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THE RUDIN KERNELS ON AN ARBITRARY DOMAIN

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Let $\{G_n\}$ $(G_0 \ni x, t)$ denote a regular exhaustion of an arbitrary domain G in the complex plane. For fixed x, $t(\in G)$, let $\hat{R}_t^{(n)}(z, x)$, $\hat{L}_t^{(n)}(z, x)$ and $L_t^{(n)}(z, x)$ denote the Rudin kernels of G_n, respectively. The convergence of the sequences $\{\hat{R}_t^{(n)}(z, x)\}$, $\{\hat{L}_t^{(n)}(z, x)\}$ and $\{L_t^{(n)}(z, x)\}$ is discussed and some properties with respect to their limit functions are investigated. In the final Section, it is pointed out that in the case of an arbitrary hyperbolic Riemann surface, the circumstances are quite different, in general.

1. **Introduction.** In [3] and [4], we have been concerned with some properties of the Rudin kernels and the associated reproducing kernels. Let G denote a domain in the extended complex plane such that ∂G is a compact subset in the finite plane and for fixed x, $t(\in G)$, let $\{G_n\}_{n=0}^{\infty}$ $(G_0 \ni x, t)$ denote a regular exhaustion of G. If there exists, let $g_G(z, t)$ denote the Green's function of G with pole at t and $W_G(z, t)$ denote $g_G(z, t) + i g_G^*(z, t)$, where g_G^* is a harmonic conjugate of g_G. For simplicity, we set $W_{G_n}(z, t) = W_n(z, t)$. Let $H_p(G)(p > 0)$ denote the analytic Hardy class on G and $\{h(z, x)\}_{p,G}$ the class of meromorphic functions $h(z, x)$ such that $h(z, x) - 1/(z - x) \in H_p(G)$ $(x \neq \infty)$. Let $H_p^d(G)$ (resp. $\{h(z, x)\}_{p,G}^d$) denote the subclass of $H_p(G)$ (resp. $\{h(z, x)\}_{p,G}$) such that $f(z) = O(|z|^{-2})$, $z \to \infty$ (resp. $h(z, x) = O(|z|^{-2})$, $z \to \infty$) (in the case that $G \ni \infty$).

For each G_n, there exist the Rudin kernel function $R_t^{(n)}(z, x)$ analytic on $\bar{G}_n$, and the adjoint L-kernel $L_t^{(n)}(z, x)\, dz$ meromorphic differential on $\bar{G}_n$ with one simple pole at x with residue 1, satisfying the following property:

$$(1) \qquad \overline{R_t^{(n)}(z, x)} i d W_n(z, t) = \frac{1}{i} L_t^{(n)}(z, x) dz \quad \text{along} \quad \partial G_n .$$

Here and in this paper, without loss of generality, we assume that $x \neq \infty$. The K-kernel $R_t^{(n)}(z, x)$ is characterized by the following reproducing property:

$$f(x) = \frac{1}{2\pi} \int_{\partial G_n} f(z) \overline{R_t^{(n)}(z, x)}\, i d W_n(z, t) \quad \text{for all} \quad f \in H_2(G_n) .$$

The L-kernel $L_t^{(n)}(z, x)$ is characterized in the class $\{h(z, x)\}_{2,G_n}^d$ by the following orthogonality condition:

$$(2) \qquad \int_{\partial G_n} f(z) dz \overline{L_t^{(n)}(z, x) dz} \frac{1}{i d W_a(z, t)} = 0 \quad \text{for all} \quad f \in H_2^d(G_n) .$$

On the other hand, there exist the conjugate Rudin kernel $\hat{R}_t^{(n)}(z, x)dz$ analytic differential on $\bar{G}_n$ and the adjoint L-kernel $\hat{L}_t^{(n)}(z, x)$ meromorphic function on $\bar{G}_n$ with one simple pole at x with residue 1, satisfying property:

$$(3) \qquad \overline{\hat{R}_t^{(n)}(z, x)dz} = \frac{1}{i}\hat{L}_t^{(n)}(z, x) i dW_n(z, t) \quad \text{along} \quad \partial G_n .$$

The K-kernel $\hat{R}_t^{(n)}(z, x)dz$ is characterized by the following reproducing property:

$$f(x) = \frac{1}{2\pi}\int_{\partial G_n} f(z)dz \; \overline{\hat{R}_t^{(n)}(z, x)dz} \; \frac{1}{i dW_n(z, t)} \quad \text{for all} \quad f \in H_2^d(G_n) .$$

The L-kernel $\hat{L}_t^{(n)}(z, x)$ is characterized in the class $\{h(z, x)\}_{2,G_n}$ by the following orthogonality condition:

$$(4) \qquad \int_{\partial G_n} f(z) \; \overline{\hat{L}_t^{(n)}(z, x)} \; i dW_n(z, t) = 0 \quad \text{for all} \quad f \in H_2(G_n) .$$

In [4], we have dealt with some properties of the Rudin kernel function on an arbitrary open Riemann surface. In the present paper we shall be concerned with some properties of other kernels on general regions. Especially, we shall show that if G is a hyperbolic region or simply-connected, the sequences $\{\hat{R}_t^{(n)}(z, x)\}$, $\{\hat{L}_t^{(n)}(z, x)\}$ and $\{L_t^{(n)}(z, x)\}$ do converge, respectively. This fact leads to natural definitions of kernels of such a domain, respectively. Our results should be compared with those of the Szegö kernel (cf. [5] and [6]) and the Rudin kernel function ([4]).

In §2, we state some preliminary facts. In §3, we deal with some fundamental properties of the kernels $\hat{R}_t^{(n)}(z, x)$, $\hat{L}_t^{(n)}(z, x)$ and $L_t^{(n)}(z, x)$ on general regions. In §4, we show that the sequence $\{\hat{R}_t^{(n)}(x, x)\}$ is not monotone as n increases, in general. In the final §5, we refer to the case of an arbitrary hyperbolic Riemann surface.

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2. Some preliminary facts.

In our investigation, the following lemma is fundamental:

LEMMA 1. *For an arbitrary $H_1(G)$-function $f(\not\equiv 0)$, the following limit exists and is determined independently of the choice of regular exhaustions $\{G_n\}$:*

$$\lim_{n\to\infty}\int_{\partial G_n} \frac{1}{z - x} \; \overline{f(z)} \; i dW_n(z, t) ,$$

if and only if G is a hyperbolic region or simply-connected.

Proof. It is sufficient to show the sufficiency. If f is a constant function, the assertion is clear. Hence we assume that f is not constant. Especially we note that in this case G is a hyperbolic region. Let $\{\Delta_{x,m}\}$ be discs such that $\Delta_{x,m} \equiv \{z \,|\, |z-x| < r_m, r_m > 0\}$, $\Delta_{x,m} \subset G_n$ for all m and n, and r_m tends to zero as m tends to infinity. Let $G_{n,m}$ denote $G_n - \bar{\Delta}_{x,m}$. Then we consider the following limit:

$$I = \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{\partial G_{n,m}} \frac{1}{z-x} \overline{(f(z)-f(x))} \, id W_{G_{n,m}}(z, t) ,$$

which is well-defined and is determined independently of the choice of regular exhaustions $\{G_n\}$ and the discs $\{\Delta_{x,m}\}$, as we see easily. Since $\overline{(f(z)-f(x))}/(z-x)$ is bounded about x, on letting m tend to infinity and then n tend to infinity, we have

$$I = \lim_{n\to\infty} \int_{\partial G_n} \frac{1}{z-x} \overline{(f(z)-f(x))} \, id W_n(z, t)$$

$$= \lim_{n\to\infty} \int_{\partial G_n} \frac{1}{z-x} \overline{f(z)} \, id W_n(z,t) - 2\pi \overline{f(x)} \left(\frac{1}{t-x} - \lim_{n\to\infty} W_n'(x, t) \right) , \quad (x \neq t) ,$$

which implies the desired result. Here if $x = t$, we can modify the above argument. Next we give the following lemma:

LEMMA 2. *Let G be a hyperbolic region. Then for each $h(z, x) \in \{h(z, x)\}_{2,G}$, we have*

$$\lim_{n\to\infty} \int_{\partial G_n} |h(z, x) \exp(-W_G(z, x))|^2 \, id W_n(z, t) \tag{5}$$

$$= \lim_{n\to\infty} \int_{\partial G_n} |h(z, x)|^2 \, id W_n(z, t) . \tag{6}$$

Proof. From the subharmonicity of $|h(z, x)\exp(-W_G(z, x))|^2$, we note that the limit (5) is well-defined and determined independently of the choice of regular exhaustions $\{G_n\}$. As to the limit (6), the similar assertion is valid, as we see from Lemma 1. Let Δ_n denote $\{z \in G \,|\, g_G(z, x) > \delta_n > 0\}$. Then Δ_n is the subdomain of G and $\partial\Delta_n$ consists of some analytic Jordan curves $g_G(z, x) = \delta_n$, except for a subset E_n of ∂G of logarithmic capacity zero, in general. Let $\tilde{\Delta}_n$ denote the regular subregion $\Delta_n \cup E_n$ of G. Without loss of generality, we can assume that $\Delta_n \ni t$. Since Cap $E_n = 0$, $h(z, x)$ can be extended on $\tilde{\Delta}_n - \{x\}$ analytically (cf. [4], Theorem 5.1). Hence we can take a regular subregion D_n of Δ_n such that for fixed $M > 0$,

$$(7)\qquad \left|\int_{\partial D_n} |h(z, x)|^2 id\, W_{D_n}(z, t) - \int_{\partial \tilde{\Delta}_n} |h(z, x)|^2 id\, W_{\tilde{\Delta}_n}(z, t)\right| \leqq M\,,$$

and

$$\left|\int_{\partial D_n} |h(z, x)|^2 (1 - \exp(-2g_G(z, x)))id\, W_{D_n}(z, t) - \int_{\partial \tilde{\Delta}_n} |h(z, x)|^2 (1 - \exp(-2g_G(z, x)))id\, W_{\tilde{\Delta}_n}(z, t)\right| < \frac{1}{n}\,;$$

that is,

$$(8)\qquad \left|\int_{\partial D_n} |h(z, x)|^2 (1 - \exp(-2g_G(z, x)))id\, W_{D_n}(z, t) - (1 - \exp(-2\cdot\delta_n))\int_{\partial \tilde{\Delta}_n} |h(z, x)|^2 id\, W_{\tilde{\Delta}_n}(z, t)\right| < \frac{1}{n}\,.$$

Next for $\delta_{n+1}(>0, <\delta_n)$, we consider $\tilde{\Delta}_{n+1}$ and D_{n+1} such that for $n+1$, (7) and (8) are valid and further $D_n \subset D_{n+1}$. Thus for a sequence $\{\delta_n\}(\delta_n > \delta_{n+1} > 0)$ converging to zero, we can obtain a regular exhaustion $\{D_n\}$ of G which has the properties (7) and (8). Since the sequence $\left\{\int_{\partial D_n} |h(z, x)|^2 id\, W_{D_n}(z, t)\right\}$ is bounded,

$$\left\{\int_{\partial \tilde{\Delta}_n} |h(z, x)|^2 id\, W_{\tilde{\Delta}_n}(z, t)\right\}$$

is also bounded. Hence from (8) we have

$$\lim_{n\to\infty}\int_{\partial D_n} |h(z, x)|^2 (1 - \exp(-2g_G(z, x)))id\, W_{D_n}(z, t) = 0\,.$$

Thus we have the desired result for $\{D_n\}$ and hence for every regular exhaustion of G.

Now we have a lemma of Fatou's type in the theory of integrals:

LEMMA 3. *Let G be a hyperbolic region. Let $\{h_n(z, x)\}$ be a sequence of $h_n(z, x) \in \{h(z, x)\}_{2,G_n}$ such that $h_n(z, x)$ converges to $h(z, x) \in \{h(z, x)\}_{2,G}$ uniformly on every compact subset of $G - \{x\}$. Then we have*

$$\lim_{n\to\infty}\int_{\partial G_n} |h(z, x)|^2 id\, W_n(z, t) \leqq \varliminf_{n\to\infty}\int_{\partial G_n} |h_n(z, x)|^2 id\, W_n(z, t)\,.$$

Proof. From Lemma 2, we have

$$\lim_{n\to\infty}\int_{\partial G_n} |h(z, x)|^2 id\, W_n(z, t) = \lim_{n\to\infty}\int_{\partial G_n} |h(z, x)\exp(-W_G(z, x))|^2 id\, W_n(z, t)$$

$$= \lim_{n\to\infty}\left(\lim_{m\to\infty}\int_{\partial G_n}|h_m(z, x)\exp(-W_m(z, x))|^2 id W_n(z, t)\right)$$
$$\leqq \lim_{n\to\infty}\left(\underline{\lim}_{m\to\infty}\int_{\partial G_m}|h_m(z, x)\exp(-W_m(z, x))|^2 id W_m(z, t)\right)$$
$$= \underline{\lim}_{m\to\infty}\int_{\partial G_m}|h_m(z, x)|^2 id W_m(z, t) .$$

3. The kernels on general regions. At first we deal with the convergence of the sequence $\{L_t^{(n)}(z, x)\}$. Let $R_t(y, z)(\in H_2(G))$ denote the Rudin kernel function of G with respect to z and $t(\in G)$ ([4]). Then we have the following theorem:

THEOREM 1. *Let G be a hyperbolic region or simply-connected. Then the sequence $\{L_t^{(n)}(z, x)\}$ converges to an analytic function $L_t(z, x)$ uniformly on every compact subset of $G-\{x\}$. Further $L_t(z, x)$ is independent of the choice of regular exhaustions $\{G_n\}$ and can be represented as follows:*

$$(9)\qquad L_t(z, x) = \frac{1}{z-x} + \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial G_n}\frac{1}{y-z}\overline{R_t(y, x)}\, id W_n(y, t) .$$

Proof. For each regular exhaustion $\{G_n\}$ of G, we consider the limit

$$H(z; x, t) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial G_n}\frac{1}{y-z}\overline{R_t(y, x)}\, id W_n(y, t) ,$$

which is well-defined and is independent of the choice of regular exhaustions $\{G_n\}$, as we see from Lemma 1. Since $\{R_t^{(n)}(y, x)\}$ converges to $R_t(y, x)$ in H_2-norm ([4], Theorem 2.1), we have

$$H(z; x, t) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial G_n}\frac{1}{y-z}\overline{R_t^{(n)}(y, x)}\, id W_n(y, t)$$
$$= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial G_n}\frac{1}{y-z}\frac{1}{i}L_t^{(n)}(y, x)dy$$
$$= \lim_{n\to\infty}\left(L_t^{(n)}(z, x) + \frac{1}{x-z}\right) .$$

Thus we set $L_t(z, x) \equiv H(z; x, t) + 1/(z-x)$.

Next we deal with the convergence of the sequence $\{\hat{L}_t^{(n)}(z, x)\}$. Since $\hat{L}_t^{(n)}(z, x) \equiv -L_t^{(n)}(x, z)$ ([3]), we conclude that if G is a hyperbolic region or simply-connected, $\{\hat{L}_t^{(n)}(z, x)\}$ converges to $-L_t(x, z)$. In these cases, we set $\hat{L}_t(z, x) \equiv -L_t(x, z)$. Since $R_t(z, x) \equiv 1 (x \neq t)$ if and only if $G \in O_{H_2}$ ([4], Corollary 2.2), from (9), we have the following fact:

COROLLARY 1. *If G is a hyperbolic region or simply-connected* *and further* $G \in O_{H_2}$, *then we have*

$$\hat{L}_t(z, x) = \frac{1}{z-x} + \lim_{n\to\infty} W_n'(x, t) - \frac{1}{t-x} \quad (x \neq t) .$$

If $x = t$, *we can modify the above representation, slightly.*

Furthermore we have the following theorem:

THEOREM 2. *Let* G *be a hyperbolic region or simply-connected. Then the sequence* $\{\hat{L}_t^{(n)}(z, x) - 1/(z - x)\}$ *converges uniformly on every compact subset of* G *to the* $H_2(G)$ *function* $\hat{L}_t(z, x) - 1/(z - x)$ *and further it converges to* $\hat{L}_t(z, x) - 1/(z - x)$ *in* H_2*-norm in the sense of Theorem 2.1 in* [4]. *The limit function* $\hat{L}_t(z, x)$ *is independent of the choice of regular exhaustions* $\{G_n\}$ *and can be represented as follows*:

$$\hat{L}_t(z, x) = \frac{1}{z-x} - \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial G_n} \frac{1}{y-x} \overline{\hat{R}_t(y, z)}\, id W_n(y, t) . \tag{10}$$

Furthermore $\hat{L}_t(z, x)$ *is characterized by each of the following orthogonality condition and the following extremal property in the class* $\{h(z, x)\}_{2,G}$:

$$\lim_{n\to\infty} \int_{\partial G_n} f(z)\overline{\hat{L}_t(z, x)} id W_n(z, t) = 0 \quad \textit{for all} \quad f \in H_2(G) , \tag{11}$$

and

$$\lim_{n\to\infty} \Big(\min_{\{h(z,x)\}_{2,G_n}} \int_{\partial G_n} |h(z, x)|^2 id W_n(z, t) \Big) \tag{12}$$

$$= \min_{\{h(z,x)\}_{2,G}} \Big(\lim_{n\to\infty} \int_{\partial G_n} |h(z, x)|^2 id W_n(z, t) \Big) \tag{13}$$

$$= \lim_{n\to\infty} \int_{\partial G_n} |\hat{L}_t(z, x)|^2 id W_n(z, t) , \tag{14}$$

respectively.

Proof. Since $\{\hat{L}_t^{(n)}(z, x) - 1/(z - x)\}$ is locally uniformly bounded (see the following (15)) and $\hat{L}_t^{(n)}(z, x)$ converges to $-L_t(x, z)$, we note that $\{\hat{L}_t^{(n)}(z, x) - 1/(z - x)\}$ converges uniformly on every compact subset of G to $\hat{L}_t(z, x) - 1/(z - x)$, independently of the choice of regular exhaustions $\{G_n\}$.

In order to see that $\hat{L}_t(z, x) - 1/(z - x)$ belongs to $H_2(G)$, we consider the following limit:

$$J = \lim_{n\to\infty} \int_{\partial G_n} \left| \hat{L}_t(z, x) - \frac{1}{z-x} \right|^2 id W_n(z, t) .$$

Then we have

$$(15)\qquad \begin{aligned} J &= \lim_{n\to\infty}\left(\lim_{m\to\infty}\int_{\partial G_n}\left|\hat{L}_t^{(m)}(z, x) - \frac{1}{z-x}\right|^2 id\,W_n(z, t)\right) \\ &\leqq \lim_{n\to\infty}\left(\varliminf_{m\to\infty}\int_{\partial G_m}\left|\hat{L}_t^{(m)}(z, x) - \frac{1}{z-x}\right|^2 id\,W_m(z, t)\right) \\ &\leqq \varliminf_{m\to\infty} 2\int_{\partial G_m}\left(|\hat{L}_t^{(m)}(z, x)|^2 + \frac{1}{|z-x|^2}\right)id\,W_m(z, t)\,. \end{aligned}$$

Since $\hat{L}_t^{(m)}(z, x)$ is the extremal function which minimizes H_2-norms among the class $\{h(z, x)\}_{2,G_m}$ ([3]), we have

$$J \leqq \lim_{m\to\infty} 4\int_{\partial G_m}\frac{1}{|z-x|^2}id\,W_m(z, t) < \infty\ ,$$

which implies the desired result.

Next we shall show that the sequence $\{\hat{L}_t^{(n)}(z, x) - 1/(z-x)\}$ converges to $\hat{L}_t(z, x) - 1/(z-x)$ in H_2-norm; that is, $\{\hat{L}_t^{(n)}(z, x)\}$ converges to $\hat{L}_t(z, x)$ in H_2-norm in the obvious sense. From the extremal property of $\hat{L}_t^{(n)}(z, x)$ on G_n, we have

$$\lim_{n\to\infty}\int_{\partial G_n}|\hat{L}_t(z, x)|^2\, id\,W_n(z, t) \geqq \varlimsup_{n\to\infty}\int_{\partial G_n}|\hat{L}_t^{(n)}(z, x)|^2 id\,W_n(z, t)\ .$$

Hence if G is a hyperbolic region, from Lemma 3, we have

$$\lim_{n\to\infty}\int_{\partial G_n}|\hat{L}_t(z, x)|^2\, id\,W_n(z, t) = \lim_{n\to\infty}\int_{\partial G_n}|\hat{L}_t^{(n)}(z, x)|^2 id\,W_n(z, t)\ .$$

Here we use the identity:

$$\begin{aligned} &\int_{\partial G_n}\hat{L}_t(z, x)\ \overline{\hat{L}_t^{(n)}(z, x)}\ id\,W_n(z, t) \\ &\quad = \int_{\partial G_n}\hat{L}_t(z, x)\frac{1}{i}\hat{R}_t^{(n)}(z, x)dz = 2\pi\hat{R}_t^{(n)}(x, x) \\ &\quad = \int_{\partial G_n}|\hat{L}_t^{(n)}(z, x)|^2 id\,W_n(z, t)\ . \end{aligned}$$

Thus we have the desired result.

On the other hand, if G is a parabolic region and simply-connected, from the proof of the following Theorem 3 for such a domain, we have the desired result, directly.

Since $\hat{L}_t^{(n)}(z, x)$ converges to $\hat{L}_t(z, x)$ in H_2-norm, from (4) we have (11). From (11) and the extremal property of $\hat{L}_t^{(n)}(z, x)$ on G_n, we obtain the extremal property of $\hat{L}_t(z, x)$ on G.

We define two functions $\hat{l}_t^{(n)}(z, x)$ and $\hat{l}_t(z, x)$ as follows:

$$\hat{l}_t^{(n)}(z, x) \equiv \hat{L}_t^{(n)}(z, x) - \frac{1}{z-x}\,,$$

and

$$\hat{l}_t(z, x) \equiv \hat{L}_t(z, x) - \frac{1}{z - x}$$

(in the cases that G is a hyperbolic region or simply connected).

At last we deal with the convergence of the sequence $\{\hat{R}_t^{(n)}(z, x)\}$. We have the following theorem:

THEOREM 3. *If G is a hyperbolic region or simply-connected, then the sequence $\{\hat{R}_t^{(n)}(z, x)\}$ converges to an analytic function $\hat{R}_t(z, x)$ uniformly on every compact subset of G. The function $\hat{R}_t(z, x)$ is independent of the choice of regular exhaustions $\{G_n\}$ and can be represented as follows:*

$$\hat{R}_t(z, x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial G_n} \frac{1}{(y - z)\overline{(y - x)}} i d W_n(y, t) - \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial G_n} \hat{l}_t(y, z)\overline{\hat{l}_t(y, x)} i d W_n(y, t) . \tag{16}$$

Proof. At first we assume that G is a hyperbolic region. Then from the identity:

$$\hat{R}_t^{(n)}(z, x) = \frac{1}{2\pi} \int_{\partial G_n} \frac{1}{(y - z)\overline{(y - x)}} \, i d W_n(y, t) - \frac{1}{2\pi} \int_{\partial G_n} \hat{l}_t^{(n)}(y, z)\overline{\hat{l}_t^{(n)}(y, x)} \, i d W_n(y, t) ,$$

which is obtained from (3) and since $\hat{l}_t^{(n)}(y, z)$ converges to $\hat{l}_t(y, z)$ in H_2-norm, we have the desired result.

Next we assume that G is not a hyperbolic region and simply-connected; that is, $\partial G = \{q\}$. Then we see that for an arbitrary regular exhaustion $\{G_n\}$, $\lim_{n\to\infty} \hat{R}_t^{(n)}(z, x) \equiv 0$, directly (cf. Theorem 4). From the identities:

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial G_n} \frac{1}{(y - z)\overline{(y - x)}} i d W_n(y, t) = \frac{1}{(q - z)\overline{(q - x)}}$$

and

$$\hat{l}_t(y, x) \equiv -\frac{1}{q - x}$$

(which is obtained from Corollary 1), we have the desired result. (We note that in this case $\{\hat{l}_t^{(n)}(y, x)\}$ converges to $\hat{l}_t(y, x)$ in H_2-norm.)

As to the degeneracy of the quantity $\hat{R}_t(x, x)$, we have the following theorem:

THEOREM 4. *If G is a hyperbolic region, then $\hat{R}_t(x, x) > 0$ for all $x, t \in G$. If G is a parabolic region, then for any fixed $x, t \in G$, there exists a regular exhaustion $\{G_n\}(G_0 \ni x, t)$ of G such that $\lim_{n\to\infty} \hat{R}_t^{(n)}(x, x) = 0$.*

Proof. We assume that G is a hyperbolic region and $\hat{R}_t(x, x) = 0$ for some x, t; that is,

$$2\pi\hat{R}_t(x, x) = \lim_{n\to\infty} 2\pi\hat{R}_t^{(n)}(x, x) = \lim_{n\to\infty} \int_{\partial G_n} |\hat{L}_t^{(n)}(z, x)|^2 id W_n(z, t)$$
$$= \lim_{n\to\infty} \int_{\partial G_n} |\hat{L}_t(z, x)|^2 id W_n(z, t) = 0 .$$

Then we have

$$\lim_{n\to\infty} \int_{\partial G_n} |\hat{L}_t(z, x) \exp(-W_G(z, x))| id W_n(z, t)$$
$$\leqq \lim_{n\to\infty} \left(\int_{\partial G_n} |\hat{L}_t(z, x)|^2 id W_n(z, t)\right)^{1/2}$$
$$\times \lim_{n\to\infty} \left(\int_{\partial G_n} |\exp(-W_G(z, x))|^2 id W_n(z, t)\right)^{1/2}$$
$$= 0 .$$

Hence $L_t(z, x) \exp(-W_G(z, x)) \equiv 0$, which is absurd.

Next we assume that G is a parabolic region. We take $q \in \partial G$. Then for an arbitrary regular region G_n, $G \supset G_n \ni x, t$, we have

$$\int_{\partial G_n} |\hat{L}_t^{(n)}(z, x)|^2 id W_n(z, t) \leqq \int_{\partial G_n} \left|\frac{1}{z - x} - \frac{1}{q - x}\right|^2 id W_n(z, t) .$$

Thus we have the desired assertion.

REMARK 1. We assume that G is a parabolic region and not simply-connected. Then from the identities

$$L_t^{(n)}(z, t) \equiv -\hat{L}_t^{(n)}(t, z) \equiv -W_n'(z, t)$$

(which is obtained from (1) and $R_t^{(n)}(z, t) \equiv 1$ ([3])) and

$$-W_n'(z, t) + \frac{1}{t - z} = \frac{1}{2\pi} \int_{\partial G_n} \frac{1}{y - z}\, id W_n(y, t) \quad (z \neq t)$$

we see that the limits $\lim_{n\to\infty} L_t^{(n)}(z, x)$ and $\lim_{n\to\infty} \hat{L}_t^{(n)}(z, x)$ do not exist, in general.

REMARK 2. From the theory of the Szegö kernel function, it is well-known that the following limit exists:

$$\lim_{n\to\infty}\left(\min_{\{h(z,x)\}_{2,G_n}}\int_{\partial G_n}|h(z,x)|^2 ds_z\right)\ (G\not\ni\infty)\ .$$

Further the degeneracy of such a quantity was investigated by P. R. Garabedian [1].

On the other hand, the convergence of the sequence of the Szegö kernel functions was investigated by N. Suita [6] and E. P. Smith [5].

4. An example. Here we shall show that the sequence $\{\hat{R}_t^{(n)}(x, x)\}$ is not monotone as n increases, in general.

For fixed $x = 0$ and $t(>0)$, let G_r be the disc $\{z\,||z| < r\}$. Then we can take as a complete orthonormal system (which defines the kernel $\hat{R}_t^{(r)}(z, x)$ of G_r)

$$\left\{\left(r\frac{z-t}{r^2-tz}\right)^n r\frac{r^2-t^2}{(r^2-tz)^2}\right\}_{n=0}^{\infty},$$

as we see by the simple computation. Hence we have

$$\hat{R}_t^{(r)}(z, 0) = \sum_{n=0}^{\infty}\left(r\frac{z-t}{r^2-tz}\right)^n r\frac{r^2-t^2}{(r^2-tz)^2}\left(-\frac{t}{r}\right)^n\frac{r^2-t^2}{r^3}$$
$$= \frac{r^2-t^2}{r^2(r^2-tz)},$$

and so

$$\hat{R}_t^{(r)}(0, 0) = \frac{r^2-t^2}{r^4}.$$

Thus

for r: $t < r \leqq \sqrt{2}\,t$, $\hat{R}_t^{(r)}(0, 0)$ increases as r increases,

for r: $\sqrt{2}\,t \leqq r$, $\hat{R}_t^{(r)}(0, 0)$ decreases as r increases .

Now let E be a compact subset of G such that $G - E$ is connected. Then we note that a property of E for which for fixed x, $t(\in G - E)$, $\hat{R}_t^{(G)}(x, x) = \hat{R}_t^{(G-E)}(x, x)$ is valid is quite different from the properties of E in the cases of other various kernel functions (cf. [4], §5).

5. The case of an arbitrary hyperbolic Riemann surface. Let S denote a hyperbolic Riemann surface. For simplicity, we shall use the same notation for a point on $\bar{S}$ and a fixed local parameter around there. For fixed local parameter around x and fixed $t(\in S)$, let $\{\tilde{h}(z, x)\}_{p,S}^{\sim}(p > 0)$ denote the class of meromorphic functions $\tilde{h}(z, x)$ such that $\tilde{h}(z, x)$ are analytic on S except for a simple at x with residue 1 and for some compact subset $K \ni x$, $\left\{\int_{\partial S_r} |\tilde{h}(z, x)|^p i d W_{S_r}(z, t)\right\}$ is bounded for all regular subregions S_r, $S \supset \bar{S}_r \supset K^{\cup}\{t\}$.

In this paper we needed essentially a function $\tilde{h}(z, x)$ and the class $\{\tilde{h}(z, x)\}_{2,S}^{\sim}$ instead of the Cauchy kernel and the class $\{h(z, x)\}_{2,G}$, respectively. But there exist hyperbolic Riemann surfaces S such that $\{\tilde{h}(z, x)\}_{p,S}^{\sim}(p > 0)$ are all void.

We introduce two copies S_1 and S_2 of $\{z \mid |z| > 1\}$ and distinguish the segments $[2n, 2n+1]$ $(n = 1, 2, \cdots)$. We construct the desired surface S by joining S_1 and S_2 along their common distinguished slits in the usual manner. Let $\{Q_1, Q_2\}$ be the preimage of $z(|z| > 1)$ with respect to the projection map φ. Then we see that $S \notin O_{H_p}$ $(p > 0)$ and $f(Q_1) = f(Q_2)$ for all $f \in H_p(S)$ (cf. [2], pp. 36-37). Now we assume that for fixed $x, t \in S_1(\operatorname{Im}\varphi(x), \operatorname{Im}\varphi(t) > 0)$, there exists an $\tilde{h}(z, x) \in \{\tilde{h}(z, x)\}_{p,S}^{\sim}$. Then we see that for a regular exhaustion $\{S_n\}(S_0 \ni x, t)$ of S,

$$\lim_{n\to\infty} \int_{\partial S_n} |\tilde{h}(z, x) \exp(-W_S(z, x))|^p i d W_{S_n}(z, t) < \infty .$$

Hence there exists a harmonic majorant $u(z)$ satisfying

$$|\tilde{h}(z, x) \exp(-W_S(z, x))|^p \leqq u(z) \quad \text{on} \quad S .$$

Let D denote a small disc such that $S_1 \supset D \ni x$ $(\operatorname{Im}\varphi(D) > 0)$ and $\exp(-W_S(z, x))$ is univalent on $\bar{D}$. Then there exists δ such that $|\exp(-W_S(z, x))| \geqq \delta > 0$ on $S - \bar{D}$. Hence we get

$$|\tilde{h}(z, x)|^p \leqq \delta^{-p} \cdot u(z) \quad \text{on} \quad S - \bar{D} .$$

Hence $\tilde{h}(z, x)$ must be extended on D analytically, which is absurd.

Therefore we see that for such Riemann surfaces, all the theorems in this paper are not valid. For example, Theorems 2 and 3 are not valid.

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