollaries are presented in the third section. n denotes always an arbitrary but fixed positive integer.

2. Two lemmas. If $S \subset E^n$, $T \subset E^n$ we denote by $S + T$ the vector sum $\{s + t : s \in S, t \in T\}$. For $x \in E^n$ we write $S + x$ instead of $S + \{x\}$. Lebesgue measure in E^n is denoted by m. The closed ball in E^n of radius r and with center at the origin is denoted by $U(r)$. Instead of $U(1)$ we write simply U. By κ_n we denote the volume of U.

LEMMA 1. *Let M_1 and M_2 be two measurable subsets of E^n such that M_1 is contained in a translate of $U(r_1)$ and M_2 in a translate of $U(r_2)$. Then, there exists a point $p \in E^n$ so that*

$$m(M_1 \cap (M_2 + p)) \geqq \frac{1}{\kappa_n(r_1 + r_2)^n} mM_1 mM_2 .$$

Proof. We can certainly assume that $mM_1 \neq 0$, $mM_2 \neq 0$ and $M_1 \subset U(r_1)$, $M_2 \subset U(r_2)$. If m is viewed as a Haar measure on the translation group of E^n it is a well-known result of measure theory (cf. Halmos [8], p. 261) that

$$\int_{E^n} m(M_1 \cap (M_2 + x))dx = mM_1 mM_2 . \tag{1}$$

The integration in (1) can obviously be restricted to those points x that satisfy $M_1 \cap (M_2 + x) \neq \phi$, which is equivalent to $x \in \{p - q : p \in M_1, q \in M_2\}$. Because of $M_1 \subset U(r_1)$ and $M_2 \subset U(r_2)$ it follows that the integration in (1) can be restricted to $U(r_1 + r_2)$. Using this fact we deduce from (1) that

$$\kappa_n(r_1 + r_2)^n \sup_x m(M_1 \cap (M_2 + x)) \geqq mM_1 mM_2 . \tag{2}$$

Here, if the equality sign would hold we could infer, taking also into account the assumption $mM_1 mM_2 > 0$, that for almost all $y \in U(r_1 + r_2)$

$$m(M_1 \cap (M_2 + y)) = \sup_x m(M_1 \cap (M_2 + x)) > 0 .$$

But this relation is clearly not satisfied if y belongs to a set of positive measure sufficiently close to the boundary of $U(r_1 + r_2)$. Hence, (2) holds with strict inequality and this implies obviously Lemma 1.

To state our second lemma we need the following concepts which have also been used extensively in [3]. For $0 \leqq d \leqq n$ we denote by $\mathscr{C}^d$ the class of compact convex subsets of E^d. For $C \in \mathscr{C}^d$ let $D(C)$ be a line segment of maximal length that is contained in C.

By $N(C)$ we denote the orthogonal projection of C onto a $(d-1)$-flat that is orthogonal to $D(C)$. Furthermore, if $C \in \mathscr{C}^n$ we define

$$N^0(C) = C$$
$$N^k(C) = N(N^{k-1}(C)) \quad (k = 1, 2, \cdots, n)\ ,$$

and

$$D^k(C) = D(N^k(C)) \quad (k = 0, 1, \cdots, n)\ .$$

Here $N^j(C)$ has to be viewed as a subset of an E^{n-j}. By a *truncated k-cylinder* we mean a set of form $K + I_1 + I_2 + \cdots + I_{n-k}$ where K is a compact convex subset of a k-flat H, and $I_1, I_2, \cdots, I_{n-k}$ are mutually orthogonal compact line segments contained in an $(n-k)$-flat $H^\perp$ that is orthogonal to H. The analogously defined set $K + L_1 + L_2 + \cdots + L_{n-k}$ where $L_1, \cdots, L_{n-k}$ are mutually orthogonal lines in $H^\perp$ will be simply referred to as a *k-cylinder*. K is called the base of the k-cylinder. Every truncated k-cylinder is compact and convex, every k-cylinder is closed and convex.

LEMMA 2. *Let k be one of the integers $0, 1, \cdots, n$ and $C \in \mathscr{C}^n$. If Z is the truncated $(n-k)$-cylinder defined by*

$$Z = 3^{-k}N^k(C) + 3^{-1}D^0(C) + 3^{-2}D^1(C) + \cdots + 3^{-k}D^{k-1}(C)$$

($Z = C$ if $k = 0$) then there is a translate Z' of Z so that

$$Z' \subset C\ . \tag{3}$$

Proof. We prove the lemma by induction with respect to k. The case $k = 0$ is trivial. If (3) is true for a given k (and every n) it can be applied to $N(C)$ and we find that there is a translation vector x so that

$$3^{-k}N^{k+1}(C) + 3^{-1}D^1(C) + \cdots + 3^{-k}D^k(C) + x \subset N(C)\ .$$

This implies obviously

$$3^{-(k+1)}N^{k+1}(C) + 3^{-1}D^0(C) + 3^{-2}D^1(C) + \cdots$$
$$+ 3^{-(k+1)}D^k(C) + x \subset \frac{1}{3}(N(C) + D^0(C))\ .$$

Hence (3) holds for $k+1$ if it can be shown that a translate of $1/3(N(C) + D^0(C))$ is contained in C. However, this fact has been proved in [3] (Lemma 1).

3. Theorems. First we prove a theorem whose analogue for isometric coverings has been proved in [4]. The volume, considered as a functional on $\mathscr{C}^d$, will be denoted by v_d. But instead of v_n we write often simply v; hence $v = m$ (the Lebesgue measure) on $\mathscr{C}^n$.

THEOREM 1. *Let (C_i) be a sequence of compact convex subsets of E^n such that the sequence of the diameters of the sets C_i is bounded. Then, (C_i) permits a translative covering of almost all points of E^n if and only if*

$$\sum v(C_i) = \infty . \tag{4}$$

Proof. Since (4) is obviously a necessary condition we have only to show that it is also sufficient. Moreover, it suffices to show that almost all points of the unit ball U can be covered by translates of the sets C_i. The validity of this remark follows from the fact that E^n can be written as a union of countably many unit balls, and that it is possible to partition the positive integers into infinitely many subsequences so that (4) holds for each corresponding subsequence of (C_i) (for more details on this matter see the proof of Theorem 2 in [3]).

To prove that almost all points of U can be covered we note first that there exists a number r so that each C_i is contained in some sphere of radius r. We show now inductively that there exist translates $C'_1, C'_2, \cdots$ of $C_1, C_2, \cdots$ so that for every positive integer k

$$m\Big(U\Big\backslash\bigcup_{i=1}^{k} C'_i\Big) \leqq \kappa_n \prod_{i=1}^{k} (1 - cmC_i) \tag{5}$$

where $c = 1/(\kappa_n(1 + r)^n)$. An application of Lemma 1 to the case $M_1 = U$, $M_2 = C_1$, $r_1 = 1$, $r_2 = r$ yields a set $C'_1 = C_1 + p$ so that

$$m(U \cap C'_1) \geqq cmUmC_1 .$$

This shows that

$$m(U\backslash C'_1) \leqq \kappa_n(1 - cmC_1) ,$$

which is (5) for $k = 1$. If $C'_1, C'_2, \cdots, C'_k$ have already been constructed so that (5) holds we apply again Lemma 1, this time with $M_1 = U\backslash\bigcup_{i=1}^{k} C_2$, $M_2 = C_{k+1}$, $r_1 = 1$, $r_2 = r$. Then we obtain a set $C'_{k+1} = C_{k+1} + p$ so that

$$m\Big(\Big(U\Big\backslash\bigcup_{i=1}^{k} C'_i\Big) \cap C'_{k+1}\Big) \geqq cm\Big(U\Big\backslash\bigcup_{i=1}^{k} C'_i\Big)mC_{k+1} .$$

It follows that

$$m\left(\left(U\setminus\bigcup_{i=1}^{k} C'_i\right)\setminus C'_{k+1}\right) \leqq m\left(U\setminus\bigcup_{i=1}^{k} C'_i\right) - cm\left(U\setminus\bigcup_{i=1}^{k} C'_i\right)mC_{k+1} .$$

Because of (5) this shows that

$$m\left(U\setminus\bigcup_{i=1}^{k+1} C'_i\right) \leqq \kappa_n \prod_{i=1}^{k+1}(1 - cmC_i)$$

which is (5) with k replaced by $k + 1$. Due to the fact that $m(U\setminus C'_1)$ is finite and $m(U\setminus\bigcup_{i=1}^{k} C'_i)$, considered as a sequence in k, is decreasing it follows from (5) that

$$m\left(U\setminus\bigcup_{i=1}^{\infty} C'_i\right) \leqq \kappa_n \prod_{i=1}^{\infty}(1 - cmC_i) . \tag{6}$$

Since (4) implies $\prod_{i=1}^{\infty}(1 - cmC_i) = 0$ we obtain from (6) $m(U\setminus\bigcup_{i=1}^{\infty} C'_i) = 0$, which is the desired result.

Theorem 1 can be generalized if one considers for a given $k = 1, 2, \cdots, n$ a sequence of k-cylinders $Z_i = B_i + H_i$ where each B_i is a compact convex subset of some k-flat G_i in E^n, and H_i is an $(n - k)$-flat orthogonal to G_i. The k-flats G_i are not assumed to be parallel. If Q_i is a unit cube in H_i and if we define $Z_i^* = B_i + Q_i$ then $v(Z_i^*) = v_k(B_i)$. Therefore, if the sequence of the diameters of B_i is bounded the condition $\sum_i v_k(B_i) = \infty$ is sufficient in order that (Z_i^*), and consequently (Z_i), permits a translative covering of almost all points of E^n. Conversely if such a covering, say (Z'_i), exists then $m(U(r) \cap \bigcup_i Z'_i) = \kappa_n r^n$. But it is easily proved (see [3], Lemma 2) that for $k = 1, 2, \cdots, n$

$$m(U(r) \cap Z'_i) \leqq \kappa_{n-k} r^{n-k} v_k(B_i) \tag{7}$$

$(\kappa_0 = 1)$. Hence,

$$\kappa_n r^n = m\left(U(r) \cap \bigcup_i Z'_i\right) \leqq \sum_{i=1}^{\infty} m(U(r) \cap Z'_i) \leqq \kappa_{n-k} r^{n-k} \sum_{i=1}^{\infty} v_k(B_i) .$$

If we let r tend to infinity this implies $\sum_{i=1}^{\infty} v_k(B_i) = \infty$. Thus, we can state the following corollary.

COROLLARY 1. *Let k denote one of the integers $1, 2, \cdots, n$, and let (Z_i) be a sequence of k-cylinders Z_i in E^n whose bases B_i have volume $v_k(B_i)$. Moreover, it is assumed that the sequence of the diameters of the sets B_i is bounded. Then, (Z_i) permits a translative covering of almost all points of E^n if and only if*

$$\sum_{i=1}^{\infty} v_k(B_i) = \infty . \tag{8}$$

In the case of 1-cylinders $Z_i = B_i + H_i$ each set Z_i is a strip (or "slab") consisting of all points between two parallel hyperplanes of distance $v_1(B_i)$. Again, for different subscripts i the segments B_i are not assumed to be parallel. We call $v_1(B_i)$ the *width* of Z_i and denote it by w_i. The condition of Corollary 1 that the sequence of the diameters of B_i be bounded is of no importance for strips. In the proof of the necessity of condition (8) the boundedness was not used. On the other hand, if (8) is satisfied there are two possibilities. If $w_i > 1$ for only finitely many strips one can remove these strips from the given sequence without changing the validity of (8). But if $w_i > 1$ for infinitely many strips, each of these strips Z_i can be replaced by a strip contained in Z_i and of width equal to 1, again without disturbing (8). Hence, Corollary 1 yields the following result:

COROLLARY 2. *If (Y_i) is a sequence of strips of widths w_i, then (Y_i) permits a translative covering of almost all points of E^n if and only if $\sum w_i = \infty$.*

The problem regarding translative coverings of all E^n by strips is much more difficult. In [7] it has been shown that for $n = 2$ the condition $\sum w_i^{3/2} = \infty$ is sufficient.

We can now prove our main theorem concerning the relationship between isometric coverings of all E^n and translative coverings of almost all points of E^n.

THEOREM 2. *A sequence (C_i) of compact convex subsets of E^n permits a translative covering of almost all points of E^n if and only if it permits an isometric covering of E^n.*

Proof. It can be assumed that $C_i \neq \phi$ for all C_i. We use the projections $N^k(C)$ and the diameter sets $D^k(C)$ that have been used in Lemma 2. Moreover, for $k = 0, 1, \cdots, n$ we denote for any non-empty $C \in \mathscr{C}^n$ the diameter of $N^k(C)$ by $d^k(C)$, and define $v^n(C) = 1$, $v^k(C) = v_{n-k}(N^k(C))$ $(k = 0, 1, \cdots, n-1)$. As in [3] we introduce also classes S^k consisting of sets C_i by

$$S^0 = \{C_i: d^0(C_i) \leqq 1\}$$
$$S^k = \{C_i: d^k(C_i) \leqq 1, d^{k-1}(C_i) > 1\} \quad (k = 1, 2, \cdots, n) .$$

It has been shown in [3] (Theorem 2) that (C_i) permits an isometric covering of E^n if and only if there is an integer k $(0 \leqq k \leqq n)$ so that

$$\sum_{C_i \in S^k} v^k(C_i) = \infty . \tag{9}$$

Hence, our theorem is proved if we can show that (C_i) permits a translative covering of almost all points of E^n if and only if (9) holds.

Let us first assume that (9) is satisfied. It follows from Lemma 2 that each $C_i \in S^k$ contains a truncated $(n-k)$-cylinder Z_i that is a translate of $3^{-k}N^k(C_i) + 3^{-1}D^0(C_i) + \cdots + 3^{-k}D^{k-1}(C_i)$. Because of $C_i \in S^k$ we have $d^{k-1}(C_i) \geqq 1$ and this implies that the length of each of the segments $D^0(C_i)$, $D^1(C_i)$, $\cdots$, $D^{k-1}(C_i)$ is at least 1. It follows that every $C_i \in S^k$ contains a translate of a truncated $(n-k)$-cylinder T_i of the form $T_i = 3^{-k}(N^k(C_i) + Q_i)$ where Q_i is a k-dimensional unit cube. Since $C_i \in S^k$ implies also that the diameter of each $N^k(C_i)$ is not greater than 1 it follows that the sequence of the diameters of the cylinders T_i is bounded. Taking into account that the volume of T_i is $3^{-kn}v^k(C_i)$ we deduce from (9) and Theorem 1 that the sets C_i from S^k permit a translative covering of almost all points of E^n.

Let us now suppose that (C_i) permits a translative covering of almost all points of E^n. We may assume that the sets C_i have already been so translated that

$$m(E^n \setminus \cup C_i) = 0 . \tag{10}$$

If $k = 0, 1, \cdots, n-1$ and $C_i \in S^k$ we note that C_i is contained in an $(n-k)$-cylinder of the form $X_i = N^k(C_i) + L$ where L is a k-flat orthogonal to $N^k(C_i)$. To each X_i with $C_i \in S^k$ we may apply (7) (replacing the k-cylinders Z'_i by the $(n-k)$-cylinders X_i) which yields

$$m(U(r) \cap X_i) \leqq r^k \kappa_k v^k(C_i) \quad (k = 0, 1, \cdots, n-1) .$$

From this and (10) we can deduce that

$$\begin{aligned} \kappa_n r^n &= m\Big(U(r) \cap \Big(\bigcup_{C_i \notin S^n} X_i \cup \bigcup_{C_i \in S^n} C_i\Big)\Big) \\ &\leqq \sum_{k=0}^{n-1} r^k \kappa_k \sum_{C_i \in S^k} v^k(C_i) + \sum_{C_i \in S^n} v(C_i) . \end{aligned}$$

For $r \to \infty$ it follows that (9) holds for some $k = 0, 1, \cdots, n-1$ or that $\sum_{C_i \in S^n} v(C_i) = \infty$. In the latter case the class S^n must be infinite and this implies $\sum_{C_i \in S^n} v^n(C_i) = \infty$. Hence, (9) holds for some $k = 0, 1, \cdots, n$, and we have obtained the desired conclusion.

As a consequence of Theorem 2 we can transfer all the necessary and sufficient conditions stated in [3] to translative coverings. We mention only the following result:

A sequence (C_i) with $C_i \in \mathscr{C}^n$ permits a translative covering of almost all points of E^n if and only if

$$\sum_{i=1}^{\infty} \frac{v(C_i)}{v(C_i + U) - v(C_i)} = \infty \ .$$

References

1. G. D. Chakerian, *Covering space with convex bodies,* Lecture Notes in Math., Vol. **490,** The geometry of metric and linear spaces. Mich., 1974, Springer-Verlag, (1975), 187–193.

2. G. D. Chakerian and H. Groemer, *On classes of convex sets that permit plane coverings,* Israel J. Math., **19** (1974), 305–311.

3. ———, *On coverings of euclidean space by convex sets,* Pacific J. Math., **75** (1978), 77–86.

4. H. Groemer, *On a covering property of convex sets,* Proc. Amer. Math. Soc., **59** (1976), 346–352.

5. ———, *Some packing and coverings problems,* Amer. Math. Monthly, **83** (1976), 726–727.

6. ———, *On finite classes of convex sets that permit space coverings,* Amer. Math. Monthly, (in print).

7. ———, *On coverings of plane convex sets by translates of strips,* (to appear).

8. P. R. Halmos, *Measure theory,* D. Van Nostrand, Princeton, N. J., 1950.

9. E. Hlawka, *Ausfüllung und Überdeckung konvexer Körper durch konvexe Körper,* Monatsh. Math., **53** (1949), 81–131.

Received November 1, 1978. Supported by National Science Foundation Research Grant MCS 76-06111 A01.

THE UNIVERSITY OF ARIZONA
TUSCON, AZ 85721