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SOME ABSTRACT GENERALIZATIONS OF THE LJUSTERNIK-SCHNIRELMANN-BORSUK COVERING THEOREM

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Ljusternik and Schnirelmann and independently Borsuk proved the following well known result: Let $H_1, \cdots, H_k$ be closed subsets of the sphere S^n such that $\bigcup_{i=1}^k H_i = S^n$ and $H_i \cap (-H_i) = \emptyset$ for $i = 1, \cdots, k$, then $k \geqq n+2$.

In this paper, this result is considered from an abstract topological viewpoint: We develope methods for the proof of generalizations of this result in the context of the genus in the sense of A. S. Švarc.

1. Introduction The main concept, which is used in this paper, is the "genus" in the sense of A. S. Švarc (cf. [6, 7]).

DEFINITION 1. (cf. [6, 7, 8]; for another way to introduce this notion cf. [6, 7].) Let M be a topological Hausdorff space, p a prime number and $f: M \to M$ a free $\boldsymbol{Z}_p$-action (i.e., f is continuous, $f^p = id$ and $f(x) \neq x$ for all $x \in M$). Then

$$\mathscr{C}(M, f) := \{G \subset M \mid \text{There exist disjoint closed sets } G_0, \cdots, G_{p-1} \subset M \text{ with } \bigcup_{i=0}^{p-1} G_i = G \text{ and } f^i(G_0) = G_i \text{ for } i = 1, \cdots, p-1\},$$

and the genus $g(M, f)$ is defined by

$$g(M, f) := \min \{\text{card } \mathscr{G} \mid \mathscr{G} \subset \mathscr{C}(M, f), \quad \cup \mathscr{G} = M\}.$$

The genus has several very nice properties (cf. [6, 7, 8]). It is closely related to the earlier notions of the Ljusternik-Schnirelmann category [5] and the Yang index [9]. In general, it is difficult to compute the genus, but there are various estimates in terms of the dimension, connectivity, or (co-)homology of the space.

As for the Ljusternik-Schnirelmann-Borsuk result, it is interesting that, independently of the prime number p and the action f, we always have $g(S^n, f) = n+1$ (this result is mainly due to Krasnosel'skii [4]). Thus, in the Ljusternik-Schnirelmann-Borsuk theorem, we could replace the estimate $k \geqq n+2$ by $k \geqq g(S^n, -id) + 1$, and with this estimate, the result holds in a trivial way in a much more general setting.

THEOREM. (cf. [9, 8].) *Let M be a Hausdorff space, $f: M \to M$ a free $\boldsymbol{Z}_2$-action (i.e., a fixed-point-free involution) and let $M_1, \cdots, M_k \subset M$*

be closed sets such that $\bigcup_{i=1}^{k} M_i = M$ *and* $M_i \cap f(M_i) = \emptyset$ *for* $i = 1, \cdots, k$. *Then* $k \geqq g(M, f) + 1$.

On the other hand, the analogous question for Z_p-actions with $p \geqq 3$ seems to be much more complicated. I formulate it only for normal spaces, since I have no idea how one could treat the general case of Hausdorff spaces.

Problem 1. Let M be a normal topological space, $p \geqq 3$ a prime number, $f: M \to M$ a free Z_p-action and $M_1, \cdots, M_k \subset M$ closed sets such that $\bigcup_{i=1}^{k} M_i = M$ and $M_i \cap f(M_i) = \emptyset$ for $i = 1, \cdots, k$. What is the best estimate of $g(M, f)$ in terms of k and p?

There is some motivation for this problem. If one could prove an estimate $g(M, f) \leqq r(k, p)$ with $r(k, p) = o(p)$ for every fixed k, this would imply that the following long standing conjecture in asymptotic fixed point theory is true (cf. [8]).

Conjecture. Let E be a normed space, $H \subset E$ a nonempty closed convex set and $f: H \to H$ a continuous map such that $f^{m_0}(H)$ is relatively compact for some $m_0 \in N$. Then f has a fixed point (?).

At present, instead of the needed $o(p)$-estimate, only a $O(p)$-estimate is known: In [8], $g(M, f) \leqq (p-1)/2(k-2)$ was proved for compact spaces M, a result which will be slightly improved in this paper.

The main result of this paper (Theorem 2) is a reduction of Problem 1 to the equivalent problem of computing the genus of nice space $L_{k,p}$ with nice actions $\varphi_{k,p}$ on it. It will be shown that $g(M, f) \leqq g(L_{k,p}, \varphi_{k,p})$, where $(L_{k,p}, \varphi_{k,p})$ is a prototype for (M, f) in Problem 1.

To date, only for $p = 2$ or for $k = 3$ have the values of $g(L_{k,p}, \varphi_{k,p})$ been computed and only rough estimates are available for the general case. But the spaces $L_{k,p}$ and the actions $\varphi_{k,p}$ seem to be nice enough to allow numerical computations of $g(L_{k,p}, \varphi_{k,p})$ for small numbers k and p (e.g., $k, p \leqq 7$), which might suggest the general result one should expect. My own (a little vage) conjecture is $g(L_{k,p}, \varphi_{k,p}) = k - s(k, p)$ with $s(k, p) \in \{1, 2, 3\}$.

2. The reduction of Problem 1. Let $N := \{1, 2, 3, \cdots\}$ and $R^\infty := \{x: N \to R \mid x(n) = 0$ for almost every $n \in N\}$, equipped with the usual Euclidean topology. Let $E_i \in R^\infty$, $E_i(n) := \delta_{in}$ for all $n \in N$, and for $q \in N$, $I \subset \{1, \cdots, q\}$ and $i \in \{1, \cdots, q\}$ let

$$\begin{aligned} \Delta_{q-1} &:= \mathrm{co}\{E_1, \cdots, E_q\} , \\ \Delta_{q-1}^I &:= \mathrm{co}\{E_j \mid j \in I\} , \\ \Delta_{q-1;i} &:= \Delta_{q-1}^{\{1,\cdots,q\}\setminus\{i\}} = \mathrm{co}\{E_j \mid j \in \{1, \cdots, q\}\setminus\{i\}\} , \\ \partial\Delta_{q-1} &:= \bigcup_{i=1}^{q} \Delta_{q-1;i} . \end{aligned}$$

Thus Δ_{q-1} is the closed $(q-1)$-dimensional simplex spanned by $E_1, \cdots, E_q$ and Δ_{q-1}^I and $\Delta_{q-1;i}$ are (closed) faces of Δ_{q-1}. We denote by $[\sigma]$ the barycenter of a simplex σ.

Now we are able to state our first theorem:

THEOREM 1. *Let M be a normal space, $k \in N$, p a prime number, $f: M \to M$ a free Z_p-action, and $M_1, \cdots, M_k \subset M$ closed sets such that $\bigcup_{i=1}^k M_i = M$ and $M_i \cap f(M_i) = \emptyset$ for $i = 1, \cdots, k$. Then there exists a continuous map $h: M \to \partial\Delta_{k-1}$ such that $h(M_i) \subset \Delta_{k-1;i}$ and*

$$h(f(h^{-1}(\Delta_{k-1;i}))) \subset \bigcup_{\substack{j=1 \\ j\neq i}}^{k} \mathrm{co}\{[\Delta_{k-1}^K] \mid \{i\} \subset K \subset \{1, \cdots, k\}\setminus\{j\}\} ,$$

in particular $h(f(h^{-1}(\Delta_{k-1;i}))) \cap \Delta_{k-1;i} = \emptyset$ for $i = 1, \cdots, k$.

Proof. Because of $M_i \cap f(M_i) = \emptyset$ and the normality of the space M, there exist open $N_i \subset M$ with $M_i \subset N_i$ and $N_i \cap f(N_i) = \emptyset$ $(i = 1, \cdots, k)$. For $I, J \subset \{1, \cdots, k\}$, let $W_{I,J} := \bigcap_{i \in \{1,\cdots,k\}\setminus I} M_i \setminus \bigcup_{j \in J} N_j$.

We want to define $h: M \to \partial\Delta_{k-1}$ such that for $\emptyset \neq J \subset I \subset \{1, \cdots, k\}$ we have

$$h(W_{I,J}) \subset \mathrm{co}\{[\Delta_{k-1}^K] \mid J \subset K \subset I\} \tag{1}$$

(i.e., roughly speaking, h maps $W_{I,J}$ into the traverse $\mathrm{Tr}(\Delta_{k-1}^J)$ in the complex Δ_{k-1}^I; cf. [2]). The existence of such a map h can be proved as follows:

We proceed by induction on card I, starting with the trivial case card $I = 0$, i.e., $I = \emptyset$. In this case we have $J = \emptyset$ and hence

$$W_{I,J} = \bigcap_{i \in \{1,\cdots,k\}} M_i = \emptyset$$

(observe that $f(\bigcap_{i \in \{1,\cdots,k\}} M_i) \cap M_j \subset f(M_j) \cap M_j = \emptyset$ for every $j \in \{1, \cdots, k\}$ and hence $\bigcap_{i \in \{1,\cdots,k\}} M_i = \emptyset$).

Let $n \in \{0, \cdots, k-2\}$ and assume that we could define h on

$$M^{(n)} := \bigcup_{\substack{I \subset \{1,\cdots,k\} \\ \mathrm{card}\, I \leq n}} \bigcap_{i \in \{1,\cdots,k\}\setminus I} M_i$$

such that (1) holds for $\emptyset \neq J \subset I \subset \{1, \cdots, k\}$ with $\operatorname{card} I \leqq n$ and such that h is continuous on $M^{(n)}$.

Since for $I_1, I_2 \subset \{1, \cdots, k\}$ with $I_1 \neq I_2$ and $\operatorname{card} I_1 = \operatorname{card} I_2 = n+1$, we have

$$\bigcap_{i \in \{1,\cdots,k\} \setminus I_1} M_i \cap \bigcap_{i \in \{1,\cdots,k\} \setminus I_2} M_i = \bigcap_{i \in \{1,\cdots,k\} \setminus (I_1 \cap I_2)} M_i \subset M^{(n)} ,$$

it suffices to extend h independently to all the sets $M^{(n)} \cup \bigcap_{i \in \{1,\cdots,k\} \setminus I} M_i$ with $\operatorname{card} I = n+1$ according to our conditions. The union of all these extensions will be an extension of h to $M^{(n+1)}$ with all the desired properties.

Thus we choose a fixed $I_0 \subset \{1, \cdots, k\}$ with $\operatorname{card} I_0 = n+1$. We define the extension of h to $M^{(n)} \cup \bigcap_{i \in \{1,\cdots,k\} \setminus I_0} M_i$ by induction on $\operatorname{card} J$, where $J \subset I_0$: We start with $\operatorname{card} J = n+1$, i.e., $J = I_0$, and define

$$h(x) := [\Delta_{k-1}^{I_0}] \quad \text{for all} \quad x \in W_{I_0, I_0} .$$

Since $M^{(n)} \cap W_{I_0,I_0} = \emptyset$, this extension is justified and of course continuous.

Let $m \in \{2, \cdots, n+1\}$ and assume that we have defined h on

$$M_{I_0}^{(m)} := M^{(n)} \cup \bigcup_{\substack{J \subset I_0 \\ \operatorname{card} J \geqq m}} W_{I_0, J}$$

such that (1) holds for all $\emptyset \neq J \subset I \subset \{1, \cdots, k\}$ with $\operatorname{card} I \leqq n$ or $\operatorname{card} J \geqq m$ and $I = I_0$ and such that h is continuous on $M_{I_0}^{(m)}$.

Since for $J_1, J_2 \subset I_0$ with $J_1 \neq J_2$ and $\operatorname{card} J_1 = \operatorname{card} J_2 = m-1$ we have

$$W_{I_0,J_1} \cap W_{I_0,J_2} = W_{I_0, J_1 \cup J_2} \subset M_{I_0}^{(m)} ,$$

it suffices to extend h independently to all the sets $M_{I_0}^{(m)} \cup W_{I_0,J}$ with $\operatorname{card} J = m-1$ according to our conditions. The union of all these extensions will be an extension of h to $M_{I_0}^{(m-1)}$ with all the desired properties.

Accordingly, let $J_0 \subset I_0$ with $\operatorname{card} J_0 = m-1$. Then we have

$$\begin{aligned} & W_{I_0,J_0} \cap M_{I_0}^{(m)} \\ & = W_{I_0,J_0} \cap \Big(M^{(n)} \cup \bigcup_{\substack{J \subset I_0 \\ \operatorname{card} J \geqq m}} W_{I_0,J} \Big) \\ & = (W_{I_0,J_0} \cap M^{(n)}) \cup \bigcup_{\substack{J \subset I_0 \\ \operatorname{card} J \geqq m}} (W_{I_0,J_0} \cap W_{I_0,J}) \\ & = \bigcup_{\substack{I \subset \{1,\cdots,k\} \\ \operatorname{card} I \leqq n}} \Big(W_{I_0,J_0} \cap \bigcap_{i \in \{1,\cdots,k\} \setminus I} M_i \Big) \cup \bigcup_{\substack{J \subset I_0 \\ \operatorname{card} J \geqq m}} W_{I_0, J \cup J_0} \end{aligned}$$

$$= \bigcup_{\substack{I\subset\{1,\cdots,k\}\\ \mathrm{card}\, I\leqq n}} W_{I\cap I_0, J_0} \cup \bigcup_{\substack{J\subset I_0\\ \mathrm{card}\, J\geqq m}} W_{I_0, J\cup J_0}$$
$$= \bigcup_{\substack{J_0\subset I\subset I_0\\ \mathrm{card}\, I\leqq n}} W_{I,J_0} \cup \bigcup_{\substack{J_0\subset J\subset I_0\\ \mathrm{card}\, J\geqq m}} W_{I_0,J}\ ,$$

and hence

$$h(W_{I_0,J_0}\cap M_{I_0}^{(m)}) = \bigcup_{\substack{J_0\subset I\subset I_0\\ \mathrm{card}\, I\leqq n}} h(W_{I,J_0}) \cup \bigcup_{\substack{J_0\subset J\subset I_0\\ \mathrm{card}\, J\geqq m}} h(W_{I_0,J})$$
$$\subset \bigcup_{\substack{J_0\subset I\subset I_0\\ \mathrm{card}\, I\leqq n}} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J_0\subset K\subset I\} \cup \bigcup_{\substack{J_0\subset J\subset I_0\\ \mathrm{card}\, J\geqq m}} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J\subset K\subset I_0\}$$
$$\subset \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J_0\subset K\subset I_0\}\ .$$

Since every closed convex subset of a finite dimensional normed space is an AR(normal), we can extend $h|_{W_{I_0,J_0}\cap M_{I_0}^{(m)}}$ continuously to W_{I_0,J_0} such that

$$h(W_{I_0,J_0})\subset \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J_0\subset K\subset I_0\}\ .$$

By this iterative construction, we finally obtain an extension of h to the set $M_{I_0}^{(1)}$, which is equal to $M^{(n)}\cup\bigcap_{i\in\{1,\cdots,k\}\setminus I_0} M_i$, since for every $x\in M$ there is a $j\in\{1,\cdots,k\}$ with $x\notin N_j$.

This shows that we can extend h continuously to $M^{(n+1)}$ such that (1) holds for $\varnothing\neq J\subset I\subset\{1,\cdots,k\}$ with $\mathrm{card}\, I\leqq n+1$ and such that

$$h(M^{(n+1)})\subset \bigcup_{\substack{I\subset\{1,\cdots,k\}\\ \mathrm{card}\, I\leqq n+1}} \bigcup_{\varnothing\neq J\subset I} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J\subset K\subset I\}$$
$$\subset \bigcup_{i=1}^{k} \bigcup_{\substack{I\subset\{1,\cdots,k\}\setminus\{i\}\\ \mathrm{card}\, I\leqq n+1}} \bigcup_{\varnothing\neq J\subset I} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J\subset K\subset I\}$$
$$\subset \bigcup_{i=1}^{k} \varDelta_{k-1;i} = \partial\varDelta_{k-1}\ .$$

Thus we have proved the existence of a continuous map $h\colon M\to\partial\varDelta_{k-1}$, which fulfills (1) for all $\varnothing\neq J\subset I\subset\{1,\cdots,k\}$. We have to prove that (1) implies $h(M_i)\subset\varDelta_{k-1;i}$ and

$$h(f(h^{-1}(\varDelta_{k-1;i})))\subset \bigcup_{\substack{j=1\\ j\neq i}}^{k} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,\{i\}\subset K\subset\{1,\cdots,k\}\setminus\{j\}\}$$

for $i=1,\cdots,k$.

Let $I_i := \{1,\cdots,k\}\setminus\{i\}$. Then we have

$$h(M_i)\subset \bigcup_{\varnothing\neq J\subset I_i} h(W_{I_i,J})\subset \bigcup_{\varnothing\neq J\subset I_i} \mathrm{co}\,\{[\varDelta_{k-1}^K]\,|\,J\subset K\subset I_i\}$$
$$\subset \varDelta_{k-1}^{I_i} = \varDelta_{k-1;i}\ .$$

In addition,

$$h(M\backslash N_i) = \bigcup_{\substack{j=1\\ j\neq i}}^{k} h(M_j\backslash N_i)$$
$$\subset \bigcup_{\substack{j=1\\ j\neq i}}^{k} \operatorname{co}\{[\Delta_{k-1}^K] \mid \{i\} \subset K \subset \{1, \cdots, k\}\backslash\{j\}\} \subset \partial\Delta_{k-1}\backslash\Delta_{k-1;i}$$

and hence

$$h(f(h^{-1}(\Delta_{k-1;i}))) \subset h(f(N_i)) \subset h(M\backslash N_i)$$
$$\subset \bigcup_{\substack{j=1\\ j\neq i}}^{k} \operatorname{co}\{[\Delta_{k-1}^K] \mid \{i\} \subset K \subset \{1, \cdots, k\}\backslash\{j\}\} .$$

For every $k \in \boldsymbol{N}$ and every prime number p we define

$$L_{k,p} := \{(x_1, \cdots, x_p) \in (\partial\Delta_{k-1})^p \mid \text{If } m, n \in \{1, \cdots, p\},\ n \equiv m + 1 (\operatorname{mod} p) \text{ and } x_m \in \Delta_{k-1;i}, \text{ then } x_n \notin \Delta_{k-1;i}\}$$

and

$$\tilde{L}_{k,p} := \{(x_1, \cdots, x_p) \in (\partial\Delta_{k-1})^p \mid \text{If } m, n \in \{1, \cdots, p\},\ n \equiv m + 1 (\operatorname{mod} p) \text{ and } x_m \in \Delta_{k-1;i}, \text{ then } x_n \in \bigcup_{\substack{j=1\\ j\neq i}}^{k} \operatorname{co}\{[\Delta_{k-1}^K] \mid \{i\} \subset K \subset \{1, \cdots, k\}\backslash\{j\}\}\} .$$

Obviously, $\tilde{L}_{k,p} \subset L_{k,p}$, and the map $\varphi_{k,p}: L_{k,p} \to L_{k,p}$, $\varphi_{k,p}(x_1, \cdots, x_p) := (x_2, \cdots, x_p, x_1)$ is a free $\boldsymbol{Z}_p$-action on $L_{k,p}$ and on $\tilde{L}_{k,p}$. Now we can prove

THEOREM 2. *Let M be a normal space, $k \in \boldsymbol{N}$, p a prime number and $f: M \to M$ a free $\boldsymbol{Z}_p$-action. Let $M_1, \cdots, M_k \subset M$ be closed sets such that $\bigcup_{i=1}^k M_i = M$ and $M_i \cap f(M_i) = \emptyset$ for $i = 1, \cdots, k$. Then we have $g(M, f) \leqq g(\tilde{L}_{k,p}, \varphi_{k,p}) = g(L_{k,p}, \varphi_{k,p})$.*

Proof. By Theorem 1, there exists a continuous map $h: M \to \partial\Delta_{k-1}$ such that $h(M_i) \subset \Delta_{k-1;i}$ and such that

$$h(f(h^{-1}(\Delta_{k-1;i}))) \subset \bigcup_{\substack{j=1\\ j\neq i}}^{k} \operatorname{co}\{[\Delta_{k-1}^K] \mid \{i\} \subset K \subset \{1, \cdots, k\}\backslash\{j\}\}$$
$$\subset \partial\Delta_{k-1}\backslash\Delta_{k-1;i} .$$

Let $P: M \to \tilde{L}_{k,p}$, $P(x) := (h(x), h(f(x)), \cdots, h(f^{p-1}(x)))$. Obviously, P is an equivariant map (i.e., $P \circ f = \varphi_{k,p} \circ P$) and hence $g(M, f) \leqq g(\tilde{L}_{k,p}, \varphi_{k,p}) \leqq g(L_{k,p}, \varphi_{k,p})$ (cf. [7, 8]).

Conversely, $g(L_{k,p}, \varphi_{k,p}) \leqq g(\tilde{L}_{k,p}, \varphi_{k,p})$ follows from the fact that $L_{k,p}$ can be covered by the closed subsets $\hat{M}_i := \{(x_1, \cdots, x_p) \in L_{k,p} \mid x_1 \in \Delta_{k-1;i}\}$ $(i = 1, \cdots, k)$, which obviously have the property

$\hat{M}_i \cap \varphi_{k,p}(\hat{M}_i) = \varnothing$, and hence the estimate $g(M, f) \leqq g(\tilde{L}_{k,p}, \varphi_{k,p})$ applies to $(L_{k,p}, \varphi_{k,p})$ instead of (M, f).

REMARKS. 1. Theorem 2 reduces Problem 1 to the following equivalent problem:

Problem 2. Let $k \in N$ and p a prime number. What is the value of $g(L_{k,p}, \varphi_{k,p}) = g(\tilde{L}_{k,p}, \varphi_{k,p})$?

The end of the proof of Theorem 2 shows that, in fact, the value of $g(L_{k,p}, \varphi_{k,p})$ gives the *best* estimate for $g(M, f)$.

2. Since the $\tilde{L}_{k,p}$ are finite dimensional compact sets, Theorem 2 shows that for Problem 1 one cannot expect a better estimate for finite dimensional compact spaces M than for the larger class of normal spaces.

3. Computing $g(L_{k,p}, \varphi_{k,p})$: First results. I can give here the exact value of $g(L_{k,p}, \varphi_{k,p})$ only for the special cases $p = 2$ and $k = 3$. For the rest, only rough estimates are available.

THEOREM 3. (cf. [9] and [8], Satz 8.) *Let* $k \in N$. *Then* $g(L_{k,2}, \varphi_{k,2}) = k - 1$.

Proof. Let $M_i := \{(x_1, x_2) \in L_{k,2} \mid x_1 \in \Delta_{k-1;i}\}$ $(i = 1, \cdots, k)$. Then we have $M_i \cap \varphi_{k,2}(M_i) = \varnothing$ and hence $M_k \subset \bigcup_{i=1}^{k-1} \varphi_{k,2}(M_i)$, which implies

$$L_{k,2} = \bigcup_{i=1}^{k} M_i = \bigcup_{i=1}^{k-1} M_i \cup M_k = \bigcup_{i=1}^{k-1} (M_i \cup \varphi_{k,2}(M_i)) .$$

Since $M_i \cup \varphi_{k,2}(M_i) \in \mathscr{C}(L_{k,2}, \varphi_{k,2})$, we have $g(L_{k,2}, \varphi_{k,2}) \leqq k - 1$.

It is a well known fact that the sphere S^{k-2} can be covered by closed sets $M_1, \cdots, M_k$ such that $M_i \cap (-M_i) = \varnothing$ for $i = 1, \cdots, k$ (cf. [1]). Thus, by Theorem 2 we have $g(L_{k,2}, \varphi_{k,2}) \geqq g(S^{k-2}, -id) = k - 1$.

A less trivial result is

THEOREM 4. *Let* $p \geqq 3$ *be a prime number. Then*

$$g(L_{3,p}, \varphi_{3,p}) = \begin{cases} 1 & if \quad p = 3 \\ 2 & if \quad p \geqq 5 . \end{cases}$$

Proof. I. Obviously, $L_{3,3} \neq \varnothing$ and hence $g(L_{3,3}, \varphi_{3,3}) \geqq 1$. On the other hand, for every $x \in L_{3,3}$, the set $M_1 := \{(x_1, x_2, x_3) \in L_{3,3} \mid x_1 \in \Delta_{2;1}\}$ contains exactly one of the points x, $\varphi_{3,3}(x)$, $\varphi_{3,3}^2(x)$, which shows that $\varphi^j_{3,3}(M_1) \cap \varphi^k_{3,3}(M_1) = \emptyset$ for $j, k = 0, 1, 2, j \neq k$ and $\bigcup_{j=0}^2 \varphi^j_{3,3}(M_1) = L_{3,3}$. Hence $g(L_{3,3}, \varphi_{3,3}) \leqq 1$.

II. Let $p \geqq 5$. To show that $g(L_{3,p}, \varphi_{3,p}) \geqq 2$, we consider the space $S^1(\subset \boldsymbol{C})$ with the $\boldsymbol{Z}_p$-action $f: S^1 \to S^1$, $f(z):= e^{((p-1)/p)\pi i}z$. We cover S^1 by the sets $M_j:= \{e^{i\alpha} \mid 2\pi(j-1)/3 \leqq \alpha \leqq 2\pi j/3\}$ for $j = 1, 2, 3$. By the definition of f, it follows that $M_j \cap f(M_j) = \emptyset$. Hence, by Theorem 2, we have $2 = g(S^1, f) \leqq g(L_{3,p}, \varphi_{3,p})$.

It remains to prove that $g(L_{3,p}, \varphi_{3,p}) \leqq 2$. For every $x = (x_1, \cdots, x_p) \in L_{3,p}$, we define

$$T_x:= \{(a_1, \cdots, a_p) \in \{1, 2, 3\}^p \mid x_j \in \Delta_{2;a_j} \text{ for } j = 1, \cdots, p\} .$$

For $a, b \in \{1, 2, 3\}$, $a \neq b$, let

$$r(a, b):= \begin{cases} 1 & \text{if} \quad b \equiv a + 1 (\bmod 3) \\ 2 & \text{if} \quad b \equiv a + 2 (\bmod 3) , \end{cases}$$

and for each $j \in \{1, \cdots, p\}$, let

$$j^+:= \begin{cases} j+1 & \text{if} \quad j \leqq p-1 \\ 1 & \text{if} \quad j = p \end{cases} \quad \text{and} \quad j^-:= \begin{cases} j-1 & \text{if} \quad j \geqq 2 \\ p & \text{if} \quad j = 1 . \end{cases}$$

Then, for $x \in L_{3,p}$, we define

$$v(x):= \frac{1}{3} \sum_{i=1}^p r(a_j, a_{j^+}) ,$$

where $(a_1, \cdots, a_p)$ is an arbitrary element of T_x. We have to show that this definition does not depend on the special choice of $(a_1, \cdots, a_p) \in T_x$. Let $(a_1, \cdots, a_p), (b_1, \cdots, b_p) \in T_x$ and let $j_1, \cdots, j_l \in \{1, \cdots, p\}$ with $j_1 < j_2 < \cdots < j_l$ such that $a_{j_k} \neq b_{j_k}$ for $k = 1, \cdots, l$, but $a_j = b_j$ for $j \in \{1, \cdots, p\} \setminus \{j_1, \cdots, j_l\}$. Then, by the definition of $L_{3,p}$, we have

$$a_{j_k^+} = a_{j_k^-} = b_{j_k^+} = b_{j_k^-} \in \{1, 2, 3\} \setminus \{a_{j_k}, b_{j_k}\} \quad \text{for} \quad k = 1, \cdots, l .$$

Hence we have, setting $J:= \{j_1^-, \cdots, j_l^-, j_1, \cdots, j_l\}$,

$$\begin{aligned} \frac{1}{3} \sum_{j=1}^p r(a_j, a_{j^+}) &= \frac{1}{3} \sum_{j \in \{1, \cdots, p\} \setminus J} r(a_j, a_{j^+}) + l \\ &= \frac{1}{3} \sum_{j \in \{1, \cdots, p\} \setminus J} r(b_j, b_{j^+}) + l = \frac{1}{3} \sum_{j=1}^p r(b_j, b_{j^+}) . \end{aligned}$$

Obviously, $v(x) \in \boldsymbol{N}$, $p/3 \leqq v(x) \leqq 2p/3$ and $v(x) = v(\varphi_{3,p}(x))$ for all $x \in L_{3,p}$. Furthermore, all the sets $W_n:= v^{-1}(n)$ $(n \in \boldsymbol{N})$ are closed. Since $L_{3,p}$ is the finite, disjoint union of the closed sets $W_n (n \in \boldsymbol{N}, p/3 \leqq n \leqq 2p/3)$, which are invariant under $\varphi_{3,p}$, it suffices to show that $g(W_n, \varphi_{3,p}) \leqq 2$ for all $n \in \boldsymbol{N}$, $p/3 \leqq n \leqq 2p/3$.

We assume that there exists such an n with $g(W_n, \varphi_{3,p}) \geqq 3$. Without loss of generality, we may assume that $g(W_n, \varphi_{3,p}) = 3$, otherwise we could replace W_n by a subset $\tilde{W}_n$ with $\varphi_{3,p}(\tilde{W}_n) = \tilde{W}_n$ and $g(\tilde{W}_n, \varphi_{3,p}) = 3$.

Let $h: \partial\Delta_2 \to S^1 (\subset C)$ be a homeomorphism such that

$$h(\Delta_{2;j}) = \left\{e^{i\alpha} \mid (j-1)\frac{2\pi}{3} \leqq \alpha \leqq j\frac{2\pi}{3}\right\} \quad \text{for} \quad j = 1, 2, 3 .$$

We want to construct a map $P: W_n \to S^1$ via a homotopy argument, such that P is equivariant with respect to $\varphi_{3,p}$ and $f: S^1 \to S^1$,

$$f(z) := e^{((2\pi i)/p)n} z , \quad \text{i.e.,} \quad P(\varphi_{3,p}(x)) = e^{((2\pi i)/p)n} P(x) = f(P(x))$$

for all $x \in W_n$. This will imply that $g(W_n, \varphi_{3,p}) \leqq g(S^1, f) = 2$ in contradiction to $g(W_n, \varphi_{3,p}) = 3$ (cf. [7] and [8], Hilfssatz 10).

Since $g(W_n, \varphi_{3,p}) = 3$, there exist closed subsets $W_n^{(j,k)}$, $W_n^{(j)}$ $(j = 1, 2, 3; k = 0, \cdots, p-1)$ such that $W_n^{(j)} = \bigcup_{k=0}^{p-1} W_n^{(j,k)}$, $\bigcup_{j=1}^{3} W_n^{(j)} = W_n$, $W_n^{(j,k_1)} \cap W_n^{(j,k_2)} = \emptyset$ for $k_1, k_2 = 0, \cdots, p-1$, $k_1 \neq k_2$ and $\varphi_{3,p}^k(W_n^{(j,0)}) = W_n^{(j,k)}$ for $k = 1, \cdots, p-1$ $(j = 1, 2, 3)$. We have to construct a special homotopy

$$H: (W_n^{(1)} \cup W_n^{(2)} \cup W_n^{(3,0)}) \times [0, 1] \longrightarrow S^1:$$

(a) We define

$$H(x, t) := h(x_1) \quad \text{for} \quad (x, t) = ((x_1, \cdots, x_p), t)$$
$$\in ((W_n^{(1)} \cup W_n^{(2)} \cup W_n^{(3,0)}) \times \{0\}) \cup (W_n^{(1,0)} \times [0, 1]) ,$$

and

$$H(x, 1) := f^k(H(\varphi_{3,p}^{p-k}(x), 1)) = e^{((2\pi i)/p)nk} h(x_{p+1-k})$$

for $x = (x_1, \cdots, x_p) \in W_n^{(1,k)}$ with $k \in \{1, \cdots, p-1\}$. Thus, $H_1(\cdot) := H(\cdot, 1)$ is equivariant on $W_n^{(1)}$.

(b) Let $d_1: W_n^{(1)} \times [0, 1] \to (0, 2\pi)$,

$$d_1(x, t) := \arg\left(\frac{H(\varphi_{3,p}(x), t)}{H(x, t)}\right) \quad \text{for} \quad (x, t) \in W_n^{(1)} \times \{0, 1\}$$

and

$$d_1(x, t) := t d_1(x, 1) + (1-t) d_1(x, 0) \quad \text{for} \quad (x, t) \in W_n^{(1)} \times (0, 1) .$$

Observe that we used here the fact that for $x = (x_1, \cdots, x_p) \in W_n^{(1)}$ we have $x_2 \neq x_1$, which implies $H(\varphi_{3,p}(x), 0) = h(x_2) \neq h(x_1) = H(x, 0)$. Now we can define

$$H(x, t) := H(\varphi_{3,p}^{p-k}(x), t) \prod_{m=1}^{k} e^{i d_1(\varphi_{3,p}^{p-m}(x), t)}$$

for $(x, t) \in W_n^{(1,k)} \times (0, 1)$, $k \in \{1, \cdots, p-1\}$.

(c) H is now given in particular on $(W_n^{(1)} \times [0, 1]) \cup (W_n^{(2,0)} \times \{0\})$. By a well known homotopy extension theorem (cf. [3], p. 14), we can extend H continuously to the set $(W_n^{(1)} \cup W_n^{(2,0)}) \times [0, 1]$ such that $H((W_n^{(1)} \cup W_n^{(2,0)}) \times [0, 1]) \subset S^1$. Furthermore, we can define for $x \in W_n^{(2,k)}$ with $k \in \{1, \cdots, p-1\}$:

$$H(x, 1) := f^k(H(\varphi_{3,p}^{p-k}(x), 1)) = e^{((2\pi i)/p)nk} H(\varphi_{3,p}^{p-k}(x), 1) .$$

(d) Let $d_2 : (W_n^{(1)} \cup W_n^{(2)}) \times [0, 1] \to (0, 2\pi)$ be defined analogously to d_1. Since, for $x \in W_n^{(1)} \cup W_n^{(2)}$, $(a_1, \cdots, a_p) \in T_x$ and $s \in \{1, \cdots, p\}$, we have

$$\left| \frac{2\pi}{3} \sum_{m=1}^{s} r(a_m, a_{m^+}) - \sum_{m=1}^{s} d_2(\varphi_{3,p}^{m-1}(x), 0) \right| \leqq \frac{2\pi}{3} ,$$

which implies

$$\sum_{m=1}^{p} d_2(\varphi_{3,p}^{m-1}(x), 0) = \frac{2\pi}{3} \sum_{m=1}^{p} r(a_m, a_{m^+}) = 2\pi n ,$$

it follows for every $(x, t) \in (W_n^{(1)} \cup W_n^{(2)}) \times [0, 1]$ that

$$\begin{aligned} &\sum_{m=1}^{p} d_2(\varphi_{3.p}^{p-m}(x), t) \\ &\quad = t \sum_{m=1}^{p} d_2(\varphi_{3,p}^{p-m}(x), 1) + (1-t) \sum_{m=1}^{p} d_2(\varphi_{3,p}^{p-m}(x), 0) \\ &\quad = t \sum_{m=1}^{p} \frac{2\pi}{p} n + (1-t) 2\pi n = 2\pi n . \end{aligned}$$

Hence, for $(x, t) \in W_n^{(1)} \times [0, 1]$ and $k \in \{1, \cdots, p-1\}$, we have

$$\begin{aligned} &H(\varphi_{3,p}^{p-k}(x), t) \prod_{m=1}^{k} e^{id_2(\varphi_{3,p}^{p-m}(x), t)} \\ &\quad = H(x, t) \prod_{m=k+1}^{p} e^{id_1(\varphi_{3,p}^{p-m}(x), t)} \prod_{m=1}^{k} e^{id(\varphi 2_{p,3}^{p-m}(x), t)} \\ &\quad = H(x, t) \prod_{m=k+1}^{p} e^{id_2(\varphi_{3,p}^{p-m}(x), t)} \prod_{m=1}^{k} e^{id_2(\varphi_{3,p}^{p-m}(x), t)} \\ &\quad = H(x, t) \prod_{m=1}^{p} e^{id_2(\varphi_{3,p}^{p-m}(x), t)} = H(x, t) e^{i2\pi n} = H(x, t) . \end{aligned}$$

This justifies the definition

$$H(x, t) := H(\varphi_{3,p}^{p-k}(x), t) \prod_{m=1}^{k} e^{id_2(\varphi_{3,p}^{p-m}(x), t)}$$

for $(x, t) \in W_n^{(2,k)} \times (0, 1)$, $k \in \{1, \cdots, p-1\}$.

(e) To obtain H on $(W_n^{(1)} \cup W_n^{(2)} \cup W_n^{(3,0)}) \times [0, 1]$, we apply the same homotopy extension theorem as in (c). Finally, we obtain

$P: W_n \to S^1$ by

$$P(x) := \begin{cases} H(x, 1) & \text{for} \quad x \in W_n^{(1)} \cup W_n^{(2)} \cup W_n^{(3,0)} \\ f^k(H(\varphi_{3,p}^{p-k}(x), 1)) & \text{for} \quad x \in W_n^{(3,k)} \text{ with } k \in \{1, \cdots, p-1\} . \end{cases}$$

For $k \geqq 4$ and $p \geqq 3$, only estimates of $g(L_{k,p}, \varphi_{k,p})$ are known, which seem to be not best possible in most cases. However, we can prove a new result, which yields, in conjunction with Theorem 2, a slight improvement of Satz 10 in [8]:

THEOREM 5. *Let $p \geqq 3$ be a prime number and $k \in \{3, 4, 5, \cdots\}$. Then we have*

$$g(L_{k,p}, \varphi_{k,p}) = g(\tilde{L}_{k,p}, \varphi_{k,p}) \leqq \frac{p-1}{2}(k-3) + \begin{cases} 1 & \text{if} \quad p = 3 \\ 2 & \text{if} \quad p \geqq 5 . \end{cases}$$

Proof. Let $M_i := \{(x_1, \cdots, x_p) \in \tilde{L}_{k,p} \mid x_1 \in \Delta_{k-1:i}\}$ and $F_i := \bigcup_{j=0}^{p-1} \varphi_{k,p}^j(M_i)$ $(i = 1, \cdots, k-3)$, and let

$$G := \tilde{L}_{k,p} \cap \left(\bigcup_{j=k-2}^{k} \Delta_{k-1:j}\right)^p .$$

Then we have

$$\tilde{L}_{k,p} = \bigcup_{i=1}^{k-3} F_i \cup G .$$

As a consequence of Theorems 2 and 4, we have

$$g(G, \varphi_{k,p}) \leqq \begin{cases} 1 & \text{if} \quad p = 3 \\ 2 & \text{if} \quad p \geqq 5 . \end{cases}$$

Furthermore, in the proof of Satz 10 in [8], it was shown that $g(F_i, \varphi_{k,p}) \leqq (p-1)/2$. It follows that

$$\begin{aligned} g(\tilde{L}_{k,p}, \varphi_{k,p}) &\leqq \sum_{i=1}^{k-3} g(F_i, \varphi_{k,p}) + g(G, \varphi_{k,p}) \\ &\leqq \frac{p-1}{2}(k-3) + \begin{cases} 1 & \text{if} \quad p = 3 \\ 2 & \text{if} \quad p \geqq 5 . \end{cases} \end{aligned}$$

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