PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

A NOTE ON COMPACT OPERATORS WHICH ATTAIN THEIR NORM

J. M. BAKER

For Banach spaces X having the unit cell of $X^{**}w^*$-sequentially compact, the compact operators from X into a Banach space Y attain their norm in X^{**}. The same holds for weakly compact operators if, in addition, X has the strict Dunford-Pettis property. For Banach spaces X such that the quotient space X^{**}/X is separable and Y the space of absolutely summable sequences, a proper subset P_σ of the finite rank operators from X into Y is exhibited. The set P_σ is shown to consist of operators which attain their norm and to be norm-dense in the operator space.

Throughout, X and Y will be Banach spaces and $\mathscr{L}(X, Y)$ the space of bounded linear operators from X into Y. An operator $T \in \mathscr{L}(X, Y)$ attains its norm on the unit cell $S_{X^{**}}$ of X^{**} if $\|T\| = \|T^{**}x^{**}\|$ for some $x^{**} \in X^{**}$ of norm one. For general results on norm attaining operators and their density in $\mathscr{L}(X, Y)$, see [2]. A space X is said to have the strict Dunford-Pettis property [4 p. 137] if for all Banach spaces Y an arbitrary weakly compact operator $T \in \mathscr{L}(X, Y)$ maps weakly Cauchy sequences to strongly Cauchy sequences.

THEOREM 1. *Let X be a Banach space with $S_{X^{**}}$ sequentially compact in the $\sigma(X^{**}, X^*)$ topology. Then*

(i) *if $T \in \mathscr{L}(X, Y)$ is compact, T attains its norm on $S_{X^{**}}$. Thus, every compact operator with reflexive domain X attains its norm on S_X.*

(ii) *if $T \in \mathscr{L}(X, Y)$ is weakly compact and X has the strict Dunford-Pettis property, T attains its norm on $S_{X^{**}}$. In addition, therefore, if Y is reflexive, all operators attain their norms on $S_{X^{**}}$.*

Proof. There is a sequence $\{x_n\}$ in S_X satisfying $\|T\| < \|Tx_n\| + 1/n$. Let J_x be the canonical embedding of X into X^{**}. Since $\{J_x x_n\} \subseteqq S_{X^{**}}$ there exists a subsequence $\{x_{n_j}\}$ and an $x^{**} \in S_{X^{**}}$ such that $J_x x_{n_j} \xrightarrow{j} x^{**}$ in the $\sigma(X^{**}, X^*)$-topology. The sequence $\{x_{n_j}\}$ is weakly Cauchy in X, whence under either hypothesis there exists a subsequence $\{w_j\}$ of $\{x_{n_j}\}$ such that $\{Tw_j\}$ is norm-convergent to some $y \in Y$. Since $\{J_x w_j\}$ is $\sigma(X^{**}, X^*)$-convergent to x^{**} and $\{Tw_j\}$ is weakly convergent to y, we have $T^{**}x^{**} = J_y y$. Thus,

$$\|Tw_j\| \to \|y\| = \|T^{**}x^{**}\|, \quad \text{whence } \|T\| = \|T^{**}x^{**}\|. \quad \Delta$$

A consequence of [9, Theorem 3] is that if S_X is an RNP set and $\delta > 0$ every $T \in \mathscr{L}(X, Y)$ may be written as $T = T_1 + T_2$, where T_2 attains its norm on S_X and T_1 is rank one (thus attaining its norm on $S_{X^{**}}$) with $\|T_1\| < \delta$. A similar weaker result comes from [5] (proof of Theorem 1 and Remark p. 142) and Theorem 1.

COROLLARY. *If X is a Banach space with $S_{X^{**}}\sigma(X^{**}, X^*)$-sequentially compact and $\delta > 0$, every $T \in \mathscr{L}(X, Y)$ may be written as $T = T_1 + T_2$, where both attain their norm on $S_{X^{**}}$ and T_1 is compact with $\|T_1\| < \delta$.*

Let Y be a weakly sequentially complete space and $X = c$ the Banach space of convergent sequences. Every $T \in \mathscr{L}(X, Y)$ is compact [3, p. 515], and since X^* is separable, part (i) of the theorem gives that every operator attains its norm on $S_{X^{**}}$. Moreover, the same can occur under the hypotheses of part (ii). Such cases render the central result Theorem 1 of [5] trivial, making it desirable to find useful subsets of norm attaining operators which are dense in the operator space. Such is the purpose of the remainder of this note for the case of Banach spaces X having X^{**}/J_xX separable and $Y = l_1$, the space of absolutely summable sequences. For such spaces X, $\mathscr{L}(X, l_1)$ consists entirely of compact operators [6, Theorem 5].

LEMMA. *If X is a Banach space for which X^{**}/J_xX is separable, then $S_{X^{**}}$ is $\sigma(X^{**}, X^*)$-sequentially compact.*

Proof. Let $\{x_n^{**}\} \subseteq S_{X^{**}}$. Since the $\sigma(X^{**}, X^*)$-sequential closure of X is X^{**} [6], for each positive integer n there exists a sequence $\{x_{ni}\}_{i=1}^{\infty}$ in S_X such that $J_xx_{ni} \xrightarrow{i} x_n^{**}$ in the $\sigma(X^{**}, X^*)$ topology. Let Z be the closed linear span of the set $\{x_{ni}\}$ and apply the lemma in §1 of [6] to deduce that Z^{**}, whence Z^*, is separable. This gives $S_{Z^{**}}$ to be $\sigma(Z^{**}, Z^*)$-sequentially compact. The remainder of the proof is straight forward using the Hahn-Banach theorem.

Let $X^* \otimes_\lambda Y$ denote the tensor product of X^* and Y equipped with the least crossnorm λ [8]. The assignment $(\Sigma f_i \otimes y_i)(x) = \Sigma f_i(x)y_i$ defines an isometric isomorphism of $X^* \otimes_\lambda Y$ onto the subspace of compact operators in $\mathscr{L}(X, Y)$ of finite rank. In the following we let $\{e_i\}$ be the usual unit vector basis of l_1 and put

$$P_o(X, l_1) = \left\{ \sum_{i=1}^{n} f_i \otimes e_i : f_i \in X^* \right\}$$

where n is arbitrary and f_i attains its norm on S_X. $P_\sigma(X, l_1)$ is not equivalent to the tensor product of two norm dense subsets.

THEOREM 2. *If* $X^{**}/J_x X$ *is separable,* $P_\sigma(X, l_1)$ *is norm dense in* $\mathscr{L}(X, l_1)$.

Proof. Let $X^{**}/J_x X$ be separable, $\varepsilon > 0$ be given, and $T \in \mathscr{L}(X, l_1)$. Since every operator in $\mathscr{L}(X, l_1)$ is compact and l_1 has the approximation property [7, p. 115], we have $\mathscr{L}(X, l_1) = X^* \hat{\otimes}_\lambda l_1$, where $\hat{\otimes}$ denotes the closure in $\mathscr{L}(X, l_1)$ of $X^* \otimes_\lambda l_1$. Thus, there exists $T_1 \in X^* \otimes_\lambda l_1$ such that $\|T - T_1\| < \varepsilon/3$, where $T_1 = \sum_{s=1}^k x_s^* \otimes y_s$ for appropriate $x_s^* \in X^*$ and $y_s = (\xi_{1s}, \xi_{2s}, \cdots) \in l_1$. Hence, $T_1 x = \sum_{s=1}^k x_s^*(x) y_s = ([\sum_{s=1}^k \xi_{1s} x_s^*](x), [\sum_{s=1}^k \xi_{2s} x_s^*](x), \cdots)$.

The series $\sum_{j=1}^\infty \|\sum_{s=1}^k \xi_{js} x_s^*\|$ converges: if $\eta > 0$ and m, n are positive integers, $n > m$, there exists $N > 0$ such that $n, m > N$ imply $\sum_{j=m+1}^n |\xi_{js}| < \eta/k\|x_s^*\|$, for $s = 1, 2, \cdots, k$, because for each s $\sum_{j=1}^\infty |\xi_{js}|$ converges. Thus, $\sum_{j=m+1}^n \|\sum_{s=1}^k \xi_{js} x_s^*\| \leqq \sum_{s=1}^k \|x_s^*\| \sum_{j=m+1}^n |\xi_{js}| < \eta$.

For each $j = 1, 2, \cdots$, there exists a norm attaining $f_j \in X^*$ such that $\|f_j - \sum_{s=1}^k \xi_{js} x_s^*\| < \varepsilon/3^{j+1}$[1], whence the series $\sum_{j=1}^\infty \|f_j\|$ converges since $\|f_j\| < \varepsilon|3^{j+1} + \|\sum_{s=1}^k \xi_{js} x_s^*\|$. We define $T_2 x = (f_1(x), f_2(x), \cdots)$. Since T_2 is clearly a bounded linear operator from X into l_1, we note $\|T_2 x - T_1 x\| \leqq \sum_{j=1}^\infty \|f_j - \sum_{s=1}^k \xi_{js} x_s^*\| \|x\| < \varepsilon\|x\|/3$, whence $\|T_2 - T_1\| < \varepsilon/3$. Since there exists $N > 0$ such that $\sum_{j=N+1}^\infty \|f_j\| < \varepsilon/3$, we have $\|T - \sum_{j=1}^N f_j \otimes e_j\| \leqq \|T - T_1\| + \|T_1 - T_2\| + \|T_2 - \sum_{j=1}^N f_j \otimes e_j\| < 2\varepsilon/3 + \sup_{S_X} \sum_{j=N+1}^\infty |f_j(x)| < \varepsilon$.

REFERENCES

1. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc., **67** (1961), 97-98.
2. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Amer. Math. Soc. Survey, **15** (1977).
3. H. Dunford and J. T. Schwartz, *Linear Operators* I, Inter-Science Pub., N. Y., 1958.
4. A. Grothendieck, *Sur les applications lineaires faiblement compactes d'espaces du type C(K)*, Canad. J. Math., **5** (1953), 129-173.
5. J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math., **1** (1963), 139-148.
6. R. D. McWilliams, *On certain Banach spaces which are w*-sequentially dense in their second duals*, Duke J. Math., **37** (1970), 121-126.
7. H. H. Schaefer, *Topological vector spaces*, 2d. ed., McMillan, N. Y., 1967.
8. R. Shatten, *A theory of cross spaces*, Annals of Math. Studies, **26** (1950).
9. Charles Stegall, *Optimization of functions on certain subsets of Banach spaces*, Math. Annalen, **236** (1978), 171-176.

Received April 4, 1978.

WESTERN CAROLINA UNIVERSITY
CULLOWHEE, NC 28723