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INTEGRAL REPRESENTATION FOR ELEMENTS OF THE DUAL OF $ba(S, \Sigma)$

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D. Mauldin presented conditions under which the dual of $ca(S, \Sigma)$ could be given an integral representation. W.D.L Appling gave a "pseudo-representation" for the dual of $ba(S, \Sigma)$, but the latter was not the type obtained by Mauldin. This paper gives conditions that are necessary and sufficient for the existence of the classical type integral representation for the dual of $ba(S, \Sigma)$.

1. Introduction. Mauldin [11] found that an integral representation is possible for the elements of the dual of $ca(S, \Sigma)$ provided the cardinality of $ca(S, \Sigma)$ is $\leqq 2^{\aleph_0}$ and the continuum hypothesis holds. Edwards and Wayment noted in [7] that for the function space of real-valued, absolutely continuous functions on [0, 1], AC, the dual cannot be represented using Riemann or Lebesgue-type integrals. Using a different type of integral, a "v-integral", the latter authors then presented an integral representation for elements of AC^*. W. D. L. Appling [1] observed that an analogous result held for the subspace of $ba(S, \Sigma)$ made up of functions absolutely continuous with respect to a given member μ of $ba(S, \Sigma)$. The latter paper went on to give a "pseudo-representation" of elements of the dual of $ba(S, \Sigma)$. The representation was not the same type as obtained, conditionally, by Mauldin for $ca(S, \Sigma)$. Dunford and Schwarz [6], p. 374, noted that no "satisfactory representation" for members of $ba(S, \Sigma)^*$ was known.

In this paper it is proven that a representation such as that obtained by Mauldin for $ca(S, \Sigma)$ is possible for $ba(S, \Sigma)$ if and only if $ba(S, \Sigma)$ does not contain a "continuous" element, a condition which is shown by application of a theorem of Horn and Tarski [8], to be equivalent to the existence of a subfield of Σ that is isomorphic to the smallest field of subsets of (0, 1] containing the intervals of the form $(k/2^n, (k+1)/2^n]$, for nonnegative integers k and n, hereafter denoted Σ_0.

The integral here will be the same as that used by both Mauldin and Appling. The reader is referred to [2] for its definition and properties.

2. Filters. Properties of filters make them very convenient for several of the arguments. Filters are discussed in [12], and the first two lemmas below, stated without proof, can be found there.

The terminology used here differs slightly, so definitions are given first.

DEFINITION. Let $\sum$ be a field of subsets of S. $F \subseteqq \sum$ is a *filter* iff $v, w \in F$ implies $v \cap w \neq \emptyset$, $v \cap w \in F$ and for $z \in \sum$, $v \cup z \in F$. A filter F is an *ultrafilter* if it is contained properly in no other filter. B is a *base* for a filter F if $B \subseteqq F$ and for $v \in F$ there is $w \in B$ such that $w \subseteqq v$.

LEMMA 1. *Every filter is contained in an ultrafilter.*

LEMMA 2. *F is an ultrafilter iff $v \in \sum$ implies $v \in F$ or $v' \in F$ (where v' is the complement of v).*

For an ultrafilter F we can define $\mu(v) = 1$, if $v \in F$, and $\mu(v) = 0$, if $v \notin F$. It easily seen that $\mu \in ba(S, \sum)$. Therefore the lemma below can be used to construct some special members of $ba(S, \sum)$. Note, however, that there is no guarantee that the μ constructed above is in $ca(S, \sum)$, should $\sum$ happen to be a σ-algebra.

LEMMA 3. *Let Q be a property that if possessed by $v \in \sum$, then for a subdivision D of v, there is $w \in D$ having property Q. A filter F, with a base of elements having property Q, is contained in an ultrafilter with a base of elements having property Q.*

Proof. Let $\mathscr{L}$ be the family of all filters having a base of elements having property Q. $\mathscr{L}$ is partially ordered by inclusion, and by Kuratowski's lemma [9] there is a maximal chain $\mathscr{H}$ in $\mathscr{L}$ containing F. Clearly $F^* = \cup \mathscr{H}$ is a filter, and since $v \in F^*$ implies $G \in \mathscr{H}$ such that $v \in G$, and hence a w with property Q in G such that $w \subseteqq v$, then it follows that $F^* \in \mathscr{L}$. Suppose $z \in \sum$ and neither z nor z' is in F^*. Then $z \cap v \neq \emptyset$, for $v \in F^*$, and it follows that $M = \{y \in \sum \mid y \supseteqq z \cap v,$ for some $v \in F^*\}$ is a filter containing F^*. The maximality of $\mathscr{H}$ implies $M \notin \mathscr{L}$, and hence there is y_M in M such that $k \in M$ and $k \subseteqq y_M$ implies k does not have property Q. Similarly for $N = \{y \in \sum \mid y \supseteqq z' \cap v,$ for some $v \in F^*\}$. If $w \in F^*$ such that $z \cap w \subseteqq y_M$ and $z' \cap w \subseteqq y_N$, then $\{z \cap w, z' \cap w\}$ subdivides w, from which it follows that w does not have property Q. Since F^* is in $\mathscr{L}$, the latter must be false, and we have $z \in F^*$ or $z' \in F^*$; i.e., by Lemma 2, F^* is an ultrafilter.

3. Continuous and discrete elements. There are various uses of the words "atom" and "continuous" in the literature. Their use here is generally consistent with the literature, but we provide definitions for clarity.

DEFINITION. Let $\mu \in ba(S, \sum)$. μ is said to be an *atom* if $\{v \mid \mu(v) \neq 0\}$ is an ultrafilter.

DEFINITION. Let $\mu \in ba(S, \sum)$. μ is said to be *continuous* if for $\varepsilon > 0$ there is a subdivision D of S such that if E refines D, then for $v \in E$, $|\mu(v)| < \varepsilon$.

DEFINITION. Let $\mu \in ba(S, \sum)$. μ is said to be *discrete* if for λ such that $0 < \lambda$ and $\lambda \leqq \int |\mu|$, then λ is not continuous.

The set of nonnegative members of $ba(S, \sum)$ will be denoted $ba(S, \sum)^+$. The next lemma is a restatement of results by Appling, [4], and is a consequence of a decomposition theorem in [5] since the continuous elements form a normal subspace.

LEMMA 4. *Let $\mu \in ba(S, \sum)^+$. There is μ_0 continuous and μ' discrete such that $\mu = \mu_0 + \mu'$, and $\mu'(v) = \int_v \mu^*$, for $v \in \sum$, where $\mu^*(v) = \inf \{\sup \{\mu(w) \mid w \in D\} \mid D$ is a subdivision of $v\}$, for $v \in \sum$.*

Lemma 5 reveals the relationship between atoms and discrete elements.

LEMMA 5. *Let $\mu \in ba(S, \sum)^+$. μ is discrete iff there is a sequence of atoms, $\{\mu_n\}$, such that $\mu = \sum \mu_n$.*

Proof. Lemma 4 implies that if μ is discrete, then $\mu = \int \mu^*$. For $v \in \sum$, if D is a subdivision of v, then there is w in D such that $\mu^*(w) = \mu^*(v)$. Thus for $v \in \sum$, by Lemma 3, there is an ultrafilter F_v with a base of elements on which μ^* equals $\mu^*(v)$; let μ_v be the atom on F_v such that $\mu_v(S) = \mu^*(v)$. It is immediate that $\mu^*(w) = \mu^*(v)$, for w in F_v and $w \subseteqq v$. Therefore, $\mu_v \neq \mu_w$ implies $F_v \neq F_w$, and hence there is z such that $z \in F_v$ and $z' \in F_w$. Thus $\mu_v \vee \mu_w = \int \max \{\mu_v, \mu_w\} = \mu_v + \mu_w$. Since $ba(S, \sum)$ is a complete vector lattice, [5], and $\mu \geqq \mu_v$, for every v, it follows that $\lambda = \sup M$, where $M = \{\mu_v \mid \mu_v > 0\}$, exists, equals $\sum M$, and $\lambda \leqq \mu$. M summable implies that M is countable. And since $\lambda(v) \geqq \mu_v(v) = \mu^*(v)$, for $v \in \sum$, it follows that $\lambda \geqq \mu$.

For the converse, suppose $\mu = \sum \mu_n$, for some sequence $\{\mu_n\}$ of atoms. For $v \in \sum$, D a subdivision of v, and n a positive integer, $\mu_n(v) = \sup \{\mu_n(w) \mid w \in D\}$; thus $\sup \{\mu(w) \mid w \in D\} \geqq \mu_n(v)$. It follows that $\int \mu^* \geqq \sum^N \mu_n$, for every N, since there is a subdivision D such that for $w \in D$, $\sum^N \mu_n(w) = \sup \{\mu_n(w)\}^N$. Therefore, $\int \mu^* \geqq \sum \mu_n = \mu$. From Lemma 4 we conclude that $\int \mu^* = \mu$, and hence μ is discrete.

4. On integral representation. Discrete and continuouse elements play important roles insofar as integral representations of the dual are concerned. One of the most familiar examples of a field of sets, that generated by the half open intervals in $(0, 1]$, gives rise to a continuous element, the interval length measure. On the other hand, it is not difficult to construct a field for which no continuous elements exist; given a set S, let $\sum$ be the field consisting of all finite subsets of S and their complements, then $ba(S, \sum)$ contains no nonzero continuous elements. Theorem 2 demonstrates that the first mentioned example has no classical integral representation for members of the dual, and Theorem 3 shows that the second example does have such a representation for members of the dual.

THEOREM 1. *$\sum$ contains a subfield $\sum'$ isomorphic to $\sum_0$ iff there is a positive, continuous $\mu \in ba(S, \sum)$.*

Proof. The "interval length" function can be extended to a member μ' of $ba(S, \sum')$ and, by Theorem 1.22 of [8], μ' can be extended to a $\mu \in ba(S, \sum)$. Clearly μ is a positive and continuous element of $ba(S, \sum)$. Conversely, a positive and continuous $\mu \in ba(S, \sum)$ can be used via an induction argument to construct a subfield $\sum'$ of $\sum$ isomorphic to $\sum_0$.

DEFINITION. A member T of $ba(S, \sum)^*$ will be said to be *representable* provided there is $f: \sum \rightarrow R$ such that $T(\mu) = \int_S f\mu$, for $\mu \in ba(S, \sum)$. For nonnegative $\mu \in ba(S, \sum)$, P_μ represents the Lebesque decomposition projection operator (see [2] for definition and properties).

THEOREM 2. *If there is a positive $\mu \in ba(S, \sum)$ for which $I \in \sum$ such that $\mu(I) > 0$ implies there are disjoint $J, K \in \sum$ such that $I = J \cup K$ and $\mu(I) \notin \{\mu(J), \mu(K)\}$, then $T(\lambda) = P_\mu(\lambda)(S)$, for $\lambda \in ba(S, \sum)$, defines a nonrepresentable member of $ba(S, \sum)^*$.*

Proof. Suppose T is representable, with $f: \sum \rightarrow R$ such that $\int_S f\lambda = T(\lambda)$, for each $\lambda \in ba(S, \sum)$. If for $I \in \sum$, $\mu^I(V) = \mu(I \cap V)$, for $V \in \sum$, then $T(\mu^I) = \int_S f\mu^I = \int_I f\mu$ and since $P_\mu(\mu^I)(S) = \mu(I)$, we have $\mu(I) = \int_I f\mu$. Applying a theorem of Kolmogoroff, [10],

$$0 = \int_S \left| f\mu - \int f\mu \right| = \int_S \left| f\mu - \mu \right| = \int_S |f - 1| \mu \, .$$

Let $Q = \{I \in \sum \mid$ each subdivision of I contains J such that $\mu(J) >$

0 and $f(J) \geqq 1/2\}$, and note that if $V \in Q$ and D subdivides V, there is $J \in D \cap Q$. Suppose $V \in \Sigma$ such that $\mu(V) > 0$. Since $\int_V \left|f - 1\right| \mu = 0$, there is a subdivision D of V such that if E refines D, then $\sum_E |f - 1| \mu < \mu(V)/2$, from which it follows that there is $I \in D \cap Q$. Since to each $V \in \Sigma$ on which $\mu(V) > 0$ there are disjoint $J, K \in \Sigma$ such that $V = J \cup K$, $\mu(J) > 0$ and $\mu(K) > 0$, then we may construct, inductively, a sequence $\{D_n\}$ of disjoint subsets of Q such that $I \in D_n$ implies there are $J, K \in D_{n+1}$ such that $I \supseteqq J \cup K$ and $I \cap V = \emptyset$ for $V \in D_{n+1}$ and $V \notin \{J, K\}$. Then the collection of infinite maximal chains in $\cup D_n$ has cardinality $\geqq 2^{\aleph_0}$ and any two contain disjoint sets. Each gives rise, by Lemma 3, to an ultrafilter with a base in Q and since $\mu \not\to 0$ on at most $\aleph_0$ ultrafilters, we conclude there is ultrafilter F with a base in Q (implying $f \not\to 0$) and on which $\mu \to 0$. Thus $\lambda(I) = 1$, for $I \in F$, and $= 0$, otherwise, defines $\lambda \in ba(S, \Sigma)$ for which $P_\mu(\lambda) = 0$, although $\int_S f\lambda \geqq 1/2$! Therefore T is not representable.

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $(0, 1)$. For $I \in \Sigma_0$, let $\mu_n(I) = 1/2^n$, if $r_n \in I$, and $= 0$, if $r_n \notin I$. $\{\sum_{n=1}^{N} \mu_n\}_{N=1}^{\infty}$ forms a Cauchy sequence in the Banach space $ba(S, \Sigma_0)$ (with variation norm). Let $\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \mu_n$, and note that μ is a discrete function satisfying the hypothesis of Theorem 2. Thus the latter can produce non-representable members of $ba(S, \Sigma)^*$ through both continuous and discrete elements. However, applying it to continuous elements alone yields the following result.

COROLLARY. *If each member of $ba(S, \Sigma)^*$ is representable, then there is no positive continuous member of $ba(S, \Sigma)$.*

Lemma 4 assures us that if a $ba(S, \Sigma)$ contains no positive continuous elements, then all members of $ba(S, \Sigma)$ are discrete and can be written as in Lemma 5. This representation will be useful in Theorem 3.

THEOREM 3. *If $ba(S, \Sigma)$ contains no positive continuous element, then each T in $ba(S, \Sigma)^*$ is representable.*

Proof. For $B \subseteqq U = \{$all ultrafilters in $\Sigma\}$, let $B^0 = \{F \in B \mid B(V) = \{F' \in B \mid V \in F'\}$ is infinite for $V \in F\}$. Let $g(1) = U$ and if g is defined for all ordinals less than an ordinal x, then let

$$g(x) = (\cap \{g(y) \mid y < x\})^0 .$$

By transfinite induction (see [13]) g can be defined for all ordinals $\leqq 2^{2^U}$. Since $x < y$ implies $g(x) \supseteqq g(y)$, the domain of g has cardinality greater than that of its range, and since the ordinals are well ordered, it follows that there are ordinals x and x' such that $x < x'$ and $g(x) = g(x')$. Thus $g(x) = g(x+1)$ and if $B = g(x)$, then $B = B^0$.

If $F \in B$ and $J \in F$, then $B(J)$ must be infinite (since $F \in B^0$!) and hence we may select disjoint I_0 and I_1 such that each belongs to a member of B and $J = I_0 \cup I_1$. Thus if $B \neq \emptyset$, then we may, by induction, construct a sequence $\{D_n\}$ of subdivisions S such that for each n, every $I \in D_n$ is contained in a member of B, and there are distinct $J, K \in D_{n+1}$ such that $I = J \cup K$. The smallest field $\sum'$ containing $\cup D_n$ is isomorphic to $\sum_0$. Theorem 1 implies there is a positive continuous member of $ba(S, \sum)$. Thus $B = \emptyset$.

Let $A = \{\mu \mid \mu$ is an atom and $\mu(S) = 1\}$ and $F_\mu = \{V \mid \mu(V) = 1\}$, for $\mu \in A$. For $V \in \sum$, let $V^* = \{F \in \cap \{g(y) \mid y < z(V)\} \mid V \in F\}$, where $z(V) = \inf \{y \mid V \notin \cup g(y)\}$ (which exists since $g(x) = \emptyset$). Finally, define $f(V) = \sup \{T(\mu) \mid \mu \in A$ and $F_\mu \in V^*\}$, $V^* \neq \emptyset$, and $f(V) = 0$, if $V^* = \emptyset$. For $\mu \in A$, let $h(\mu) = \inf \{y \mid F_\mu \notin g(y)\}$ and note that $F_\mu \notin g(h(\mu))$ and $F_\mu \in \cap \{g(y) \mid y < h(\mu)\}$. Thus there is $V \in F_\mu$ such that $M = \{F \in \cap \{g(y) \mid y < h(\mu)\} \mid V \in F\}$ is finite. In fact, we can find V such that $M = \{F_\mu\}$, and since $V \notin \cup g(h(\mu))$, it follows that $h(\mu) = z(V)$ and hence $V^* = M$. Thus $f(V) = T(\mu)$, and for $W \in F_\mu$ and $\subseteq V$, $f(W) = T(\mu)$, from which it follows that $\int_S f\mu = T(\mu)$. Since each $\mu \in ba(S, \sum)$ is discrete, there is is a sequence $\{x_n\} \subseteq R$ and a sequence $\{\mu_n\} \subseteq A$ such that $\mu = \sum x_n \mu_n$. Since T is continuous, $T(\mu) = \sum x_n T(\mu_n)$. And since $\|\lambda\| = 1$, for $\lambda \in A$, it follows that f is bounded by $\|T\|$. A result in [3] implies that $\int_S f\mu$ exists and equals $\lim \int_S f \sum^N x_n \mu_n = \lim \sum^N x_n \int_S f\mu_n = \lim \sum^N x_n T(\mu_n) = T(\mu)$.

Theorems 1, 2 and 3 may be combined to produce the following characterization of those $ba(S, \sum)$ with representable duals.

THEOREM 4. *The following are equivalent.*

(i) *Each* $T \in ba(S, \sum)^*$ *is representable.*

(ii) *Each element of* $ba(S, \sum)$ *is discrete.*

(iii) *There is no positive continuous member of* $ba(S, \sum)$.

(iv) *There is no subfield of* $\sum$ *isomophic to* $\sum_0$.

Thus for spaces with a dual representable in the sense of this paper one must turn to subspaces of $ba(S, \sum)$ (as Mauldin [11] has done with $ca(S, \sum)$, where $\sum$ is a σ-algebra, and as Appling [1] has done with the space of functions absolutely continuous with respect to a given member of $ba(S, \sum)$) or be content to deal with a $ba(S, \sum)$ having no positive continuous element.

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