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A CHARACTERIZATION OF THE REPRESENTABLE LEBESGUE DECOMPOSITION PROJECTIONS

WAYNE C. BELL AND MICHAEL KEISLER

Finitely additive measures whose Lebesgue decomposition projections have refinement integral representations are characterized in terms of certain atomic properties.

In [2] it was shown that under certain fairly weak conditions the dual of $ba(S, F)$ does not have a refinement integral representation since these conditions lead to a $\mu \in ba(S, F)$ for which T_μ, the linear functional associated with the Lebesgue decomposition projection P_μ, is not representable. In [1] it was shown that for any $\mu > 0$ there is a maximal "nonrepresentable" part. Here we combine these results to give a necessary and sufficient condition under which T_μ is representable.

A nonnegative μ in $ba(S, F)$ is *atomic* if $I \in F$ such that $\mu(I) > 0$ implies I contains a *μ-atom* J (i.e., $J \in F$, $\mu(J) > 0$ and for $K \in F$ and $K \subseteqq J$, $\mu(K) \in \{0, \mu(J)\}$. In this paper we are interested in a stronger notion of atomic.

DEFINITION. If $\mu \in ba(S, F)$ and $\mu \geqq 0$, then μ is *totally atomic* if each $\lambda \in ba(S, F)$ such that $0 \leqq \lambda \leqq \mu$ is atomic.

A totally atomic μ is a sum of two-valued measures, but the converse statement is false. It was noted in [3] that for the σ-field $P(N)$ one may select a $\mu' \geqq 0$ which is a sum of two-valued measures and for which there are no μ'-atoms in $P(N)$. Letting $\lambda \geqq 0$ be a sum of λ_n, where λ_n is two-valued and $\{n\}$ is a λ_n-atom, for every n, we then have that $\mu' + \lambda$ is a sum of two-valued measures which is also atomic but still not totally atomic.

It may be noted that $T_\mu(\lambda) = P_\mu(\lambda)(S)$, for $\lambda \in ba(S, F)$, defines a member of the dual of $ba(S, F)$. If $\eta \in ba(S, F)$, and $f: F \to R$, then $T_\mu(\lambda) = \int_S f\lambda$, for $\lambda \in ba(S, F)$("T_μ is representable") iff $P_\mu(\lambda) = \int f\lambda$, for $\lambda \in ba(S, F)$("μ is representable" in [1]). The equivalence is an immediate consequence of the definition of T_μ, and is useful below.

THEOREM. *Let $\mu \in ba(S, F)$ and $\mu \geqq 0$. The following are equivalent.*

(i) *T_μ is representable.*

(ii) *μ is totally atomic.*

Proof. In Theorem 2 of [1] it was shown that $\mu=\lambda+\eta$, where λ is representable and if $I \in F$, then η^I is representable iff $\eta^I = 0$ ($\eta^I(V) = \eta(I \cap V)$ for I, $V \in F$). If I is an η-atom, then clearly η^I is representable. Thus η has no atoms. If μ is totally atomic, then $\eta = 0$. Therefore $\mu = \lambda$ and we have that μ is representable.

If μ is not totally atomic, then there is $\lambda \in ba(S, F)$ such that $0 \leqq \lambda \leqq \mu$ and $I \in F$ such that $\lambda(I) > 0$ and I contains no λ-atom. Thus if $J \in F$ and $\lambda^I(J) > 0$, then there is $K \in F$ for which $K \subseteqq J$ and $\lambda^I(K) \notin \{0, \lambda^I(J)\}$. By Theorem 2 of [2], λ^I is not representable, and by 3.c.1. of [1] it follows that μ is not representable.

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MURRAY STATE UNIVERSITY
MURRAY, KY 42071
AND
ARKANSAS TECH UNIVERSITY
RUSSELLVILLE, AR