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GENERALIZATION OF LENTIN'S THEORY OF PRINCIPAL SOLUTIONS OF WORD EQUATIONS IN FREE SEMIGROUPS TO FREE PRODUCT OF COPIES OF POSITIVE REALS UNDER ADDITION

MOHAN S. PUTCHA

Word equations in free semigroups have been studied by many authors. However, word equations in geometrically arising semigroups lead to word equations in free product of copies of positive reals under addition (see "Word equations in some geometric semigroups" by Putcha). In this paper we generalize Lentin's theory of word equations in free semigroups to free product of copies of positive reals under addition.

1. Preliminaries. Throughout this paper Z_0, Z^+, Z, Q_0, Q^+, Q, $\boldsymbol{R}_0$, $\boldsymbol{R}^+$, $\boldsymbol{R}$ will denote the sets of nonnegative integers, positive integers, integers, nonnegative rational numbers, positive rational numbers, rational numbers, nonnegative real numbers, positive real numbers and reals, respectively. If S is a semigroup without an identity element, then $S^1 = S \cup \{1\}$ with obvious multiplication. If $a, b \in S$, then $a|_f b$ (a is a final segment of b) if $b = xa$ for some $x \in S^1$.

If Γ is a nonempty set, then let $\mathscr{F} = \mathscr{F}(\Gamma)$ denote the free semigroup on Γ. If $w \in \mathscr{F}$, then let $l(w) =$ length of w. Let $\mathscr{F}_{\boldsymbol{R}} = \mathscr{F}_{\boldsymbol{R}}(\Gamma)$ denote the set of all nonempty finite sequences (also called words) of the type $w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ where $n \in Z^+$, $\alpha_1, \cdots, \alpha_n \in \boldsymbol{R}^+$, $A_1, \cdots, A_n \in \Gamma$ and $A_i \neq A_{i+1}$ for $i, i+1 \in \{1, \cdots, n\}$. We define $e(w) = n$ and $l(w) = \alpha_1 + \cdots + \alpha_n$. Let $w_1, w_2 \in \mathscr{F}_{\boldsymbol{R}}$. Suppose $w_1 = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, $w_2 = B_1^{\beta_1} \cdots B_m^{\beta_m}$. Then we define

$$w_1 w_2 = \begin{cases} A_1^{\alpha_1} \cdots A_n^{\alpha_n + \beta_1} B_2^{\beta_2} \cdots B_m^{\beta_m} & \text{if} \quad A_n = B_1 \\ A_1^{\alpha_1} \cdots A_n^{\alpha_n} B_1^{\beta_1} \cdots B_m^{\beta_m} & \text{if} \quad A_n \neq B_1 \,. \end{cases}$$

Now of course expressions of the type $w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$($\alpha_1, \cdots, \alpha_n \in \boldsymbol{R}^+$; $A_1, \cdots, A_n \in \Gamma$) make sense even when $A_i = A_{i+1}$ for some i, $i+1 \in \{1, \cdots, n\}$. But note that if $n = e(w)$, then $A_i \neq A_{i+1}$ for any $i, i+1 \in \{1, \cdots, n\}$. In such a case we call $A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, the *standard form* of w. $\mathscr{F}_{\boldsymbol{R}}(\Gamma)$ is a semigroup and is just the free product of $|\Gamma|$ copies of $\boldsymbol{R}^+$ under addition (see for example [3; p. 411]). If $w \in \mathscr{F}_{\boldsymbol{R}}^1$, then we let $w^0 = 1$. Let $\bar{\Gamma} = \{A^\alpha | A \in \Gamma, \alpha \in \boldsymbol{R}^+\}$. If $u, v \in \bar{\Gamma}$, define $u \sim v$ if $u = A^\alpha$, $v = A^\beta$ for some $\alpha, \beta \in \boldsymbol{R}^+$, $A \in \Gamma$. Let $w \in \mathscr{F}_{\boldsymbol{R}}$,

$w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ in standard form. Then we say that w starts with A_1 and ends with A_n. Let $A \in \Gamma$. Then A *appears* in w if $A = A_i$ for some $i \in \{1, \cdots, n\}$. A appears *integrally* in w if for each $i \in \{1, \cdots, n\}$, $A_i = A$ implies $\alpha_i \in Z^+$. Otherwise A appears *nonintegrally* in w. Note that if A does not appear in w, then A appears integrally in w. A appears *rationally* in w if for each $i \in \{1, \cdots, n\}$, $A_i = A$ implies $\alpha_i \in Q^+$. Let $\mathscr{F}_Q(\Gamma) = \{w | w \in \mathscr{F}_R(\Gamma), A$ appears rationally in w for each $A \in \Gamma\}$. If $\Lambda \subseteqq \Gamma$, then let $\mathscr{F}_R(\Gamma|\Lambda) = \{w | w \in \mathscr{F}_R(\Gamma)$, A appears integrally in w for each $A \in \Lambda\}$. So $\mathscr{F}_R(\Gamma|\Gamma) = \mathscr{F}(\Gamma)$ and $\mathscr{F}_R(\Gamma|\varnothing) = \mathscr{F}_R(\Gamma)$. Also let $\mathscr{F}_Q(\Gamma|\Lambda) = \mathscr{F}_R(\Gamma|\Lambda) \cap \mathscr{F}_Q(\Gamma)$.

Let $\Lambda_1 \subseteqq \Gamma_1$, $\Lambda_2 \subseteqq \Gamma_2$, $\phi: \Gamma_1 \to \mathscr{F}_R(\Gamma_2|\Lambda_2)$ such that $\phi(\Gamma_1 \backslash \Lambda_1) \subseteqq \overline{\Gamma_2 \backslash \Lambda_2}$. Then we can extend ϕ to a homomorphism $\hat{\phi}: \mathscr{F}_R(\Gamma_1|\Lambda_1) \to \mathscr{F}_R(\Gamma_2|\Lambda_2)$ by letting $\hat{\phi}(A^\alpha) = [\phi(A)]^\alpha$ for $A \in \Gamma_1 \backslash \Lambda_1$, $\alpha \in \boldsymbol{R}^+$. We call $\hat{\phi}$ the *natural extension* of ϕ. Let $\psi: \mathscr{F}_R(\Gamma_1 \backslash \Lambda_1) \to \mathscr{F}_R(\Gamma_2 \backslash \Lambda_2)$ be a homomorphism. Let $A \in \Gamma_1 \backslash \Lambda_1$. Then A and hence $\psi(A)$ has nth roots for all $n \in Z^+$. It follows that $\psi(A) \in \overline{\Gamma_2 \backslash \Lambda_2}$. We say that ψ is a *natural* homomorphism if $\psi(A^\alpha) = \psi(A)^\alpha$ for all $A \in \Gamma_1 \backslash \Lambda_1$, $\alpha \in \boldsymbol{R}^+$. In such a case $\psi = \hat{\phi}$ where ϕ is the restriction of ψ to Γ_1. In this paper we only consider natural homomorphisms.

DEFINITION 1.1. By a word equation in variables $X_1, \cdots, X_n$ we mean $\{w_1, w_2\}$ where $w_1 = w_1(X_1, \cdots, X_n)$, $w_2 = w_2(X_1, \cdots, X_n) \in \mathscr{F}(X_1, \cdots, X_n)^1$. It is not necessary that each X_i appears in $w_1 w_2$. Let S be a semigroup and $a_1, \cdots, a_n \in S$. Then $(a_1, \cdots, a_n)$ is a solution of $\{w_1, w_2\}$ if $w_1(a_1, \cdots, a_n) = w_2(a_1, \cdots, a_n)$.

Detailed study of word equations leads to the concept of a generalized word equation defined below. This concept is similar but not identical to the concept of a constrained word equation defined in [5].

DEFINITION. By a generalized word equation in variables $X_1, \cdots, X_n$ we mean $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ where $w_1 = w_1(X_1, \cdots, X_n)$, $w_2 = w_2(X_1, \cdots, X_n) \in \mathscr{F}(X_1, \cdots, X_n)^1$ and $T_1, \cdots, T_s$ are pairwise disjoint nonempty subsets of $\{X_1, \cdots, X_n\}$. ($s = 0$ means that $\mathscr{W}$ is the word equation $\{w_1, w_2\}$.) Let $X \in \{X_1, \cdots, X_n\}$. Then X is a constrained variable if $X \in T_i$ for some i. Otherwise X is a free variable of $\mathscr{W}$.

Let $u_1, \cdots, u_n \in \mathscr{F}_R(\Omega)$. Then $\mu = (u_1, \cdots, u_n)$, is a solution of $\mathscr{W}$ if (i) $w_1(u_1, \cdots, u_n) = w_2(u_1, \cdots, u_n)$, (ii) $X_i \in T_j$ implies $e(u_i) = 1$, and (iii) $X_i, X_k \in T_j$ implies $u_i \sim u_k$. Let $\Omega_1 = \{A | A \in \Omega, A$ appears in $u_1 \cdots u_n\}$ and $\Omega_2 = \{A | A \in \Omega, A$ appears integrally in each u_i, $i = 1, \cdots, n\}$. We define the symbol $\Omega_1 | \Omega_2$ to be the alphabet of μ. If $\Omega_1 = \Omega_2$ we also write Ω_1 instead of $\Omega_1 | \Omega_1$. Clearly $u_1, \cdots, u_n \in \mathscr{F}_R(\Omega_1 | \Omega_2)$. In this paper we only consider solutions of (generalized) word equations in $\mathscr{F}_R(\Gamma)$, Γ a nonempty set.

DEFINITION 1.2. Let $\nu = (a_1, \cdots, a_n)$, $\mu = (b_1, \cdots, b_n)$ be solutions of a generalized word equation $\mathscr{W}$ such that ν has alphabet $\Gamma_1|\Lambda_1$ and μ has alphabet $\Gamma_2|\Lambda_2$. Then $\nu \leqq \mu$ (μ follows from ν) if there exists a natural homomorphism $\psi: \mathscr{F}_R(\Gamma_1|\Lambda_1) \to \mathscr{F}_R(\Gamma_2 \mid \Lambda_2)$ such that $\psi(a_i) = b_i$, $i = 1, \cdots, n$. We write $\psi: \nu \leqq \mu$ if we want to take into account the corresponding homomorphism ψ. $u \approx \nu$ if $\mu \leqq \nu \leqq \mu$.

REMARK 1.3. $\leqq$ is transitive in the class of all solutions of a given generalized word equation $\mathscr{W}$. When we restrict the above definition to solutions in free semigroups of word equations, we are led to Lentin's concepts in [1] (some of the incorrect concepts in [1] are corrected in [2]).

2. Principal solutions.

We now generalize Lentin's concept of a principal solution [1, 2].

DEFINITION 2.1. Let μ be a solution of a generalized word equation $\mathscr{W}$. Then μ is a principal solution if for any solution ν of $\mathscr{W}$, $\nu \leqq \mu$ implies $\nu \approx \mu$.

Let $\mu = (b_1, \cdots, b_n)$ be a solution of a word equation $\mathscr{W}$. Then $\mu \in \mathscr{S}$ (as a solution of $\mathscr{W}$) if there exists a solution $\nu = (a_1, \cdots, a_n)$ of $\mathscr{W}$ such that the following properties are true:

(2.2) There exists a unique $f: \nu \leqq \mu$.

(2.3) If δ is a solution of $\mathscr{W}$ such that $\delta \leqq \mu$, then $\nu \leqq \delta$.

(2.4) Let $\Gamma_1|\Gamma_2$ be the alphabet of ν. Then for each $A \in \Gamma_1$, there exists $i \in \{1, \cdots, n\}$ such that a_i ends with A.

REMARK 2.5. ν above is necessarily principal. We will show (Theorem 2.19) that if μ is a solution of any generalized word equation: then $\mu \in \mathscr{S}$. We will also show (Theorem 2.23) that any principal solution ν satisfies both (2.4) and its right-left dual.

In Lemmas 2.6–2.18 we develop the machinery for replacing the solution of a given word equation with a simpler solution of a related word equation.

LEMMA 2.6. *Let* $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ *be a generalized word equation in variables* $X_1, \cdots, X_n$, $n > 1$. *Suppose* X_1 *is a free variable not appearing in* w_1w_2. *Let* $\mu = (u_1, \cdots, u_n)$ *be a solution of* $\mathscr{W}$. *Let* $\mathscr{W}' = \{w_1, w_2; T_1, \cdots, T_s\}$ *considered as a generalized word equation in variables* $X_2, \cdots, X_n$. *Then* $\mu' = (u_2, \cdots, u_n)$ *is a solution of* $\mathscr{W}'$. *If* $\mu' \in \mathscr{S}$, *then* $\mu \in \mathscr{S}$.

Proof. Suppose $\mu' \in \mathscr{S}$. Correspondingly there exists a solution ν' of $\mathscr{W}$ satisfying (2.2), (2.3), (2.4). Let $\nu' = (v_2, \cdots, v_n)$ with alphabet $\Gamma'|\Lambda'$. There exists unique $f' : \nu' \leqq \mu'$. Let $A \notin \Gamma'$, set $\Gamma = \Gamma' \cup \{A\}$, $\Lambda = \Lambda' \cup \{A\}$. Let $v_1 = A$, $\nu = (v_1, v_2, \cdots, v_n)$. Then ν is a solution of $\mathscr{W}$ and has alphabet $\Gamma|\Lambda$. Also $f: \nu \leqq \mu$ where f is an extension of f' with $f(A) = u_1$. Suppose $g: \nu \leqq \mu$. Let g' be the restriction of g to $\mathscr{F}_R(\Gamma'|\Lambda')$. Then $g': \nu' \leqq \mu'$. So $f' = g'$. Since $f(A) = f(v_1) = g(v_1) = g(A)$, $f = g$. Let $B \in \Gamma$. If $B = A$, then u_1 ends with A. Otherwise $B \in \Gamma'$ and some u_i, $i > 1$ ends with B. Finally suppose δ is a solution of $\mathscr{W}$, $\delta \leqq \mu$. Let $\delta = (a_1, \cdots, a_n)$. Set $\delta' = (a_2, \cdots, a_n)$. Then δ' is a solution of $\mathscr{W}'$. So there exists $g': \nu' \leqq \delta'$. g' extends to $g: \nu \leqq \delta$ where $g(A) = a_1$. This proves that $\mu \in \mathscr{S}$.

LEMMA 2.7. *Let $\mathscr{W} = \{uw_1, uw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$. Suppose μ is a solution of $\mathscr{W}$. Then μ is a solution of $\mathscr{W}' = \{w_1, w_2; T_1, \cdots, T_s\}$ in variables $X_1, \cdots, X_n$. If $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}'$, then $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}$.*

Proof. This follows trivially since the solutions of $\mathscr{W}$ are exactly the same as the solutions of $\mathscr{W}'$.

LEMMA 2.8. *Let $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$. Suppose $T_1 = \{X_1, \cdots, X_k\}$, $u, v \in \mathscr{F}(T_1)$. Let $\mu = (b_1, \cdots, b_n)$, $\nu = (a_1, \cdots, a_n)$ be solutions of $\mathscr{W}$, $\nu \leqq \mu$. Let $c_1 = u(b_1, \cdots, b_k)$, $c_2 = v(b_1, \cdots, b_k)$, $d_1 = u(a_1, \cdots, a_k)$, $d_2 = v(a_1, \cdots, a_k)$. Then $c_1 = c_2$ if and only if $d_1 = d_2$; $l(c_1) < l(c_2)$ if and only if $l(d_1) < l(d_2)$.*

Proof. Let $f: \nu \leqq \mu$. Let $\Gamma|\Lambda$ be the alphabet of ν. There exists $A \in \Gamma$ such that $a_1, \cdots, a_k \in \bar{A}$. So $c_1, c_2 \in \bar{A}$. Suppose $l(c_1) = l(c_2)$. Then $c_1 = c_2$ and so $d_1 = f(c_1) = f(c_2) = d_2$. Next suppose $l(c_1) < l(c_2)$. Then $c_2 = c_1 x$ for some $x \in \mathscr{F}_R(\Gamma|\Lambda)$. So $d_2 = f(c_2) = f(c_1)f(x) = d_1 f(x)$. Thus $l(d_1) < l(d_2)$. Similarly $l(c_2) < l(c_1)$ implies $l(d_2) < l(d_1)$. Since these are mutually exclusive cases, we are done.

LEMMA 2.9. *Let $\mathscr{W} = \{uw_1, vw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$ such that $uv \in \mathscr{F}(T_1)$. Suppose $\mu = (b_1, \cdots, b_n)$ is a solution of $\mathscr{W}$ such that $u(b_1, \cdots, b_n) = v(b_1, \cdots, b_n)$. Then μ is a solution of $\mathscr{W}' = \{w_1, w_2; T_1, \cdots, T_s\}$ in variables $X_1, \cdots, X_n$. Moreover, if $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}'$, then $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}$.*

Proof. Let $\mathscr{A} = \nu|\nu$ is a solution of $\mathscr{W}$, $\nu \leqq \mu\}$, $\mathscr{B} = \{\nu|\nu$ is a solution $\mathscr{W}'$, $\nu \leqq \mu\}$. It suffices to show that $\mathscr{A} = \mathscr{B}$. First let $\nu = (a_1, \cdots, a_n) \in \mathscr{A}$. Then by Lemma 2.8, $u(a_1, \cdots, a_n) = v(a_1, \cdots, a_n)$ and so $\nu \in \mathscr{B}$. Conversely if $\nu \in \mathscr{B}$, then the same argument shows that $\nu \in \mathscr{A}$.

LEMMA 2.10. *Let $\mathscr{W} = \{uw_1, vX_jw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$ such that $uv \in \mathscr{F}(T_1)$, $X_j \in T_2$. Let $\mu = (b_1, \cdots, b_n)$ be a solution of $\mathscr{W}$ such that $l(v(b_1, \cdots, b_n)) < l(u(b_1, \cdots, b_n))$. Then μ is a solution of $\mathscr{W}' = \{uw_1, vX_jw_2; T_1 \cup T_2, T_3, \cdots, T_s\}$. If $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}'$, then $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}$.*

Proof. Let $c_1 = u(b_1, \cdots, b_n)$, $c_2 = v(b_1, \cdots, b_n)$. Then $c_1 \sim c_2$, $l(c_2) < l(c_1)$, $c_1 \cdots = c_2b_j \cdots$. Since X_j is constrained, it follows that $e(b_j) = 1$ and $b_j \sim c_1 \sim b_1$. So for all $X_k \in T_2$, $b_k \sim b_j \sim b_1$. It follows that μ is a solution of $\mathscr{W}'$. It is clear that every solution of $\mathscr{W}'$ is a solution of $\mathscr{W}$. So it suffices to show that if ν is a solution $\mathscr{W}$ and $\nu \leqq \mu$, then ν is a solution of $\mathscr{W}'$. Let $\nu = (a_1, \cdots, a_n)$. By Lemma 2.8, $l(v(a_1, \cdots, a_n)) < l(u(a_1, \cdots, a_n))$ and the above argument shows that ν is a solution of $\mathscr{W}'$.

LEMMA 2.11. *Let $\mathscr{W} = \{w, w; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$. Let $\mu = (u_1, \cdots, u_n)$ be a solution of $\mathscr{W}$. Then $\mu \in \mathscr{S}$.*

Proof. Since the solutions of $\mathscr{W}$ are the same as those of $\{1, 1; T_1, \cdots, T_s\}$ we can assume without loss of generality that $w = 1$. First assume $s = 0$. Then by Lemma 2.6, we can assume that $n = 1$. Introducing a new symbol A we see that $\nu = (A)$ is a solution of $\mathscr{W}$ and satisfies (2.2), (2.3), and (2.4).

Next assume $s \geqq 1$. Then by Lemma 2.6, we can assume that $\mathscr{W}$ has no free variables. Let $\Omega = \{A_1, \cdots, A_n\}$ where $A_i = A_k$ if and only if X_i, X_k lie in same T_j. We define a solution $\nu = (v_1, \cdots, v_n)$ of $\mathscr{W}$ as follows: Let $T_j = \{X_{j(1)}, \cdots, X_{j(t)}\}$. Let the alphabet of $\mu = \Gamma_1|\Gamma_2$. There exist $A \in \Gamma_1$, $r_1, \cdots, r_t \in \boldsymbol{R}^+$ such that $u_{j(i)} = A^{r_i}$. We wish to distinguish two cases. First suppose there exists $d \in \boldsymbol{R}^+$, $k_1, \cdots, k_t \in Z^+$ such that $(r_1, \cdots, r_t) = d(k_1, \cdots, k_t)$. Then we can assume $k_1, \cdots, k_t$ are relatively prime. Also, if $r_1, \cdots, r_t \in Z^+$, then $d \in Z^+$. In such a case set $v_{j(i)} = C^{k_i}$ where $C = A_{j(1)} = \cdots = A_{j(t)}$, $\phi(C) = A^d$, $i = 1, \cdots, t$. Next assume there is no such d. Then set $v_{j(i)} = A^{r_i}_{j(i)}$, $\phi(A_{j(i)}) = A$, $i = 1, \cdots, t$. Then $\nu = (v_1, \cdots, v_n)$ is a solution of $\mathscr{W}$. Let $\Omega_1 = \{A_i|A_i$ appear integrally in each v_j, $j = 1, \cdots, n\}$. Then $\phi(\Omega\backslash\Omega_1) \subseteq \overline{\Gamma_1\backslash\Gamma_2}$. Also if $\hat{\phi}$ is the natural extension of ϕ

to $\mathscr{F}_R(\Omega|\Omega_1)$, then $\hat{\phi}: \nu \leqq \mu$. Since $e(u_i) = e(v_i) = 1$ for all i, it is easy to see that $\hat{\phi}$ is unique. Also it is clear that for each $B \in \Omega$ there exists v_i such that v_i ends with B.

Now let $\delta = (a_1, \cdots, a_n)$ be a solution of $\mathscr{W}$, with alphabet $\Lambda_1|\Lambda_2$, $g: \delta \leqq \mu$. We wish to define $\hat{\psi}: \nu \leqq \delta$. Let $T_j = \{X_{j(1)}, \cdots, X_{j(t)}\}$, $u_{j(i)} = A^{r_i}$, $a_{j(i)} = B^{p_i}$, $i = 1, \cdots, t$, $v_{j(i)} = C^{k_i}$, $C \in \Omega$. Let $\phi(C) = A^d$, $g(B) = A^q$. Then $d(k_1, \cdots, k_t) = (r_1, \cdots, r_t) = q(p_1, \cdots, p_t)$. Define $\psi(C) = B^{d/q}$. We must show that if $B \in \Lambda_2$, then $d/q \in Z^+$. So let $B \in \Lambda_2$. Then $p_1, \cdots, p_t \in Z^+$. Then by above, $k_1, \cdots, k_t \in Z^+$ and are relatively prime. Also $p_i/k_i = d/q$, $i = 1, \cdots, t$. There exist $\pi_1, \cdots, \pi_t \in Z$ such that $\pi_1 k_1 + \cdots + \pi_t k_t = 1$. So $d/q = \pi_1 k_1 d/q + \cdots + \pi_t k_t d/q = \pi_1 p_1 + \cdots + \pi_t p_t \in Z$. So $d/q \in Z^+$. It is clear that if $\hat{\psi}$ is the natural extension of ψ to $\mathscr{F}_R(\Omega|\Omega_1)$, then $\hat{\psi}: \nu \leqq \delta$. So ν satisfies (2.2), (2.3) and (2.4); proving the lemma.

LEMMA 2.12. *Let $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$. Suppose $\mathscr{W}$ has no free variables and $\mu = (b_1, \cdots, b_n)$ is a solution of $\mathscr{W}$. Then $\mu \in \mathscr{S}$.*

Proof. We prove by induction on $l(w_1w_2) + s$. If $w_1 = w_2 = 1$, then we are done by Lemma 2.11. So assume $w_1 \neq 1$ (and hence $w_2 \neq 1$). Without loss of generality we can assume that w_1 starts with a letter in T_1. We can write $w_1 = u_1u_2$ such that $u_1 \in \mathscr{F}(T_1)$, $l(u_1)$ maximal. Similarly write $w_2 = v_1v_2$ such that $v_1 \in \mathscr{F}(T_1)^1$, $l(v_1)$ maximal. If $u_1(b_1, \cdots, b_n) = v_1(b_1, \cdots, b_n)$, then we are done by Lemma 2.9 and the induction hypothesis. So by symmetry assume $l(v_1(b_1, \cdots, b_n)) < l(u_1(b_1, \cdots, b_n))$. Let $v_2 = X_2v_3$. Without loss of generality we can assume $X_2 \in T_2$. We are again done by the induction hypothesis and Lemma 2.10.

LEMMA 2.13. *Let $\mathscr{W} = \{uw_1, vw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$. Let $\mu = (b_1, \cdots, b_n)$, $\nu = (a_1, \cdots, a_n)$ be solutions of $\mathscr{W}$, $\nu \leqq \mu$. Set $c_1 = u(a_1, \cdots, a_n)$, $c_2 = v(a_1, \cdots, a_n)$, $d_1 = u(b_1, \cdots, b_n)$, $d_2 = v(b_1, \cdots, b_n)$. Then $c_1 = c_2$ if and only if $d_1 = d_2$; $l(c_1) < l(c_2)$ if and only if $l(d_1) < l(d_2)$.*

Proof. Let $f: \nu \leqq \mu$. Then $f(c_1) = d_1$, $f(c_2) = d_2$. Now $c_1 \cdots = c_2 \cdots$. If $l(c_1) = l(c_2)$, then $c_1 = c_2$ and $d_1 = d_2$. If $l(c_1) < l(c_2)$; then $c_2 = c_1c_2'$ for some $c_2' \neq 1$ and so $d_2 = d_1d_2'$ where $d_2' = f(c_2') \neq 1$. Hence $l(d_1) < l(d_2)$. Similarly $l(c_2) < l(c_1)$ implies $l(d_2) < l(d_1)$. Since these are mutually exclusive cases, we are done.

LEMMA 2.14. *Let $\mathscr{W} = \{uw_1, vX_jw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$ such that $uv \in \mathscr{F}(T_1)$*

and X_j is a free variable. Let $\mu = (b_1, \cdots, b_n)$ be a solution of $\mathscr{W}$. Set $c_1 = u(b_1, \cdots, b_n)$, $c_2 = v(b_1, \cdots, b_n)$. Suppose $l(c_2 b_j) \leqq l(c_1)$. Let $\mathscr{W}' = \{uw_1, vX_jw_2; T_1 \cup \{X_j\}, T_2, \cdots, T_s\}$ in variables $X_1, \cdots, X_n$. Then μ is a solution of $\mathscr{W}'$. If $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}'$, then $\mu \in \mathscr{S}$ as a solution of $\mathscr{W}$.

Proof. $l(c_2b_j) \leqq l(c_1)$ and $c_1 \cdots = c_2 b_j \cdots$, and so $e(b_j) = 1$, $b_j \sim c_1$. It follows that μ is a solution of $\mathscr{W}'$. It is also clear that every solution of $\mathscr{W}'$ is a solution of $\mathscr{W}$. We need only show that if ν is a solution of $\mathscr{W}$ and $\nu \leqq \mu$, then ν is a solution of $\mathscr{W}'$. So let $\nu = (a_1, \cdots, a_n) \leqq \mu$. Let $d_1 = u(a_1, \cdots, a_n)$, $d_2 = v(a_1, \cdots, a_n)$. Then by Lemma 2.13, $l(d_2a_j) \leqq l(a_1)$. So by the above argument ν is a solution of $\mathscr{W}'$.

LEMMA 2.15. *Let $\mathscr{W} = \{uw_1, vX_jw_2; T_1, \cdots, T_s\}$ be a generalized word equation in variables $X_1, \cdots, X_n$ such that $uv \in \mathscr{F}(T_1)$ and X_j is a free variable. Let $\mu = (b_1, \cdots, b_n)$ be a solution of $\mathscr{W}$. Set $c_1 = u(b_1, \cdots, b_n)$, $c_2 = v(b_1, \cdots, b_n)$. Suppose $l(c_2b_j) > l(c_1) > l(c_2)$. Then there exist b_{n+1}, $b'_j \neq 1$ such that $b_{n+1} \sim c_1 \sim c_2$, $b_j = b_{n+1}b'_j$ and $c_1 = c_2b_{n+1}$. Introduce new variables X'_j, X_{n+1}. Set $T'_1 = T_1 \cup \{X_{n+1}\}$. Let w'_1, w'_2 be the words obtained by replacing X_j by $X_{n+1}X'_j$ in w_1, w_2 respectively. Let $\mathscr{W}'$ be the generalized word equation $\{w'_1, X'_jw'_2; T'_1, T_2, \cdots, T_s\}$ in variables $X_1, \cdots X_{j-1}, X'_j, X_{j+1}, \cdots, X_n, X_{n+1}$. Then $\mu' = (b_1, \cdots, b_{j-1}, b'_j, b_{j+1}, \cdots, b_n, b_{n+1})$ is a solution $\mathscr{W}'$. Moreover, if $\mu' \in \mathscr{S}$, then $\mu \in \mathscr{S}$.*

Proof. Now $c_1 \cdots = c_2b_j \cdots$ and so the existence of b_{n+1}, b'_j follows. It is clear that μ' is a solution of $\mathscr{W}'$. Suppose $\mu' \in \mathscr{S}$. Then there exists a solution $\nu' = (a_1, \cdots, a_{j-1}, a'_j, a_{j+1}, \cdots, a_n, a_{n+1})$ of $\mathscr{W}'$, $f\colon \nu' \leqq \mu'$ such that (2.2), (2.3), (2.4) are true. Let $a_j = a_{n+1}a'_j$, $\nu = (a_1, \cdots, a_{j-1}, a_j, a_{j+1}, \cdots, a_n)$. By Lemma 2.8, ν' is a solution of the word equation $\{u, vX_{n+1}\}$. It now follows easily that ν is a solution of $\mathscr{W}$. Let $\Gamma | \Lambda =$ alphabet of ν'. Then alphabet of $\nu = \Gamma | \Lambda$ and $f\colon \nu \leqq \mu$. If $g\colon \nu \leqq \mu$, then by Lemma 2.13 $g\colon \nu' \leqq \mu'$ and so $g = f$. Let $A \in \Gamma$. If a_{n+1} ends with A, then so does a_k where $X_k \in T_1$. If a'_j ends with A, then so does a_j. It follows that ν satisfies (2.4). Finally let $\delta = (d_1, \cdots, d_n)$ be a solution of $\mathscr{W}$, $h\colon \delta \leqq \mu$. Let $c_3 = u(d_1, \cdots, d_n)$, $c_4 = v(d_1, \cdots, d_n)$. By Lemma 2.13, $l(c_4d_j) > l(c_3) > l(c_4)$. As above there exist d_{n+1}, $d'_j \neq 1$, such that $d_{n+1} \sim c_3 \sim c_4$, $d_j = d_{n+1}d'_j$, $c_3 = c_4d_{n+1}$. It follows that $\delta' = (d_1, \cdots, d_{j-1}, d'_j, d_{j+1}, \cdots, d_n, d_{n+1})$ is a solution $\mathscr{W}'$. It is also easily seen that $h\colon \delta' \leqq \mu'$. So there exist $h_1\colon \nu' \leqq \delta'$. It follows as above that $h_1\colon \nu \leqq \delta$. Thus $\mu \in \mathscr{S}$, proving the lemma.

LEMMA 2.16. *Let $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ be a generalized* *word equation in variables* $X_1, \cdots, X_n$. *Suppose* X_j *is a free variable of* $\mathscr{W}$. *Let* $\mu = (b_1, \cdots, b_n)$ *be a solution* $\mathscr{W}$ *such that for any solution* $\nu = (a_1, \cdots, a_n)$ *of* $\mathscr{W}$, $\nu \leqq \mu$ *implies* $e(a_j) = 1$. *Let* $\mathscr{W}'$ *be the generalized word equation* $\{w_1, w_2; T_1, \cdots T_s, \{X_j\}\}$ *in variables* $X_1, \cdots, X_n$. *Then* μ *is a solution of* $\mathscr{W}'$. *If* $\mu \in \mathscr{S}$ *as a solution of* $\mathscr{W}'$, *then* $\mu \in \mathscr{S}$ *as a solution of* $\mathscr{W}$.

Proof. Clearly, every solution of $\mathscr{W}'$ is a solution of $\mathscr{W}$. Let ν be a solution of $\mathscr{W}$ such that $\nu \leqq \mu$. Then by hypothesis ν is a solution $\mathscr{W}'$. Then lemma now follows trivially.

LEMMA 2.17. *Let* $\mathscr{W} = \{X_i w_1, X_j w_2; T_1, \cdots, T_s\}$ *be a generalized word equation in variable* $X_1, \cdots, X_n$ *such that* X_j *is a free variable. Let* $\mu = (b_1, \cdots, b_n)$ *be a solution of* $\mathscr{W}$ *such that* $l(b_i) < l(b_j)$. *Then there exist* $b'_j \neq 1$ *such that* $b_j = b_i b'_j$. *Introduce a new variable* X'_j *and let* w'_1, w'_2 *be the words obtained by replacing* X_j *by* $X_i X'_j$. *Let* $\mathscr{W}' = \{w'_1, X'_j w'_2; T_1, \cdots, T_s\}$ *in variables* $X_1, \cdots, X_n$. *Then* $\mu' = (b_1, \cdots, b_{j-1}, b'_j, b_{j+1}, \cdots, b_n)$ *is a solution of* $\mathscr{W}'$. *Let* $\mathscr{A} = \{\nu \mid \nu$ *is a solution of* $\mathscr{W}, \nu \leqq \mu\}$, $\mathscr{B} = \{\nu' \mid \nu'$ *is a solution of* $\mathscr{W}', \nu' \leqq \mu'\}$. *Let* $\nu = (a_1, \cdots, a_n) \in \mathscr{A}$. *Then there exists* $a'_j \neq 1$ *such that* $a_j = a_i a'_j$. *If* $\Phi(\nu)$ *is obtained by replacing* a_j *by* a'_j *in* ν, *then* $\Phi(\nu) \in \mathscr{B}$. *If* $f: \nu \leqq \mu$, *then* $f: \Phi(\nu) \leqq \mu'$. *Let* $\nu' = (a_1, \cdots, a'_j, \cdots, a_n) \in \mathscr{B}$. *If* $\Psi(\nu)$ *is obtained by replacing* a'_j *by* $a_j = a_i a'_j$, *then* $\Psi(\nu') \in \mathscr{A}$. *If* $f: \nu' \leqq \mu'$, *then* $f: \Psi(\nu') \leqq \mu$. *Moreover* $\Psi = \Phi^{-1}$. *Finally* $\mu \in \mathscr{S}$ *if and only if* $\mu' \in \mathscr{S}$.

Proof. Suppose $\nu = (a_1, \cdots, a_n) \in \mathscr{A}$. By Lemma 2.13, $l(a_i) < l(a_j)$ and so there exists $a'_j \neq 1$ such that $a_j = a_i a'_j$. So Φ makes sense. Let $\nu' = \Phi(\nu)$. It is clear that ν satisfies (2.4) if and only if ν' does. Rest of the assertions are now fairly obvious.

LEMMA 2.18. *Let* $\mathscr{W} = \{X_1 w_1, X_2 w_2; T_1, \cdots, T_s\}$ *be a generalized word equation in variables* $X_1, \cdots, X_n$ *such that* X_1, X_2 *are free variables. Suppose* $\mu = (b_1, \cdots, b_n)$ *is a solution of* $\mathscr{W}$. *Suppose* $b_1 = b_2$. *Let* w'_1, w'_2 *be obtained by replacing* X_1 *by* X_2 *in* w_1, w_2, *respectively. Let* $\mathscr{W}'$ *be the generalized word equation* $\{w'_1, w'_2; T_1, \cdots, T_s\}$ *in variables* $X_2, \cdots, X_n$. *Then* $\mu' = (b_2, \cdots, b_n)$ *is a solution of* $\mathscr{W}'$. *Moreover, if* $\mu' \in \mathscr{S}$, *then* $\mu \in \mathscr{S}$.

Proof. Let $\mathscr{A} = \{\nu \mid \nu$ is a solution of $\mathscr{W}, \nu \leqq \mu\}$, $\mathscr{B} = \{\nu \mid \nu$ is a solution of $\mathscr{W}', \nu \leqq \mu'\}$. If $\nu = (a_1, \cdots, a_n) \in \mathscr{A}$, then by Lemma 2.13, $a_1 = a_2$. So $\Phi(\nu) = (a_2, \cdots, a_n) \in \mathscr{B}$. Also, $f: \nu \leqq \mu$ implies $f: \Phi(\nu) \leqq \Phi(\mu) = \mu'$. It is also clear that ν satisfies (2.4) if and only if $\Phi(\nu)$ does. If $\nu' = (a_2, \cdots, a_n) \in \mathscr{B}$, then let $\Phi(\nu') = \nu = (a_2, a_2, \cdots, a_n)$.

It is clear that $\nu \in \mathscr{A}$. Also $g: \nu' \leqq \mu'$ implies $g: \nu \leqq \mu$. Finally, it is obvious that $\Phi\Psi, \Psi\Phi$ are identity maps of $\mathscr{B}, \mathscr{A}$ respectively. So $\Psi = \Phi^{-1}$. This correspondence yields the result.

THEOREM 2.19. *Let* $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ *be a generalized word equation in variables* $X_1, \cdots, X_n$ *and let* $\mu = (b_1, \cdots, b_n)$ *be a solution of* $\mathscr{W}$. *Then* $\mu \in \mathscr{S}$.

Proof. Let d denote the number of free variables of $\mathscr{W}$. We prove by induction on d. If $d = 0$, we are done by Lemma 2.12. So assume $d > 0$. We now (for a given d) proceed by induction on s. If $w_1 = w_2$, we are done by Lemma 2.11. So assume $w_1 \neq w_2$. Let w_1 start with X_i, w_2 start with X_j. By Lemma 2.7, we can assume that $X_i \neq X_j$. Next suppose X_i, X_j are both constrained, $X_i \in T_1$. If $X_j \notin T_1$, we are done by Lemma 2.10 and our induction hypothesis on s. So let $X_j \in T_1$. Now $w_1 = uw_3$, $w_2 = vw_4$ such that $u, v \in \mathscr{F}(T_1)$, $l(uv)$ maximal. Let $c_1 = u(b_1, \cdots, b_n)$, $c_2 = v(b_1, \cdots, b_n)$. By Lemma 2.9 and an easy induction on $l(w_1w_2)$, we can assume that $l(c_1) \neq l(c_2)$. By symmetry assume $l(c_2) < l(c_1)$. If w_4 starts with a constrained variables we are done by Lemma 2.10 and our induction hypothesis on s. So assume w_4 starts with a free variable. Thus by Lemmas 2.14 and 2.15, we can assume, without loss of generality, that either X_i or X_j is free. By Lemma 2.6 and our induction hypothesis on d, we can assume that each free variable occurs in w_1w_2.

We now assume $u \in \mathscr{S}$ and obtain a contradiction. Let $\mathscr{C} = \{r | X_r \in T_q \text{ for some } q\}$. Let $\mu^{(0)} = \mu$, $b_r^{(0)} = b_r$, $r = 1, \cdots, n$, $\mathscr{W}^{(0)} = \mathscr{W}$, $w_1^{(0)} = w_1$, $w_2^{(0)} = w_2$, $X_r^{(0)} = X_r$, $r = 1, \cdots, n$. Also let $\mathscr{X}^{(0)} = \{\nu_\alpha^{(0)} = (a_1^{\alpha,0}, \cdots, a_n^{\alpha,0}) | \alpha \in \Omega\}$ denote the class of all solutions of $\mathscr{W}$ such that $\nu \leqq \mu$. Let $u^{(0)} = w_1w_2(X_1, \cdots, X_n)$, $m_r^{(0)}$ the number of times X_r appears in $u^{(0)}$. Then $m_r^{(0)} > 0$ for $r \notin \mathscr{C}$. Let $b = u^{(0)}(b_1, \cdots, b_n)$ and for $\alpha \in \Omega$, let $a_\alpha = u^{(0)}(a_1^{\alpha,0}, \cdots, a_n^{\alpha,0})$. We will construct a sequence of generalized word equations $\mathscr{W}^{(k)}$ with particular solutions $\mu^{(k)} = (b_1^{(k)}, \cdots, b_n^{(k)})$ such that the following conditions are true for all $k \in Z$. Intuitively we are looking at the sequence of word equations obtained by always truncating on the left.

(I) $\mathscr{W}^{(k)} = \{w_1^{(k)}, w_2^{(k)}; T_1, \cdots, T_s\}$ in variables $X_1^{(k)}, \cdots, X_n^{(k)}$ such that $X_r^{(k)} = X_r$ for $r \in \mathscr{C}$.

(II) $w_1^{(k)}, w_2^{(k)}$ start with different variables, at least one of which is free.

(III) The class of solutions ν of $\mathscr{W}^{(k)}$ such that $\nu \leqq \mu^{(k)}$ can be written as $\mathscr{X}^{(k)} = \{\nu_\alpha^{(k)} = (a_1^{\alpha,k}, \cdots, a_n^{\alpha,k}) | \alpha \in \Omega\}$.

(IV) There exists $u^{(k)} \in \mathscr{F}(X_1, \cdots, X_n)$ such that $b = u^{(k)}(b_1^{(k)}, \cdots, b_n^{(k)})$, $a_\alpha = u^{(k)}(a_1^{\alpha,k}, \cdots, a_n^{\alpha,k})$ for all $\alpha \in \Omega$. Let $m_r^{(k)}$ denote the number of times X_r appears in $u^{(k)}$. If $k > 0$, then $m_r^{(k)} \geqq m_r^{(k-1)}$,

$r = 1, \cdots, n$ and $\sum_{r=1}^{n} m_r^{(k)} > \sum_{r=1}^{n} m_r^{(k-1)}$.

(V) If $k > 0$, then there exist $u_1^{(k)}, \cdots, u_n^{(k)} \in \mathscr{F}(X_1, \cdots, X_n)$ such that $b_r^{(k-1)} = u_r^{(k)}(b_1^{(k)}, \cdots, b_n^{(k)})$, $a_r^{\alpha,k-1} = u_r^{(k)}(a_1^{\alpha,k}, \cdots, a_n^{\alpha,k})$, $r = 1, \cdots, n$, $\alpha \in \Omega$.

(VI) Suppose $k > 0$. Then there exists $q \in \{1, \cdots, n\} q \notin \mathscr{C}$, such that $X_q^{(k)} \neq X_q^{(k-1)}$ and $X_r^{(k)} = X_r^{(k-1)}$ for all $r \in \{1, \cdots, n\}$, $r \neq q$. Also there exists $t \in \{1, 2\}$ such that $w_t^{(k)}$ starts with $X_q^{(k)}$ and $w_t^{(k-1)}$ starts with $X_q^{(k-1)}$. If $w_{3-t}^{(k-1)}$ starts with $X_p^{(k-1)}$, then $m_p^{(k)} > m_p^{(k-1)}$.

(VII) Suppose $k > 0, r \in \{1, \cdots, n\}$. If $X_r^{(k)} = X_r^{(k-1)}$, then $b_r^{(k)} = b_r^{(k-1)}$, $a_r^{\alpha,k} = a_r^{\alpha,k-1}$ for all $\alpha \in \Omega$. If $X_r^{(k)} \neq X_r^{(k-1)}$, then $b_r^{(k)} |_f b_r^{(k-1)}$, $a_r^{\alpha,k} |_f a_r^{\alpha,k-1}$ for all $\alpha \in \Omega$.

(VIII) $\mu^{(k)} \notin \mathscr{S}$.

Clearly $\mathscr{W}^{(0)}$ satisfies (I) to (VIII). We proceed by induction. So, having constructed $\mathscr{W}^{(0)}, \cdots, \mathscr{W}^{(k)}$ satisfying (I) to (VIII), we proceed to construct $\mathscr{W}^{(k+1)}$. Let $w_1^{(k)} = X_p^{(k)} w_1'$, $w_2^{(k)} = X_q^{(k)} w_2'$. Then $p \neq q$ and either $X_p^{(k)}$ or $X_q^{(k)}$ is free. $\mu^{(k)} = (b_1^{(k)}, \cdots, b_n^{(k)})$ is a solution of $\mathscr{W}^{(k)}$. If $l(b_p^{(k)}) = l(b_q^{(k)})$ then $b_p^{(k)} = b_q^{(k)}$ and by Lemmas 2.14 and 2.18, $\mu^{(k)} \in \mathscr{S}$, contradicting (VIII). By symmetry assume $l(b_p^{(k)}) < l(b_q^{(k)})$. By Lemma 2.14, $X_q^{(k)}$ is free. Also $b_q^{(k)} = b_p^{(k)} b_q^{(k+1)}$ for some $b_q^{(k+1)} \neq 1$. Introduce a new variable $X_q^{(k+)}$ and set $X_r^{(k+1)} = X_r^{(k)}$ for $r \neq q$. Also let $b_r^{(k+1)} = b_r^{(k)}$ for $r \neq q$, $\mu^{(k+1)} = (b_1^{(k+1}, \cdots, b_n^{(k+1)})$. Let $w_1^{(k+1)}$ be the word obtained by replacing $X_q^{(k)}$ by $X_p^{(k)} X_q^{(k+1)}$ in w_1' and $w_2^{(k+1)} = X_q^{(k+1)} w_2''$ where w_2'' is the word obtained by replacing $X_q^{(k)}$ by $X_p^{(k)} X_q^{(k+1)}$ in w_2'. Then $w_1^{(k+1)}$ does not start with $X_q^{(k+1)}$. Let $\mathscr{W}^{(k+1)} = \{w_1^{(k+1)}, w_2^{(k+1)}; T_1, \cdots, T_s\}$ be the generalized word equation in variables $X_1^{(k+1)}, \cdots, X_n^{(k+1)}$. So (I), (II) are satisfied. By Lemma 2.17, $\mu^{(k+1)}$ is a solution $\mathscr{W}^{(k+1)}$ and $\mu^{(k+1)} \notin \mathscr{S}$. So (VIII) is satisfied. If $\alpha \in \Omega$, then by Lemma 2.17, $a_q^{\alpha,k} = a_p^{\alpha,k} = a_p^{\alpha,k} a_q^{\alpha,k+1}$ for some $a_q^{\alpha,k+1} \neq 1$. Set $a_r^{\alpha,k} = a_r^{\alpha,k+1}$ for $r \neq q$, $\nu_\alpha^{(k+1)} = (a_1^{\alpha,k+1}, \cdots, a_n^{\alpha,k+1})$. Let $\mathscr{X}^{(k+1)} = \{\nu_\alpha^{(k+1)} | \alpha \in \Omega\}$. That $\mathscr{X}^{(k+1)}$ satisfies (III) follows from Lemma 2.17. Let $u^{(k+1)}$ be the word obtained by replacing X_q by $X_p X_q$ in $u^{(k)}$. Let $m_r^{(k+1)}$ denote the number of times $X_r^{(k)}$ appears in $u^{(k+1)}$, $r = 1, \cdots, n$. Then $m_r^{(k)} = m_r^{(k+1)}$ for $r \neq p$ and $m_p^{(k+1)} = m_p^{(k)} + m_q^{(k)} \geqq m_p^{(k)} + m_q^{(0)} > m_p^{(k)}$. It is now easy to see that (IV), (VI), (VII) are satisfied. Let $u_r^{(k+1)} = X_r$ for $r \neq q$, $u_q^{(k+1)} = X_p X_q$. It is then clear that (V) is satisfied. This completes the construction.

Let $\mathscr{B} = \{r \,|\, m_r^{(k)} \to \infty \text{ as } k \to \infty\}$. By (IV), $\sum_{r=1}^{n} m_r^{(k)} \to \infty$ and so $\mathscr{B} \neq \varnothing$. Let $\mathscr{A} = \{r|$ for any $k \in Z^+$, there exists $k' > k$ such that $X_r^{(k')} \neq X_r^{(k'+1)}\}$. We claim that $\mathscr{B} = \mathscr{A}$. First assume $r \in \mathscr{B}$. It follows from (IV) that $l(b_r^{(k)}) \to 0$ as $k \to \infty$. It then follows from (VII) that $r \in \mathscr{A}$. Hence $\mathscr{B} \subseteqq \mathscr{A}$. Conversely let $r \in \mathscr{A}$. Suppose $r \notin \mathscr{B}$. Then there exists $k \in Z^+$ such that $m_r^{(k)} = m_r^{(k')}$ for all $k' > k$. There exists $m > k$ such that $X_r^{(m)} \neq X_r^{(m+1)}$. There exists $\pi \in \mathscr{B}$. So $\pi \neq r$ and by above, $\pi \in \mathscr{A}$. So there exists $\theta \in Z^+$, $\theta > m$ such that $X_\pi^{(\theta)} \neq X^{(\theta+1)}$. By (VI) $X_r^{(\theta)} = X_r^{(\theta+1)}$. Choose $j > m$, minimal so that $X_r^{(j)} = X_r^{(j+1)}$. Then $X_r^{(j-1)} \neq X_r^{(j)}$. By (VI), there exists $t \in \{1, 2\}$ such that $w_t^{(j)}$ starts with $X_r^{(j)}$. It then follows again by (VI) that $m_r^{(j+1)} > m_r^{(j)}$, a contradiction. So $r \in \mathscr{B}$ and $\mathscr{A} = \mathscr{B}$. It follows that $\mathscr{B} \cap \mathscr{C} = \emptyset$.

Since $\mathscr{A} = \mathscr{B}$, it follows from (VII) that there exists $\varepsilon \in \boldsymbol{R}^+$ such that $l(b_r^{(k)}) \geqq \varepsilon$ for all $k \in Z^+$, $r \in \{1, \cdots, n\}\backslash\mathscr{B}$. By (IV), it follows that $l(b_r^{(k)}) \to 0$ for all $r \in \mathscr{B}$. Since $\mathscr{B}$ is finite, there exists $M \in Z^+$ such that $l(b_r^{(M)}) < \varepsilon$ for all $r \in \mathscr{B}$. Let $m > M$. Repeated application (V) shows that there exist $v_1^{(m)}, \cdots, v_n^{(m)} \in \mathscr{F}(X_1, \cdots, X_n)$ such that $a_r^{\alpha,M} = v_r^{(m)}(a_1^{\alpha,m}, \cdots, a_n^{\alpha,m})$, $r = 1, \cdots, n$, $\alpha \in \Omega$ and $b_r^{(M)} = v_r^{(m)}(b_1^{(m)}, \cdots, b_n^{(m)})$, $r = 1, \cdots, n$. The choice of M implies that for $r \in \mathscr{B}$, $v_r^{(m)}$ does not involve X_t for any $t \notin \mathscr{B}$. Thus for $r \in \mathscr{B}$, $v_r^{(m)}$ is a word in $X_t (t \in \mathscr{B})$. Let $\alpha \in \Omega$, $r \in \mathscr{B}$. We will show that $e(a_r^{\alpha,M}) = 1$. Since $\mathscr{A} = \mathscr{B}$, we see by (VI) that there exists $N > M$ such that for any $\theta > N$, $w_1^{(\theta)}, w_2^{(\theta)}$ do not start with $X_j^{(\theta)}$ for any $j \notin \mathscr{B}$. It follows from (IV) that there exists $L > N$ such that for any $\theta \geqq L$, $t \in \mathscr{B}$, $e(a_t^{\alpha,\theta}) = 1$. Since $r \in \mathscr{A}$, there exists $P > N$ such that $X_r^{(P)} \neq X_r^{(P+1)}$. So by (VI), there exists $\pi \in \{1, 2\}$, such that $w_\pi^{(P)}$ starts with $X_r^{(P)}$. Suppose $w_{3-\pi}^{(P)}$ starts with $X_j^{(P)}$. Then by above $j \in \mathscr{B}$. So $e(a_r^{\alpha,P}) = e(a_j^{\alpha,P}) = 1$ and so $a_j^{\alpha,P} \sim a_r^{\alpha,P}$. We claim that for any $p \geqq P$, if either $w_1^{(p)}$ or $w_2^{(p)}$ starts with $X_t^{(p)}$, then $a_t^{\alpha,p} \sim a_r^{\alpha,P}$. We prove the above property for $p + 1$, assuming it to be true for p. By symmetry, let $w_1^{(p)}$ start with $X_t^{(p)}$, $X_t^{(p)} \neq X_t^{(p+1)}$. Then by (VI) $w_1^{(p+1)}$ starts with $X_t^{(p+1)}$ and by (VII), $a_t^{\alpha,p+1} \sim a_t^{\alpha,p} \sim a_r^{\alpha,p}$. If $w_2^{(p+1)}$ starts with $X_q^{(p+1)}$, then clearly $a_q^{\alpha,p+1} \sim a_t^{\alpha,p+1} \sim a_r^{\alpha,P}$. Thus the asserted claim is true. Now let $t \in \mathscr{B}$. Then since $t \in \mathscr{A}$, there exists $p > P$ such that $X_t^{(p)} \neq X_t^{(p)}$. So by (VI), either $w_1^{(p)}$ or $w_2^{(p)}$ starts with $X_t^{(p)}$. By above, $a_t^{\alpha,p} \sim a_r^{\alpha,p}$. Since $p > L$ and $e(a_t^{\alpha,L}) = 1$, we see by (VII) that $a_t^{\alpha,L} \sim a_r^{\alpha,P}$. Now $a_r^{\alpha,M} = v_r^{(L)}(a_1^{\alpha,L}, \cdots, a_n^{\alpha,L})$. Since X_t does not appear in $v_r^{(L)}$ for $t \notin \mathscr{B}$, we see by the above that $e(a_r^{\alpha,M}) = 1$. Since M is independent of α, we see by Lemma 2.16 (and our induction hypothesis on d) that $\mu^{(\alpha)} \in \mathscr{S}$, contradicting (VIII). This contradiction proves the theorem.

COROLLARY 2.20. *Let $\mathscr{W}$ be a generalized word equation and μ, ν_1, ν_2 solutions of $\mathscr{W}$ such that $\nu_1 \leqq \mu$, $\nu_2 \leqq \mu$. Then there exists a solution ν of $\mathscr{W}$ such that $\nu \leqq \nu_1$ and $\nu \leqq \nu_2$. In particular, if ν_1, ν_2 are principal, then $\nu_1 \approx \nu_2$.*

LEMMA 2.21. *Let $\mathscr{W}$ be a generalized word equation, μ a principal solution of $\mathscr{W}$ such that μ satisfies* (2.4). *Then μ satisfies the right-left dual of* (2.4).

Proof. By the dual of Theorem 2.19, there exists a principal solution ν of $\mathscr{W}$ satisfying the dual of (2.4) such that $\nu \leqq \mu$. Since μ is principal, $\nu \approx \mu$. There exist $f: \nu \leqq \mu$, $g: \mu \leqq \nu$. Let the alphabet of $\mu = \Gamma_1 | \Lambda_1$, alphabet of $\nu = \Gamma_2 | \Lambda_2$. Then $f(\Gamma_2 \backslash \Lambda_2) \subseteq \overline{\Gamma_1 \backslash \Lambda_1}$, $g(\Gamma_1 \backslash \Lambda_1) \subseteq \overline{\Gamma_2 \backslash \Lambda_2}$. Let $\nu = (a_1, \cdots, a_n)$, $\mu = (b_1, \cdots, b_n)$. Let $A \in \Lambda_1$. Then there exists $i \in \{1, \cdots, n\}$, $b \in \mathscr{F}_R(\Gamma_1 | \Lambda_1)^1$ such that $b_i = bA$. So $a_i = g(b_i) = ag(A)$ where $a = g(b)$. Let $g(b_i)$ end with $B \in \Gamma_2$. We claim that $B \in \Lambda_2$. Suppose not. Now $g(A) = cB^r$ for some $r \in \boldsymbol{R}^+$, $c \in \mathscr{F}_R(\Gamma_2 | \Lambda_2)^1$. Since $B \in \Gamma_2 \backslash \Lambda_2$, $f(B) \in \overline{\Gamma_1 \backslash \Lambda_1}$. Let $f(B) = D^t$, $D \in \Gamma_1 \backslash \Lambda_1$, $t \in \boldsymbol{R}^+$. But $bA = b_i = f(a_i) = f(a)fg(A) = f(a)f(c)D^{tr}$, a contradiction since $A \in \Lambda_1$, $D \notin \Lambda_1$. Thus $B \in \Lambda_2$ and we can assume that $g(A) = cB$ for some $c \in \mathscr{F}_R(\Gamma_2 | \Lambda_2)^1$. Now $bA = b_i = f(a_i) = f(ac)f(B)$. So $f(B) = dA$ for some $d \in \mathscr{F}_R(\Gamma_1 | \Lambda_1)^1$. Also the dual of the above argument, applied to B, shows that there exist $j \in \{1, \cdots, n\}$, $x, y \in \mathscr{F}_R(\Gamma_2 | \Lambda_2)^1$, $z \in \mathscr{F}_R(\Gamma_1 | \Lambda_1)^1$, $C \in \Lambda_1$ such that $a_j = Bx$, $f(B) = Cz$, $g(C) = By$. We claim that $C = A$. Suppose not. Then since $Cz = f(B) = dA$, we see that $f(B) = CuA$ for some $u \in \mathscr{F}_R(\Gamma_1 | \Lambda_1)^1$. Then $gf(B) = g(C)g(u)g(A) = Byg(u)cB$. Thus B appears at least twice in $gf(B)$. An easy induction shows that B appear at least 2^i times in $(gf)^i(B)$. In particular $l((gf)^i(B)) \to \infty$ as $i \to \infty$. However $a_j = (gf)^i(a_j) = (gf)^i(B)(gf)^i(x)$ for all i. This contradiction shows that $A = C$. So $b_j = f(a_j) = f(B)f(x) = Czf(x) = Azf(x)$ and b_j starts with A.

Next suppose $A \in \Gamma_1 \backslash \Lambda_1$. There exists $i \in \{1, \cdots, n\}$ such that b_i ends with A. So there exists $r \in \boldsymbol{R}^+$, $b \in \mathscr{F}_R(\Gamma_1 \backslash \Lambda_1)^1$ such that $b_i = bA^r$. Now $g(A) \in \overline{\Gamma_2 \backslash \Lambda_2}$. So there exists $k \in \boldsymbol{R}^+$, $B \in \Gamma_2 \backslash \Lambda_2$ such that $g(A) = B^k$. Now $f(B) \in \overline{\Gamma_1 \backslash \Lambda_1}$. So there exist $t \in \boldsymbol{R}^+$, $C \in \Gamma_1 \backslash \Lambda_1$ such that $f(B) = C^t$. Thus

$$bA^r = b_i = f(a_i) = fg(b_i) = fg(b)C^{rtk} .$$

It follows that $C = A$. There exists $j \in \{1, \cdots, n\}$ such that a_j starts with B. Since $f(B) = A^t$, $b_j = f(a_j)$ starts with A. This completes the proof.

LEMMA 2.22. *Let $\mathscr{W}$ be a generalized word equation with solutions $\nu = (a_1, \cdots, a_n)$, $\mu = (b_1, \cdots, b_n)$ having alphabets $\Gamma_1 | \Lambda_1$, $\Gamma_2 | \Lambda_2$, respectively. Suppose ν satisfies (2.4) and its dual, and $f: \nu \leqq \mu$, $g: \mu \leqq \nu$. Then $g = f^{-1}$ and f, g are unique. If f_1, f_2 are restrictions of f to Λ_1 and $\overline{\Gamma_1/\Lambda_1}$ respectively, then $f_1: \Lambda_1 \to \Lambda_2$ and $f_2: \overline{\Gamma_1/\Lambda_1} \to \overline{\Gamma_2/\Lambda_2}$ are bijections.*

Proof. Now $f(\Gamma_1 \backslash \Lambda_1) \subseteq \overline{\Gamma_2 \backslash \Lambda_2}$, $g(\Gamma_2 \backslash \Lambda_2) \subseteq \overline{\Gamma_1 \backslash \Lambda_1}$. Let $A \in \Lambda_1$. Then there exist $i, j \in \{1, \cdots, n\}$, $c, d \in \mathscr{F}_R(\Gamma_1 \backslash \Lambda_1)^1$ such that $a_i = AC$, $a_j = dA$. So $Ac = a_i = gf(a_i) = gf(A)gf(c)$. Similarly $dA = gf(d)gf(A)$. Thus $gf(A)$ starts and ends with A. Let $f(A)$ start with B, end with C.

If $B \in \Gamma_2 \backslash \Lambda_2$, then $(B) \in \overline{\Gamma_1 \backslash \Lambda_1}$ contradicting the fact that $gf(A)$ starts with A. So $B \in \Lambda_2$. Similarly $C \in \Lambda_2$. So $f(A) = Bx = yC$ for some $x, y \in \mathscr{F}_R(\Gamma_2|\Lambda_2)^1$. Also $gf(A) = g(B)g(x) = g(y)g(C)$. So $g(B)$ starts with A and $g(C)$ ends with A. So there exist $a, b \in \mathscr{F}_R(\Gamma_1|\Lambda_1)^1$ such that $g(B) = Aa$, $g(C) = bA$. We claim that $f(A) = B$. Otherwise there exists $z \in \mathscr{F}_R(\Gamma_2|\Lambda_2)^1$ such that $f(A) = BzC$. So $gf(A) = AazbA$. An easy induction shows that A appears at least 2^{k+1} times in $(gf)^k(A)$ for all $k \in Z^+$. In particular $l(gf)^k(A) \to \infty$. But $a_i = (gf)^k(a^i) = (gf)^k(A)c$ for all $k \in Z^+$. This contradiction shows that $f(A) = B$. Now $g(B) = Aa = bA$. We claim that $A = g(B)$. Otherwise there exists $a_1 \in \mathscr{F}_R(\Gamma_1|\Lambda_1)^1$ such that $g(B) = Aa_1A$. So $fg(B) = Bf(a_1)B$. As above, this implies that $l((fg)^k(B)) \to \infty$ as $k \to \infty$. This contradicts the fact that $(fg)^k(B)$ is a segment of $(fg)^k(b_i) = b_i$ for all $k \in Z^+$. So $A = g(B)$. Thus $f(\Lambda_1) \subseteqq \Lambda_2$ and gf is the identity map on Λ_1. So f is 1-1 on Λ_1. Let $D \in \Lambda_2$. There exists $j \in \{1, \cdots, n\}$ such that D appears in b_j. Now $f(\Gamma_1 \backslash \Lambda_1) \subseteqq \overline{\Gamma_2 \backslash \Lambda_2}$. Since $b_j = f(a_j)$ it follows that b_j and hence D lie in the semigroup generated by $\overline{\Gamma_2 \backslash \Lambda_2}$ and $f(\Lambda_1)$. Hence $D \in f(\Lambda_1)$. So $f(\Lambda_1) = \Lambda_2$. Let f_1 be the restriction of f to Λ_1. $f_1 \colon \Lambda_1 \to \Lambda_2$ is a bijection.

Now let $A \in \Gamma_1 \backslash \Lambda_1$. There exists $i \in \{1, \cdots, n\}$ such that a_i starts with A. So there exist $a \in \mathscr{F}_R(\Gamma_1|\Lambda_1)^1$, $r \in \boldsymbol{R}^+$, such that $a_i = A^r a$ and a does not start with A. Then $f(A) \in \overline{\Gamma_2 \backslash \Lambda_2}$. So there exists $B \in \Gamma_2 \backslash \Lambda_2$, $t \in \boldsymbol{R}^+$ such that $f(A) = B^t$. Now $A^r a = a_i = gf(a_i) = g(B^{rt})gf(a)$. So $g(B) \in \bar{A}$. Thus $gf(A) \in \bar{A}$ for all $A \in \Gamma_1 \backslash \Lambda_1$. It follows that f is 1-1 on $\overline{\Gamma_1 \backslash \Lambda_1}$. Since $gf(\Lambda_1) = \Lambda_1$, it also follows that in the above instance, $gf(a)$ does not start with A. So $g(B^{rt}) = A^r$. Hence $g(B) = A^{1/t}$ and gf is the identity map on $\overline{\Gamma_1 \backslash \Lambda_1}$. Let $D \in \Gamma_2 \backslash \Lambda_2$ Then D appears in b_j for some $j \in \{1, \cdots, n\}$. Since $f(a_j) = b_j$ it follows that b_j and hence D is an element of the semigroup generated by $f(\overline{\Gamma_1 \backslash \Lambda_1})$ and Λ_2. So $D \in f(\overline{\Gamma_1 \backslash \Lambda_1})$. Hence $f(\overline{\Gamma_1 \backslash \Lambda_1}) = \overline{\Gamma_2 \backslash \Lambda_2}$. So if f_2 is the restriction of f to $\overline{\Gamma_1 \backslash \Lambda_1}$, then $f_2 \colon \overline{\Gamma_1 \backslash \Lambda_1} \to \overline{\Gamma_2 \backslash \Lambda_2}$ is a bijection. Also, it now follows that $f \colon \mathscr{F}_R(\Gamma_1|\Lambda_1) \to \mathscr{F}_R(\Gamma_2|\Lambda_2)$ is a bijection. Since gf is the identity map on Λ_1 and $\overline{\Gamma_1 \backslash \Lambda_1}$, it is the identity map on $\mathscr{F}_R(\Gamma_1|\Lambda_1)$. So $g = f^{-1}$. If $g' \colon \mu \leqq \nu$, then the above argument shows that $g' = f^{-1} = g$. So g is unique. Similarly f is unique.

THEOREM 2.23. *Let $\mathscr{W}$ be a generalized word equation and $\mu = (b_1, \cdots, b_n)$ a principal solution of $\mathscr{W}$ with alphabet $\Gamma|\Lambda$. Then for each $A \in \Gamma$, there exist $i, j \in \{1, \cdots, n\}$ such that b_i starts with A and b_j ends with A.*

Proof. By Theorem 2.19, there exists a principal solution ν of

$\mathscr{W}$ satisfying (2.4) such that $f: \nu \leqq \mu$ for some f. By Lemma 2.21, ν satisfies the dual of (2.4). Since μ is principal, there exists $g: \mu \leqq \nu$. Thus f has the structure given by Lemma 2.22. It immediately follows that μ satisfies (2.4) and its dual.

Lemma 2.22 and Theorem 2.23 imply the following.

THEOREM 2.24. *Let $\mathscr{W}$ be a generalized word equation with principal solutions ν, μ having alphabets $\Gamma_1|\Lambda_1, \Gamma_2|\Lambda_2$, respectively. Suppose $f: \nu \leqq \mu$. Then $f: \mathscr{F}_R(\Gamma_1|\Lambda_1) \cong \mathscr{F}_R(\Gamma_2|\Lambda_2)$ is a natural isomorphism. Moreover f is unique and $f^{-1}: \mu \leqq \nu$. If f_1, f_2 are the restrictions of f to Λ_1 and $\overline{\Gamma_1 \backslash \Lambda_1}$ respectively, then $f_1: \Lambda_1 \rightarrow \Lambda_2$, $f_2: \overline{\Gamma_1 \backslash \Lambda_1} \rightarrow \overline{\Gamma_2 \backslash \Lambda_2}$ are bijections.*

REMARK 2.25. Let ν, μ be solutions of a generalized word equation $\mathscr{W}$ such that $\Gamma_1|\Lambda_1, \Gamma_2|\Lambda_2$ are the alphabets of ν, μ, respectively. Suppose $f: \nu \leqq \mu$, $g: \mu \leqq \nu$. In free semigroups (i.e., if $\Gamma_1 = \Lambda_1, \Gamma_2 = \Lambda_2$) this implies that f, g are unique and $g = f^{-1}$. However, this is not true in general. In fact consider the word equation $\{X, X\}$. Let $\mu = (AB^{1/2})$ with alphabet $\{A, B\}|\{A\}$. Then $f: \mu \leqq \mu$ where $f(A) = AB^{1/4}$ and $f(B) = B^{1/2}$. Also $I:_{/}\mu \leqq \mu$ where $I(A) = A$ and $I(B) = B$.

THEOREM 2.26. *Let μ, ν be solutions of a generalized word equation $\mathscr{W}$. Suppose ν is principal and $g: \nu \leqq \mu$. Then g is unique.*

Proof. By Theorem 2.19 there exists a principal solution δ of $\mathscr{W}$, $f: \delta \leqq \mu$ such that f is unique and $\delta \in \mathscr{S}$. Since $\delta \in \mathscr{S}$ and ν is principal, $\nu \approx \mu$. By Theorem 2.25, there exists h such that $h: \delta \leqq \nu$, $h^{-1}: \nu \leqq \delta$. Tnen $gh: \delta \leqq \mu$. By uniqueness of f, $gh = f$. So $g = fh^{-1}$. Similarly if $g': \nu \leqq \mu$, then $g' = fh^{-1}$. So g is unique.

EXAMPLE 2.27. Theorem 2.26 is not true without the assumption that ν is principal. To see this let $\mathscr{W} = \{X, X\}$. Let $\nu = (AB)$, $\mu = (AB^2)$ in alphabet $\{A, B\}$. Let $f: \mathscr{F}(A, B) \rightarrow \mathscr{F}(A, B)$ be given by $f(A) = AB, f(B) = B$. Let $g: \mathscr{F}(A, B) \rightarrow \mathscr{F}(A, B)$ be given by $g(A) = A$, $g(B) = B^2$. Then $f: \nu \leqq \mu$, $g: \nu \leqq \mu$, but $f \neq g$.

DEFINITION 2.28. Let $\mathscr{W}$ be a generalized word equation and $\mu = (b_1, \cdots, b_n)$ a solution of $\mathscr{W}$. Then μ is integral if $b_1, \cdots, b_n \in \mathscr{F}(\Gamma)$ for some Γ. μ is rational if $b_1, \cdots, b_n \in \mathscr{F}_Q(\Gamma)$ for some Γ.

OBSERVATION 2.29. Let μ, ν be solutions of a generalized word equation $\mathscr{W}$ such that $\nu \leqq \mu$ and μ is integral. Then ν is integral.

THEOREM 2.30. *Let μ, ν be solutions of a generalized word equation $\mathscr{W}$ such that $\nu \leqq \mu$, μ is rational and ν is principal. Then ν is integral.*

Proof. Let $\mu = (b_1, \cdots, b_n)$ with alphabet $\Gamma | \Lambda$. Then $b_1, \cdots, b_n \in \mathscr{F}_Q(\Gamma)$. It follows that there exists a natural automorphism ϕ of $\mathscr{F}_R(\Gamma | \Lambda)$ such that ϕ restricted to $\mathscr{F}(\Lambda)$ is the identity map, and $\phi(b_1), \cdots, \phi(b_n) \in \mathscr{F}(\Lambda)$. Let $a_i = \phi(b_i)$, $i = 1, \cdots, n$. Then $\mu_1 = (a_1, \cdots, a_n)$ is an integral solution of $\mathscr{W}$ and $\phi^{-1}: \mu_1 \leqq \mu$. By Theorem 2.19, there exists a principal solution ν_1 of $\mathscr{W}$ such that $\nu_1 \leqq \mu$. By Observation 2.29, ν_1 is integral. Since $\nu_1 \leqq \mu, \nu \leqq \mu$, Corollary 2.20 implies that $\nu \approx \nu_1$. Again by Observation 2.29, ν is integral.

COROLLARY 2.31. *Let μ be a rational, principal solution of a generalized word equation $\mathscr{W}$. Then μ is integral.*

3. Rank of a word equation.

We now study the maximum of the cardinalities of the alphabets of the principal solutions of a generalized word equation. For word equations in free semigroups, this number has been studied in some detail by Lentin [2] and others.

DEFINITION 3.1. Let $\mathscr{W}$ be a generalized word equation and let μ be a solution of $\mathscr{W}$ having alphabet $\Gamma | \Lambda$. Then rank of μ, $R(\mu) = |\Gamma|$, the number of elements in Γ. Rank of $\mathscr{W}$, $R(\mathscr{W}) =$ Sup. $\{R(\nu) | \nu$ is a principal solution of $\mathscr{W}\}$. The integral rank of $\mathscr{W}$, $I.R(\mathscr{W}) =$ Sup. $\{R(\nu) | \nu$ is an integral principal solution of $\mathscr{W}\}$. We take Sup. $\varnothing$ to mean 0.

Let $\mathscr{W}$ be a generalized word equation and μ a principal solution of $\mathscr{W}$. Consider the following property of μ:

(3.2) There exists an integral principal solution ν of $\mathscr{W}$ such that $R(\nu) \geqq R(\mu)$.

Following a procedure similar to that of § 2 one can show that every principal solution of a generalized word equation satisfies (3.2). The following theorem then follows,

THEOREM 3.3. *Let $\mathscr{W}$ be a generalized word equation. Then $R(\mathscr{W}) = I \cdot R(\mathscr{W})$.*

The following theorems can also be proved in a similar manner.

THEOREM 3.4. *Let* $\mathscr{W} = \{w_1 w_2; T_1, \cdots, T_s\}$ *be a generalized word equation with* r *free variables. Then* $R(\mathscr{W}) \leqq r + s$.

THEOREM 3.5. *Let* $\mathscr{W} = \{w_1, w_2; T_1, \cdots, T_s\}$ *be a generalized word equation in variables* $X_1, \cdots, X_n$ *such that* $w_1 \neq w_2$. *Then* $R(\mathscr{W}) \leqq n - 1$.

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NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC 27650