PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 1, 1979

PERIODIC POINTS ON TORI

BENJAMIN HALPERN

We prove the following theorem.

THEOREM 1. **Given a continuous map $f: T^n \to T^n$ of the n-dimensional torus into itself. Each map homotopic to f has an infinite number of periodic points if and only if the Lefschetz numbers of the iterates $L(f^m)$, $m=1, 2, \cdots$, are unbounded.**

The "if" direction of Theorem 1 follows from a theorem of Brooks, Brown, Pak, and Taylor [1]. Let $N(f)$ denote the Nielsen number of the map f. Recall that each map homotopic to f must have at least $N(f)$ distinct fixed points.

THEOREM 2. (*Brook, Brown, Pak, and Taylor* [1]). *If $f: T^n \to T^n$ is a continuous map, then $N(f) = |L(f)|$.*

The converse direction of Theorem 1 is deduced from the more precise result, Theorem 3.

DEFINITION 1. Given a map $f: T^n \to T^n$. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic values of $H_1(f): H_1(T^n) \to H_1(T^n)$. If λ_i is not a root of unity, then set $a_{im} = |1 - \lambda_i^m|$. If λ_i is a root of unity, then let N be such that $\lambda_i^N = 1$ and $\lambda_i^m \neq 1$ for $1 \leqq m < N$ (i.e., λ_i is a primitive Nth root of unity), and set

$$a_{im} = \begin{cases} |1 - \lambda_i^m| & \text{if } m \not\equiv 0 \bmod N \\ \sum_{q|N} |1 - \lambda_i^q| & \text{if } m \equiv 0 \bmod N\,. \end{cases}$$

Set $a_m(f) = \prod_{i=1}^n a_{im}$.

THEOREM 3. *For each map $f: T^n \to T^n$ there exists a smooth map g homotopic to f such that for $m \geqq 1$,*

$$\sharp\{x \in T^n \mid g^m(x) = x\} \leqq a_m(f)\,.$$

Since $L(f^m) = \prod_{i=1}^n (1 - \lambda_i^m)$, we see that $\sharp\{x \in T^n \mid g^m(x) = x\} = N(f)$ for all m such that $\lambda_i^m \neq 1$ for all i. From Theorems 2 and 3, one may also deduce similarities between the asymptotic behaviors of $P_m = \sharp\{x \in T^n \mid g^m(x) = x\}$ and $Q_m = \max\{N(f^r) \mid 1 \leqq r \leqq m\}$.

In the process of proving Theorem 3 we establish a general result, Theorem 4, which concerns periodic points for maps homotopic to periodic maps.

THEOREM 4. *Given a smooth compact connected manifold* M *of dimension* $m \geqq 2$, *and a smooth map* $f: M \to M$ *such that* $f^N = 1_M$ *for some* $N \geqq 2$, *and* $f(x_0) = x_0$ *for some* $x_0 \in M$. *Also suppose* $P = \{x \in M \mid f^r(x) = x \text{ for some } r,\ 1 \leqq r < N\}$ *is finite. Then there exists a smooth map* $g: M \to M$ *which is homotopic to* f *and such that* $P =$ *the set of all periodic points of* g, *and* $g|P = f|P$.

When Theorem 4 is specialized to tori, it gives a map g homotopic to the given periodic map $f: T^n \to T^n$, whose numbers of periodic points of various periods are exactly the lower bounds implied by Theorem 2. Theorem 3 for an arbitrary map $f: T^n \to T^n$ is proved by homotoping f to a map g which with a "change of coordinates" takes the form $g: T^n = T^k \times T^{n-k} \to T^k \times T^{n-k} = T^n$, $g(x, y) = (a(x), r(x, y))$ where $a: T^k \to T^k$ is periodic. We homotopy a to an $\bar{a}$ according to Theorem 4 and then, using an induction hypothesis we homotopy r on the sets $\{x\} \times T^{n-k}$ for x a periodic point of $\bar{a}$. This gives a map whose periodic points are the same as a map of the form $\bar{a} \times \bar{b}: T^k \times T^{n-k} \to T^k \times T^{n-k}$ where $\bar{a}: T^k \to T^k$ and $\bar{b}: T^{n-k} \to T^{n-k}$. This is sufficient to prove Theorem 3 by induction, but it gives a map with possibly more periodic points than the lower bound set in Theorem 2. In special cases the lower bound in Theorem 2 can be achieved by refinements in the technique outlined above. So we make the following conjecture.

Conjecture. Given a map $f: T^n \to T^n$. Then there exists a smooth map g homotopic to f such that $\#\{x \in T^n \mid x \text{ is a periodic point of } g \text{ of least period } m\} = r_m$ where $r_1 = |L(f)|$ and for $q \geqq 2$

$$r_q = \begin{cases} 0 & \text{if } L(f^q) = 0 \\ |L(f^q)| - \sum\limits_{\substack{m<q \\ m|q}} r_m & \text{if } L(f^q) \neq 0\,. \end{cases}$$

This work was motivated by a question of Shub and Sullivan which appears on page 140 of Hirsch [3]. Shub and Sullivan ask whether every map homotopic to $g: T^2 \to T^2$ must have an infinite number of periodic points where g is the map covered by the linear map $\bar{g}: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ whose matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since the Lefschetz numbers $L(g^n)$ are easily seen to be unbounded, a positive answer follows from the theorem of Brooks, Brown, Pak, and Taylor, Theorem 2. An elementary, transparent proof of a special case of Theorem 2 is presented in Proposition 1.

2. **Preliminaries.** Denote the integers by $\boldsymbol{Z}$, the rationals by

$\boldsymbol{Q}$, and the reals by $\boldsymbol{R}$. For each $a \in \boldsymbol{R}^n$ let $T_a: \boldsymbol{R}^n \to \boldsymbol{R}^n$ denote the translation by a, $T_a b = b + a$ for $b \in \boldsymbol{R}^n$. Set $\mathscr{T} = \{T_a \mid a \in \boldsymbol{Z}^n\}$. Let $\pi: \boldsymbol{R}^n \to T^n$ denote the usual covering map which identifies T^n with $\boldsymbol{R}^n/\mathscr{T}$. Recall that an $n \times n$ matrix A is unimodular provided it has integer entries and $\det A = \pm 1$, or equivalently, it has integer entries and an inverse with integer entries. Clearly, the rows of a matrix A with integer entries form a basis for the module $\boldsymbol{Z}^n$ over $\boldsymbol{Z}$ if and only if A is unimodular.

We will use the form of Nielsen fixed point theorem which states that if $f: X \to X$ is a continuous map of a compact manifold X into itself, then each map g homotopic to f must have at least $N(f)$ fixed points, where $N(f)$ is the Nielsen number of f. Furthermore, $N(g) = N(f)$, (Brown [2]). The Nielsen number $N(f)$ is defined as follows. First an equivalence relation $\sim$ is defined on the set F of fixed points of f. Two fixed points $x, y \in F$ are equivalent, $x \sim y$, provided there is a path γ in X from x to y such that $f \circ \gamma$ is end points fixed homotopic to γ. The set of equivalence classes $F/\sim$ is known to be finite and each equivalence class is compact.

Using a fixed point index I, such as defined in [2] we may assign an index $i(A)$ to each $A \in F/\sim$ by setting $i(A) = I(U)$ for any open set U such that $F \cap U = A$. The Nielsen number $N(f)$ is the number of $A \in F/\sim$ such that $i(A) \neq 0$. If A is a singleton $\{x\}$, then $i\{x\}$ is the usual index of an isolated fixed point of f and consequently if f is differentiable and $1 - df_x$ is nonsingular, then $i\{x\} = \pm 1$ as $\det(1 - df_x)$ is positive or negative.

Let $e^1 = (1, 0, \cdots, 0)$, $e^2 = (0, 1, \cdots, 0)$, etc., denote the standard basis for $\boldsymbol{R}^n$. Set $\beta_i(t) = \pi(te^i)$ and $\alpha_i = [\beta_i] \in \pi_1(T^n, *)$, where $* = \pi(0)$. Then $\alpha_1, \cdots, \alpha_n$ form a basis for $\pi_1(T^n, *)$. Since the Hurewitz homomorphism $\rho: \pi_1(T^n, *) \to H_1(T^n)$ is an isomorphism, we can identify $\pi(T^n, *)$ with $H_1(T^n)$ via ρ and consider $\alpha_1, \cdots, \alpha_n$ as a basis for $H_1(T^n)$, which we shall call the standard basis of $H_1(T^n)$. If $L: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear map, we denote its matrix with respect to the standard basis by $\bar{L}$ and define it by $L(e^i) = \sum_j \bar{L}_{ji} e^j$. Throughout this paper we will consider $\boldsymbol{R}^n$ to be a space of column vectors. Then $\bar{L}$ satisfies $L(v) = \bar{L}v$ for all v in $\boldsymbol{R}^n$.

Consider the case where $\bar{L}_{jk} \in \boldsymbol{Z}$ for all j, i. Then for $a \in \boldsymbol{Z}^n$, $La = b \in \boldsymbol{Z}^n$. Since $L \circ T_a = T_b \circ L$, we see that L induces a map $L': T^n \to T^n$. We say that L covers L'. It is a straightforward verification that the matrix of $H_1(L'): H_1(T^n) \to H_1(T^n)$ with respect to the standard basis is equal to $\bar{L}$. Since T^n is covered by $\boldsymbol{R}^n$, T^n is an Eilenberg-MacLane space, $T^n = K(\boldsymbol{Z}^n, 1)$. Hence the homotopy class of a map $f: T^n \to T^n$ is determined by the homomorphism $H_1(f): H_1(T^n) \to H_1(T^n)$. We sum up these observations in the following lemma.

LEMMA 1. *Each map $f\colon T^n \to T^n$ is homotopic to a map $g\colon T^n \to T^n$ which is covered by a linear map $\bar{g}\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ whose matrix is the same as the matrix of $H_1(f)\colon H_1(T^n) \to H_1(T^n)$.*

LEMMA 2. *If $f\colon T^n \to T^n$ is covered by a linear map $A\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, $f \circ \pi = \pi \circ A$, and 1 is not a characteristic root of A, then the fixed points are isolated, they all have the same index, and the number of them is $|L(f)|$.*

Proof. Let x be a fixed point of f. Using an appropriate restriction of $\pi\colon \boldsymbol{R}^n \to T^n$ for a coordinate system about x, we see that df_x expressed in these coordinates is A. Since $\det(1 - A) = \prod_{i=1}^n (1 - \lambda_i)$, where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of A, we see that $\det(1 - A) \neq 0$. Hence x is an isolated fixed point. Therefore, $i(x) = \pm 1$ as $\det(1 - A)$ is positive or negative, and so $i(x)$ is independent of x. The Lefschetz fixed point formula asserts that the sum of the $i(x)$ as x ranges over the fixed points of f is $L(f)$. Hence the number of fixed points is $|L(f)|$.

PROPOSITION 1. *Given a map $f\colon T^n \to T^n$ such that 1 is not a characteristic root of $H_1(f)\colon T^n \to T^n$. Then $N(f) = |L(f)|$.*

Proof. By Lemma 1 and the homotopy invariance of $N(f)$, we see that we may assume that f is covered by a linear map $A\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ and that 1 is not a characteristic root of A. From Lemma 2 we know that the set F of fixed points of f satisfies $\# F = |L(f)|$, and $i(x) \neq 0$ for each $x \in F$. To prove the present proposition it is sufficient to show that if $x, y \in F$, and $x \neq y$, then x is not Nielsen equivalent to y. For then, each $\{x\}$ with $x \in F$ will be a distinct Nielsen equivalence class and their number, $\# F = |L(f)|$, will be equal to $N(f)$ by the definition of $N(f)$.

Assume $x, y \in F$, $x \neq y$, and $x \sim y$. Then there is a path γ in T^n from x to y such that γ is end points fixed homotopic to $f \circ \gamma$. Let $\tilde{\gamma}\colon I \to \boldsymbol{R}^n$ be a lift of γ, $\pi \circ \tilde{\gamma} = \gamma$, going from $\tilde{\gamma}(0) = \tilde{x}$ to $\tilde{\gamma}(1) = \tilde{y}$. Then $A \circ \tilde{\gamma}$ covers $f \circ \gamma$, since $\pi \circ A = f \circ \pi$. Set $a = \tilde{x} - A(\tilde{\gamma}(0)) = \tilde{x} - A(\tilde{x})$. Since $\pi(A(\tilde{x})) = f(\pi(\tilde{x})) = f(x) = x$ and $\pi(\tilde{x}) = x$ we deduce that $a \in \boldsymbol{Z}^n$. Then $\pi \circ T_a = \pi$ and so $T_a \circ A \circ \tilde{\gamma}$ is also a lift of $f \circ \gamma$, $\pi \circ T_a \circ A \circ \tilde{\gamma} = \pi \circ A \circ \tilde{\gamma} = f \circ \pi \circ \tilde{\gamma} = f \circ \gamma$. Also note that $T_a \circ A \circ \tilde{\gamma}(0) = A(\tilde{\gamma}(0)) + a = \tilde{x} = \tilde{\gamma}(0)$. Since γ is end point fixed homotopic to $f \circ \gamma$, we have $T_a \circ A \circ \tilde{\gamma}(1) = \tilde{\gamma}(1)$. Therefore $\tilde{\gamma}(1) - \tilde{\gamma}(0) = T_a A \tilde{\gamma}(1) - T_a A \tilde{\gamma}(0) = (A\tilde{\gamma}(1) + a) - (A\tilde{\gamma}(0) + a) = A(\tilde{\gamma}(1) - \tilde{\gamma}(0))$. From $x \neq y$ and $\pi\tilde{\gamma}(0) = x$ and $\pi(\tilde{\gamma})(1) = y$, we conclude that $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. Hence $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an eigenvector of A with eigenvalue 1, a contradiction.

2. Algebraic lemmas.

LEMMA 3. *Given* $v = (v_1, \cdots, v_n) \in Z^n$, $n \geqq 1$, *such that* g.c.d. $(v_1, \cdots, v_n) = 1$, *where* g.c.d. *stands for greatest common divisor. Then there exist* $v^2, v^3, \cdots, v^n \in Z^n$ *such that* $v, v^2, \cdots, v^n$ *form a basis for* Z^n.

Proof. We use induction on n. For $n = 1$ we must show that $\{v\}$ is a basis for Z^1, i.e., that $v = \pm 1$. But this follows from the fact that g.c.d. $(v) = 1$.

Now suppose $n > 1$ and that the lemma holds for $n - 1$. If $v_1 = v_2 = \cdots = v_{n-1} = 0$, then $v_n = \pm 1$ and the lemma obviously holds. So suppose that not all $v_1, v_2, \cdots, v_{n-1}$ are 0. Let $d=$g.c.d.$(v_1, \cdots, v_{n-1})$. Then g.c.d.$(d, v_n) = 1$ and so we may find $\alpha, \beta \in Z$ such that $\alpha v_n + \beta d = 1$. Apply the induction hypotheses to the vector $w = (v_1/d, \cdots, v_{n-1}/d) \in Z^{n-1}$ and obtain vectors $w^2, \cdots, w^{n-1} \in Z^{n-1}$ such that $w, w^2, \cdots, w^{n-1}$ form a basis for Z^{n-1}. Thus the matrix A with rows $w, w^2, \cdots, w^{n-1}$ is unimodular, and so $\det A = \pm 1$. Let B be the matrix with rows $dw, w^2, w^3, \cdots, w^{n-1}$. Then $\det B = d(\det A)$ and the first row of B is $dw = (v_1, v_2, \cdots, v_{n-1})$. Set $w^i = (w^i_1, \cdots, w^i_{n-1})$ for $2 \leqq i \leqq n - 1$. Form the matrix C indicated below.

$$C = \begin{bmatrix} v_1 & \cdots & v_{n-1} & v_n \\ w^2_1 & \cdots & w^2_{n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ w^{n-1}_1 & \cdots & w^{n-1}_{n-1} & 0 \\ \dfrac{i\alpha v_1}{d} & \cdots & \dfrac{i\alpha v_{n-1}}{d} & (\det A)\beta \end{bmatrix}$$

where $i = -\det A$. Then expanding $\det C$ on the last column we find $\det C = \alpha v_n + \beta d = 1$. Thus the last $n - 1$ rows satisfy the lemma.

LEMMA 4. *If* $v^1, v^2, \cdots, v^r \in Z^n$, $n \geqq 1$, *then there is a unimodular matrix* A *with rows* $A^1, \cdots, A^n$ *such that* $\mathrm{sp}\{A^1, \cdots, A^q\} = \mathrm{sp}\{v^1, \cdots, v^r\}$, *where* $\mathrm{sp}\, V =$ *the linear span in* R^n *for* $V \subset R^n$, *and* $q = \dim \mathrm{sp}\{v^1, \cdots, v^r\}$.

Proof. We may suppose $v^1, \cdots, v^r$ are linearly independent in R^n. We will use induction on r. For $r = 1$, Lemma 3 with $v = d^{-1}v^1$ where $v^1 = (v^1_1, \cdots, v^1_n)$ and $d =$ g.c.d. $(v^1_1, \cdots, v^1_n)$ gives the desired conclusion.

Suppose now that $r \geqq 2$ and that Lemma 4 holds for $r - 1$. Apply this supposition to $v^1, \cdots, v^{r-1}$ and obtain a unimodular matrix

B such that its rows $B^1, \cdots, B^n$ satisfy $\operatorname{sp}\{B^1, \cdots, B^{r-1}\} = \operatorname{sp}\{v^1, \cdots, v^{r-1}\}$. Note that $\operatorname{sp}\{B^1, \cdots, B^{r-1}, v^r\} = \operatorname{sp}\{v^1, \cdots, v^{r-1}, v^r\}$.

By considerring the linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ whose matrix is B it is easily seen that it is sufficient to prove the lemma in the special case where $B^i = e^i$. Now let $v^r = (\alpha_1, \cdots, \alpha_n)$, and set

$$w = (0, \cdots, 0, \alpha_r, \alpha_{r+1}, \cdots, \alpha_n) = v^r - (\alpha_1, \cdots, \alpha_{r-1}, 0, \cdots, 0) .$$

Then $\operatorname{sp}\{B^1, \cdots, B^{r-1}, v^r\} = \operatorname{sp}\{B^1, \cdots, B^{r-1}, w\}$. Since v^r is independent of $B^1, \cdots, B^{r-1}$, not all of $\alpha_r, \alpha_{r+1}, \cdots, \alpha_n$ can vanish. Set $d =$ g.c.d. $\{\alpha_r, \alpha_{r+1}, \cdots, \alpha_n\}$, and $u = d^{-1}w$. Then $u \in \{0\} \times \boldsymbol{Z}^{n-r+1}$ and the greatest common divisor of its coordinates is 1. Hence by Lemma 3 there is an $(n - r + 1) \times (n - r + 1)$ unimodular matrix C such that the first row is $\alpha_r/d, \cdots, \alpha_n/d$. Set

$$A = \left[\begin{array}{c|c} I^{(r-1)\times(r-1)} & 0 \\ \hline 0 & C \end{array}\right] .$$

Then A is unimodular, and its first r rows are $B^1, \cdots, B^{r-1}, u$ which have the same span in $\boldsymbol{R}^n$ as does $v^1, \cdots, v^r$.

Suppose $f: T^n \to T^n$, $n \geqq 1$, is a map and A is the matrix of $H_1(f): H_1(T^n) \to H_1(T^n)$. It is shown in [1] that $L(f) = \det(1 - A^t) = \det(1 - A)$, where $A^t =$ the transpose of $A =$ the matrix of $H^1(f)$: $H^1(T^n) \to H^1(T^n)$. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic roots of A. Then $L(f) = \prod_{i=1}^n (1 - \lambda_i)$. Since $\lambda_1^m, \cdots, \lambda_n^m$ are the characteristic roots of A^m we have proved the following formula.

$$(*) \qquad L(f^m) = \prod_{i=1}^{n} (1 - \lambda_i^m) .$$

LEMMA 5. *Let $\lambda_1, \cdots, \lambda_n$ be complex numbers, none of them 1, such that the set $\{\prod_{i=1}^n |1 - \lambda_i^m| \mid m = 1, 2, \cdots\}$ is bounded, then $|\lambda_i| \leqq 1$ for all $i = 1, \cdots, n$.*

Proof. Suppose not. Divide $\{1, \cdots, n\}$ into four sets I, J, K, and L by setting $I = \{i \mid |\lambda_i| > 1\}$, $J = \{i \mid |\lambda_i| < 1\}$, $K = \{i \mid \lambda_i$ is a root of unity$\}$, and $L = \{i \mid |\lambda_i| = 1$ and $\lambda_i^q \neq 1$ for all $q \geqq 1\}$. For any $U \subset \{1, \cdots, n\}$, set $U_m = \prod_{i \in U} |1 - \lambda_i^m|$, with the convention that if $U = \varnothing$, then $U_m = 1$ for all m. Formula $(*)$ gives $|L(f^m)| = I_m J_m K_m L_m$. Since $I \neq \varnothing$ we clearly have

$$(1) \qquad I_m \longrightarrow \infty .$$

Also

$$(2) \qquad J_m \longrightarrow 1 .$$

For each $i \in K$, let $q_i \geqq 1$ be such that $\lambda_i^{q_i} = 1$ and $\lambda_i^m \neq 1$ for $1 \leqq m < q_i$. If $K \neq \emptyset$, set $q = \prod_{i \in K} q_i$. If $K = \emptyset$, set $q = 1$. In either case, for $p \geqq 0$,

$$(3) \qquad K_{qp+1} = K_1 > 0 .$$

Here we have used the hypothesis that $\lambda_i \neq 1$ for all i.

Set $N = \# L$. From the definition of L we see that $1 \notin \{\lambda_i^{qr} \mid i \in L, r = 1, \cdots, N\}$, and hence we can find an $\varepsilon > 0$ such that $|1 - \lambda_i^{qr}| > 2\varepsilon$ for $i \in L$ and $1 \leqq r \leqq N$.

Claim. For each $i \in L$, and each positive integer a, at most one member of the sequence λ_i^{qm+1}, where $a(N+1) < m \leqq (a+1)(N+1)$, satisfies

$$|1 - \lambda_i^{qm+1}| \leqq \varepsilon .$$

Proof. Suppose not. Then there is an $i \in L$, m and r such that $1 \leqq r \leqq N$, and $|1 - \lambda_i^{qm+1}| \leqq \varepsilon$, and $|1 - \lambda_i^{q(m+r)+1}| \leqq \varepsilon$. It follows that

$$\begin{aligned} 2\varepsilon \geqq |\lambda_i^{q(m+r)+1} - \lambda_i^{qm+1}| &= |\lambda_i^{qm+1}|\,|\lambda_i^{qr} - 1| \\ &= |\lambda_i^{qr} - 1| > 2\varepsilon , \end{aligned}$$

a contradiction. This proves the claim.

Since the number of m's which satisfy $a(N+1) < m \leqq (a+1)(N+1)$ is $N+1$ and $N = \# L$, for each a there is an m such that $a(N+1) < m \leqq (a+1)(N+1)$, and $|1 - \lambda_i^{qm+1}| > \varepsilon$ for all $i \in L$. Hence $L_{mq+1} \geqq \varepsilon^N$ for an infinite number of m's. Note that this also holds when $N = 0$. Combining this with (1), (2), and (3) we see that $|L(f^m)|$ is unbounded, a contradiction. Hence $|\lambda_i| \leqq 1$ for all i.

LEMMA 6. *Given a map $f: T^n \to T^n$, $n \geqq 1$, such that 1 is not a characteristic root of $H_1(f): H_1(T^n) \to H_1(T^n)$, and $L(f^m)$, $m = 1, 2, \cdots$, are bounded. Then each nonzero characteristic root of $H_1(f)$ is a root of unity.*

Proof. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic roots of $H_1(f)$. By Lemma 5 we know that $|\lambda_i| \leqq 1$ for all i.

Next we will show that for each i, $|\lambda_i| = 0$ or 1. Suppose not. Let $U = \{i \mid \lambda_i \neq 0\}$, and $q = \# U$. Let

$$P(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

be the characteristic polynomial of $H_1(f)$. Then $a_q = \prod_{i \in U} (-\lambda_i) \neq 0$.

Since a_q is an integer, we have

$$1 \leqq |a_q| = \prod_{i \in U} |\lambda_i| < 1$$

because $|\lambda_i| \leqq 1$ for all i, and $0 < |\lambda_i| < 1$ for some i. This contradiction shows that for each i, $|\lambda_i| = 0$ or 1.

Note that $a_m = 0$ for $m > q$, and so

$$P(\lambda) = \lambda^{n-q}(\lambda^q + a_1\lambda^{q-1} + \cdots + a_q) = \lambda^{n-q}Q(\lambda) .$$

All the roots of $Q(\lambda)$ have unit modulus. It is known, [4] page 122, that if all the roots of a monic polynomial with integer coefficients have unit modulus, then they all are roots of unity. This completes the proof.

4. Geometric lemmas.

LEMMA 7. *Suppose the finite group G acts smoothly on a compact manifold M, and that $P = \{x \in M \mid gx = x \text{ for some } g \in G, g \neq 1\}$ is finite. Then there exists a Morse function $\varphi: M \to \boldsymbol{R}$ such that $\varphi \circ g = \varphi$ for all $g \in G$. Furthermore, each $x \in P$ is a critical point of φ.*

Proof. Following Milnor [5], we will say that a smooth map $f: M \to \boldsymbol{R}$ is "good" on a set $S \subset M$ if f has no degenerate critical points on S.

We begin by obtaining a first approximation, a smooth map $\Psi: M \to \boldsymbol{R}$, which is invariant (i.e., $\Psi \circ g = \Psi$ for all $g \in G$) and is good on a neighborhood V of P. Then we perturb Ψ equivariently to the desired Morse function φ.

It is easy to define a smooth map $h: M \to \boldsymbol{R}$ such that each $x \in P$ is a nondegenerate critical point of index zero, i.e., in a local coordinate system about x the first partial derivatives vanish and the matrix of second partial derivatives is positive definite at x. These same conditions also hold for each $h \circ g$, $g \in G$, and consequently $\Psi = \sum_{g \in G} h \circ g$ is a first approximation as desired. Clearly, the set V where Ψ is good, is open, contains P, and satisfies $g(V) = V$ for all $g \in G$.

Note that $gx \neq x$ for all $x \in M - V$ and $g \in G$, $g \neq 1$. Now a rather straightforward equivarient version of the argument used in Milnor [5], Theorem 2.7, to prove the existence of Morse functions serves to show that Ψ can be perturbed to an equivarient Morse function φ. A sketch of this equivarient version follows.

We may find coordinate neighborhoods $U_1, \cdots, U_r$ such that

$$M - V \subset \bigcup_{i=1}^{r} U_i , \qquad P \cap \mathrm{cl}\Big(\bigcup_{i=1}^{r} U_i\Big) = \varnothing , \quad \text{and} \quad U_i \cap g(U_i) = \varnothing$$

for $1 \leqq i \leqq r$ and $g \in G$, $g \neq 1$. Then $g'(U_i) \cap g(U_i) = \emptyset$ for $1 \leqq i \leqq r$ and $g, g' \in G$, $g \neq g'$. We can also find compact sets $C_i \subset U_i$ such that $C_1, \cdots, C_r$ cover $M - V$. We may alter Ψ in stages so that at the ith stage, the new Ψ is still equivarient and is good on

$$V \cup \left(\bigcup_{j=1}^{i} \bigcup_{g \in G} g(C_j) \right).$$

At the ith step we simply apply the procedure used in the proof of Theorem 2.7 of [5] to U_i and then alter Ψ on $g(U_i)$, for $g \in G$, $g \neq 1$, so as to preserve the property $\Psi \circ g = g$ for all $g \in G$.

LEMMA 8. *Suppose the finite group G acts smoothly on a compact connected m-dimensional manifold M, $m \geqq 2$, and that $P = \{x \in M \mid gx = x$ for some $g \in G$, $g \neq 1\}$ is finite. Given a finite set $S \subset M - P$ and a point $x_0 \in M$ such that $gx_0 = x_0$ for all $g \in G$. Then there exists a smooth embedding $\Psi: D^m \to (M - P) \cup \{x_0\}$ such that $\Psi(0) = x_0$, $S \subset \Psi(\operatorname{int} D^m)$, and $g(\Psi(D^m)) = \Psi(D^m)$ for all $g \in G$. Furthermore, for each $g \in G$, $\Psi^{-1} \circ g \circ \Psi: D^m \to D^m$ is the restriction to D^m of an orthogonal linear map.*

Proof. We will use induction on $^{\#}S$. First suppose $S = \emptyset$. It is an easy matter to embed M into $\boldsymbol{R}^n$ for some n such that for each $g \in G$ the map $x \to gx$, $x \in M$, is the restriction to M of an orthogonal map $L_g: \boldsymbol{R}^n \to \boldsymbol{R}^n$. Just start with a smooth embedding $h: M \to \boldsymbol{R}^k$, for some k. Set $E_g = \boldsymbol{R}^k$ for each $g \in G$, and define $e: M \to \prod_{g \in G} E_g = \boldsymbol{R}^{k^{\#}G}$ by $e(x) = \prod_{g \in G} h(gx)$, for all $x \in M$. Then the maps $e(x) \to e(gx)$, $x \in M$ are restrictions of maps $L_g: \boldsymbol{R}^{k^{\#}G} \to \boldsymbol{R}^{k^{\#}G}$ which simply permute the coordinates of $\boldsymbol{R}^{k^{\#}G}$.

Set $n = k^{\#}G$, and identify M with $e(M)$ via e. Let TM_x be the tangent space of M at x, considered as a subspace of $\boldsymbol{R}^n$. Let $T = x_0 + TM_{x_0}$ be the geometric tangent space through the point x_0. Let $N: \boldsymbol{R}^n \to T$ be the orthogonal projection onto T. Then $L_g(T) \subset T$ for all $g \in G$, and so $L_g \circ N = N \circ L_g$ for all $g \in G$. The restriction of N to a neighborhood of x_0 in M is a diffeomorphism onto a neighborhood W of x_0 in T. The desired Ψ is now easily constructed from the restriction of $(N \mid W)^{-1}$ to a ball about x_0.

Now assume that $\Psi: D^m \to M$ satisfies Lemma 8 as stated. We will show that for any $x \in M - (P \cup S)$, Ψ can be altered so as to satisfy Lemma 8 with S replaced by $S \cup \{x\}$. We will use a connectedness argument. Since $\dim M \geqq 2$ and $P \cup S$ is finite, the space $M - P \cup S$ is connected. Let $V = \{x \in M - P \cup S \mid$ Lemma 8 holds with S replaced by $S \cup \{x\}\}$.

The set V is clearly open in $M - (P \cup S)$. We will show that $(M - (P \cup S)) - V$ is also open. Let $x \in (M - (P \cup S)) - V$. If $gx \in S$

for some $g \in G$, then Ψ already satisfies $S \cup \{x\} \subset \Psi(\operatorname{int} D^m)$. Hence $gx \notin S$ for all $g \in G$. Since $gx \neq x$ for all $g \in G$, $g \neq 1$, we may find a coordinate neighborhood U of x which is diffeomorphic to an open m-ball and such that $g(U) \subset M - (P \cup S)$, and $g(U) \cap U = \emptyset$ for all $g \in G$, $g \neq 1$. Then $g(U) \cap h(U) = \emptyset$ for $g, h \in G$, $g \neq h$. We claim that $U \subset (M - (P \cup S)) - V$. Suppose not. Then we can find a $y \in U \cap V$. Let $k: M \to M$ be a diffeomorphism which is fixed outside of U and $k(y) = x$. Define $\bar{k}: M \to M$ by

$$\bar{k}(p) = \begin{cases} p & \text{if } p \notin \bigcup_{g \in G} g(U) \\ gkg^{-1}(p) & \text{if } p \in g(U) \, . \end{cases}$$

Then $\bar{k}$ is a well defined diffeomorphism, and Lemma 8 is satisfied with Ψ replaced by $\bar{k} \circ \Psi$ and S replaced by $S \cup \{x\}$. In fact, for each $g \in G$, $\bar{k} \circ g = g \circ \bar{k}$ and so $(\bar{k} \circ \Psi)^{-1} \circ g \circ (\bar{k} \circ \Psi) = \Psi^{-1} \circ g \circ \Psi$. This proves the claim, and hence $(M - (P \cup S)) - V$ is open.

Since $V \neq \emptyset$, we have $V = M - (P \cup S)$, and the induction step is complete.

5. Proofs of the theorems.

Proof of Theorem 4. First recall that a flow on M is a smooth map $F: M \times \boldsymbol{R} \to M$ such that with the notation $F_t(x) = F(x, t)$ we have $F_s \circ F_t = F_{s+t}$ for all $s, t \in \boldsymbol{R}$, and $F_0 = 1_M$. An orbit of F is a function of the form $F^x: \boldsymbol{R} \to M$ where $x \in M$ and $F^x(t) = F(x, t)$ for all $t \in \boldsymbol{R}$. An orbit F^x is periodic if $F^x(s) = F^x(0)$ for some $s \neq 0$. Thus the constant orbits are considered to be periodic.

The idea of the proof is to obtain a flow $H_t: M \to M$ such that the orbits H^x for $x \in P$ are constants and these are the only periodic orbits, and $H_t \circ f = f \circ H_t$. Then $g = H_1 \circ f$ will satisfy the conclusions of the theorem. The desired flow H_t is obtained in several steps.

Since $f^N = 1$, we have a smooth action of $\boldsymbol{Z}/n\boldsymbol{Z}$ on M. Let $\varphi: M \to \boldsymbol{R}$ be the Morse function given by Lemma 7. It is easy to obtain an equivarient Riemannian metric on M. Just average over $\boldsymbol{Z}/n\boldsymbol{Z}$ any Riemannian metric. Then the gradient of φ with respect to this equivarient Riemannian metric is an equivarient vector field v. The vector field v determines a flow F on M which satisfies $F_t(f(x)) = f(F_t(x))$ for all $(x, t) \in M \times \boldsymbol{R}$. The flow also satisfies $\varphi(F_t(x)) > \varphi(x)$ whenever x is not a critical point of φ and $t > 0$. Hence the only periodic orbits of F are the constant orbits at critical points of φ.

Let $S = \{x \in M \mid x \text{ is a critical point of } \varphi\} - P$. Let $\Psi: D^m \to M$ be given by Lemma 8. Pick $r \in (0, 1)$ such that $S \subset \Psi(D^m_r)$, where $D^m_r = \{x \in \boldsymbol{R}^m \mid \|x\| \leqq r\}$. Let $b: \boldsymbol{R} \to \boldsymbol{R}$ be a smooth map satisfying

$b(t) = 0$ for $t \leqq r$, $b(t) > 0$ for $t > r$, and $b(t) = 1$ for $t \geqq (1 + r)/2$. Define $\bar{b}: M \to \boldsymbol{R}$ by

$$\bar{b}(x) = \begin{cases} 1 & \text{if } x \notin \Psi(D^m_{(r+1)/2}) \\ b(\| \Psi^{-1}(x) \|) & \text{if } x \in \Psi(D^m) . \end{cases}$$

It is clear that $\bar{b}$ is well defined and smooth. Define a new vector field w by $w(x) = \bar{b}(x)v(x)$. Then w is equivarient under f and determines an equivarient flow G_t. The orbits of G, which are just the integral curves of w, are reparameterizations of portions of orbits of F. The orbits G^x for $x \in \Psi(D^m_r) \cup P$ are constant. All the other orbits are reparameterizations of portions of nonperiodic orbits of F by reparameterization functions which are strictly monotone increasing functions. Hence the orbits G^x for $x \in \Psi(D^m_r) \cup P$ are the only periodic orbits. Let $\theta: \boldsymbol{R} \to \boldsymbol{R}$ be a smooth map satisfying $\theta(t) = r + t$ for $t \leqq 1/3\,(1 - r)$, $\theta'(t) > 0$ for all t, and $\theta(t) = t$ for $t \geqq r + 2/3\,(1 - r)$. We will use later the obvious fact that $\theta^{-1}: \boldsymbol{R} \to \boldsymbol{R}$ exists and is smooth. Define $h: M - \{x_0\} \to M - \Psi(D^m_r)$ by

$$h(x) = \begin{cases} x & \text{if } x \notin \Psi(D^m) \\ \dfrac{\theta(\| x \|)x}{\| x \|} & \text{if } x \in \Psi(D^m) , \end{cases}$$

where we have identified $\Psi(D^m)$ with D^m via Ψ. Define $H_t(x)$ by

$$H_t(x) = \begin{cases} h^{-1}(G_t(h(x))) & \text{for } x \neq x_0 \\ x_0 & \text{for } x = x_0 . \end{cases}$$

We wish to show that $H_t(x)$ is a smooth flow by showing that $H_t(x)$ is determined by a smooth vector field. Since $H_s \circ H_t = H_{s+t}$, and $H_0 = 1_M$, it is sufficient to show that $\eta(x) = d/dt\, H_t(x)$ at $t = 0$ is a smooth vector field. It is clear that $\eta(x)$ is well defined for all $x \in M$ and $\bar{\eta} = \eta \mid M - \{x_0\}$ is smooth. Since $\eta(x_0) = 0$, it is sufficient to show that $\eta(x)$ and all its derivatives approach 0 as $x \to x_0$. We calculate for $x \neq x_0$

$$\begin{aligned} \eta(x) &= \frac{d}{dt}(H_t(x)) \mid_{t=0} = dh^{-1} \mid_{G_0(h(x))} \frac{d}{dt}(G_t(h(x))) \mid_{t=0} \\ &= dh^{-1} \mid_{h(x)} w(h(x)) . \end{aligned}$$

Since w and all its derivatives vanish on D^m_r, Taylor expansions show that for each derivative $a(x)$ of a component of $w(x)$ and each $n \geqq 1$ there is a constant c such that

$$(5) \qquad | a(x) | \leqq c \, | \| x \| - r |^n \qquad \text{for } x \in \Psi(D^m) .$$

The form of $h(x)$ for $x \in \Psi(D^m) - \{x_0\}$ and $\|x\| \leqq (1-r)/3$, is

$$h(x) = \frac{(r + \|x\|)x}{\|x\|}$$

and hence if $u(x)$ is a derivative of a component of h of order n, then there is a constant e such that

$$(6) \qquad |u(x)| \leqq e\|x\|^{-n-1} \qquad \text{for } 0 < \|x\| \leqq \frac{1-r}{3} .$$

The map $h^{-1} | (\Psi(D^m) - \Psi(D_r^m))$ has a smooth extention $\bar{h}^{-1}: \varphi(D^m) - \{x_0\} \to \varphi(D^m)$ given by $\bar{h}^{-1}(y) = \theta^{-1}(\|y\|)y/\|y\|$. Consequently $dh^{-1}|_y$ and all its derivatives are bounded. Using this and (5) and (6) we see that $\eta(x) = dh^{-1}|_{h(x)}\, w(h(x))$ and all its derivatives approach 0 as $x \to x_0$. Hence $\eta(x)$ is smooth and therefore so is $H_t(x)$.

It is clear from the definition of $H_t(x)$ and the properties of $G_t(x)$ that the only periodic orbits of H are the constant orbits H^x for $x \in P$. It is also clear from the fact that $x \to \Psi^{-1}(f(\Psi(x)))$ is the restriction of a orthogonal linear map to D^m, that $f \circ h = h \circ f$ and hence $f \circ H_t = H_t \circ f$ for all t. The map $(x, t) \to H_t(f(x))$ is a homotopy from $H_0 \circ f = f$ to the smooth map $g = H_1 \circ f$. It is easy to see that the set of periodic points of $H_1 \circ f$ is P and that $H_1 \circ f \mid P = f \mid P$. This completes the proof.

LEMMA 9. *Suppose there is given a map $g: T^k \to T^k$, $k \geqq 2$, which is covered by a linear map $A: \boldsymbol{R}^k \to \boldsymbol{R}^k$, and an integer $N \geqq 2$, such that $g^N = 1_{T^k}$, and $\lambda^m \neq 1$ for λ a characteristic root of A and $1 \leqq m < N$. Then there exists a smooth map $\bar{g}$ homotopic to g such that $P = \{x \in T^k \mid g^m(x) = x$ for some m, $1 \leqq m < N\}$ = the set of all periodic points of $\bar{g}$, $\bar{g} \mid P = g \mid P$, and for $m \geqq 1$ $\#\{x \in T^k \mid \bar{g}^m(x) = x\} \leqq a_m(g)$.*

Proof. We wish to apply Theorem 4. If follows from Lemma 2 that $P = \{x \in T^k \mid g^r(x) = x$ for some r, $1 \leqq r < N\}$ is finite. Since $N \geqq 2$, $\lambda_i^1 \neq 1$ for all characteristic roots λ_i and hence

$$L(g) = \prod_{i=1}^{n} (1 - \lambda_i) \neq 0$$

by formula $(*)$. Hence, we can find an $x_0 \in T^k$ such that $g(x_0) = x_0$. Therefore Theorem 4 gives a smooth map $\bar{g}$ homotopic to g such that $P =$ the set of periodic points of $\bar{g}$, and $\bar{g}|P = g|P$. It follows from Lemma 2 applied to g, and formula $(*)$, that for $1 \leqq m < N$,

$$\#\{x \in T^k \mid \bar{g}^m(x) = x\} = \#\{x \in T^k \mid g^m(x) = x\} = |L(g^m)| = a_m(g) .$$

Then

$$\begin{aligned}
\#\{x \in T^k \mid \bar{g}^N(x) = x\} \\
&= \sum_{\substack{m<N \\ m|N}} \#\{x \in T^k \mid x \text{ is a periodic point of } \bar{g} \text{ of period } m\} \\
&\leqq \sum_{m|N} |L(g^m)| \\
&= \sum_{m|N} \left| \prod_{i=1}^{k} (1 - \lambda_i^m) \right| \\
&\leqq \prod_{i=1}^{k} \sum_{m|N} |1 - \lambda_i^m| \\
&\leqq \prod_{i=1}^{k} a_{iN} \\
&= a_N(g) .
\end{aligned}$$

To complete the proof it is sufficient to show that if $m \geqq 1$, $q \geqq 1$, and $m \equiv q \bmod N$, then

$$\{x \in T^k \mid \bar{g}^m(x) = x\} = \{x \in T^k \mid \bar{g}^q(x) = x\} .$$

We may assume $m < q$ and so $q = m + pN$ for some $p \geqq 1$. Assume $\bar{g}^m(x) = x$. Then $x \in P$ and so $\bar{g}^N(x) = g^N(x) = x$. Consequently $\bar{g}^q(x) = \bar{g}^{m+pN}(x) = \bar{g}^m(x) = x$. The reverse implication, "$\bar{g}^q(x) = x$ implies $\bar{g}^m(x) = x$" follows similarly. This completes the proof.

Remark. In our application of Lemma 9 in the proof of Theorem 3, Lemma 9 needs to be augmented by the following observation. Lemma 9 also holds when $k = 1$ and $N \geqq 1$. We verify this as follows. It is easy to deduce that $A = 1_{R^1}$ or $A = -1_{R^1}$. In the first case, $A = 1_{R^1}$, we can homotopy $g = 1_{T^1}$ to a rotation $\bar{g}$ of the circle $S^1 = T^1$ by an angle which has an irrational ratio to 2π. Such a $\bar{g}$ has no periodic points and Lemma 9 is verified in this case.

In the second case, $A = -1_{R^1}$, g is a reflection and there are exactly two fixed points x_0 and x_1. It is easy to homotopy g to a map $\bar{g}$ which leaves x_0 and x_1 fixed, and moves all other points away from x_0 and closer to x_1. Then x_0 and x_1 will be the only periodic points of $\bar{g}$. It is easy to calculate that $a_m(g) = 2$ for all $m \geqq 1$, and so Lemma 9 holds in this case also.

Proof of Theorem 3. We prove Theorem 3 by induction on n. By convention $\boldsymbol{R}^0$ and T^0 are singletons. Hence the case $n = 0$ holds trivially. Now assume that $n > 0$ and the theorem holds for all $m < n$. Let $f: T^n \to T^n$ be a map.

By Lemma 1 we may assume that f is covered by a linear map $F: \boldsymbol{R}^n \to \boldsymbol{R}^n$. By Lemma 5 we have

$$L(f^m) = \prod_{j=1}^{n} (1 - \lambda_i^m)$$

where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of F.

First consider the case where $\lambda_i^m \neq 1$ for all $m \geqq 1$ and $1 \leqq i \leqq n$. Then the theorem follows from Lemmas 2 and 5.

Consider now the remaining case where $\lambda_i^m = 1$ for some $m \geqq 1$ and $1 \leqq i \leqq n$. Let N be the smallest such m. Let $\bar{F}^t$ denote the transpose of $\bar{F}$. Since $(\bar{F}^t)^N$ has integer entries, we may find a $w \in \boldsymbol{Z}^n$ such that $w \neq 0$ and $(\bar{F}^t)^N w = w$. Set

$$W = \mathrm{sp}_{\boldsymbol{R}^n}\{(\bar{F}^t)^m w \mid 0 \leqq m < N\} .$$

Then $\dim W \geqq 1$, $\bar{F}^t u \in W$ for all $u \in W$, and $(\bar{F}^t)^N x = x$ for all $x \in W$. Set $k = \dim W$. By Lemma 4, we can find a basis $w^1, w^2, \cdots, w^n$ for $\boldsymbol{Z}^n$ such that $w^1, w^2, \cdots, w^k$ form a basis for W. Let $K: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the linear transformation whose matrix satisfies $\bar{K}^t e^i = w^i$. Then, both $\bar{K}$ and $\bar{K}^{-1}$ have integer entries. Thus, both K and K^{-1} induce maps $K'\colon T^n \to T^n$, and $K^{-1\prime} = K'^{-1}\colon T^n \to T^n$. We "change coordinates" by noting that it is sufficient to prove the theorem for

$$g = K' \circ f \circ K'^{-1}$$

in place of f. The map g is covered by

$$M = K \circ F \circ K^{-1}\colon \boldsymbol{R}^n \longrightarrow \boldsymbol{R}^n .$$

From $\bar{K}^t e^i = w^i$ and $\bar{F}^t u \in W$ for $u \in W$, it follows that the matrix $\bar{M}$ of M has the form

$$\bar{M} = \left(\begin{array}{c|c} \bar{A} & 0 \\ \hline \bar{C} & \bar{B} \end{array}\right)$$

where $\bar{A}, \bar{B}, \bar{C}$, and 0 are $k \times k$, $(n-k) \times (n-k)$, $(n-k) \times k$ and $n \times (n-k)$ matrices, and all entries of 0 are zero. It follows from $\bar{F}^{t^N} x = x$ for all $x \in W$, that $\bar{A}^N = 1$. Since $\bar{M} = \bar{K} \cdot \bar{F} \cdot \bar{K}^{-1}$ is similar to $\bar{F}$, $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of $\bar{M}$. Hence, we may renumber the λ_i's so that $\lambda_1, \cdots, \lambda_k$ and $\lambda_{k+1}, \cdots, \lambda_n$ are the characteristic roots of $\bar{A}$ and $\bar{B}$ respectively. Let

$$B\colon \boldsymbol{R}^{n-k} \longrightarrow \boldsymbol{R}^{n-k}$$

be the linear map whose matrix is $\bar{B}$. Then B induces a map $b\colon T^{n-k} \to T^{n-k}$. Since $k = \dim W \geqq 1$, we may apply our induction hypothesis to b and find a smooth map $\bar{b}$ homotopic to b such that

$$\# \{x \in T^{n-k} \mid \bar{b}^m(x) = x\} \leqq a_m(b) = \prod_{i=k+1}^{n} a_{im} .$$

Let $A: \boldsymbol{R}^k \to \boldsymbol{R}^k$ be the linear map whose matrix is $\bar{A}$. Let $a: T^k \to T^k$ be the map induced by A. Since $\bar{A}^N = 1$, we have $a^N = 1_{T^k}$. Because we choose N so that $\lambda_i^m \neq 1$ for $1 \leqq m < N$ and all i, and $N = 1$ implies $k = 1$, we see that Lemma 9 or the remark which follows it applies. Hence we can find a smooth map $\bar{a}$ homotopic to a such that $P = \{x \in T^k \mid a^m(x) = x$ for some m, $1 \leqq m < N\}$ = the set of all periodic points of $\bar{a}$, $\bar{a} \mid P = a \mid P$, and for $m \geqq 1$,

$$\#\{x \in T^k \mid \bar{a}^m(x) = x\} \leqq a_m(a) = \prod_{i=1}^{k} a_{im} .$$

If we write $\boldsymbol{R}^n = \boldsymbol{R}^k \times \boldsymbol{R}^{n-k}$, then M has the form

$$M(x, y) = (\bar{A}x, \bar{C}x + \bar{B}y) .$$

Consequently, if we write $T^n = T^k \times T^{n-k}$, then $g(u, v) = (a(u), r(u, v))$ where $r: T^k \times T^{n-k} \to T^k$ is the map induced by the map $R: \boldsymbol{R}^k \times \boldsymbol{R}^{n-k} \to \boldsymbol{R}^k$ which is given by $R(x, y) = \bar{C}x + \bar{B}y$. The homotopy from a to $\bar{a}$ gives rise to a homotopy from g to $\bar{g}$ where

$$\bar{g}(u, v) = (\bar{a}(u), r(u, v)) .$$

The periodic points of $\bar{g}$ must have the form

$$(u, v) \in T^k \times T^{n-k} \qquad \text{where } u \in P .$$

Partition P into orbits under $\bar{a}$. Let

$$X = \{u_i = \bar{a}^i(u_0) \mid i = 0, 1, \cdots, m-1\}$$

be one such orbit consising of m distinct point, where $1 \leqq m < N$ and $\bar{a}^m(u_0) = u_0$. Consider the maps

$$g_i = \bar{g} \mid u_i \times T^{n-k}: u_i \times T^{n-k} \longrightarrow u_{i+1} \times T^{n-k} ,$$

which are covered by the maps

$$M_i = M \mid x_i \times \boldsymbol{R}^{n-k}: x_i \times \boldsymbol{R}^{n-k} \longrightarrow x_{i+1} \times \boldsymbol{R}^{n-k}$$

where $u_m = \bar{a}^m(u_0) = u_0$ and x_0 is chosen so that $\pi(x_0) = u_0$, and $x_i = \bar{A}^i x_0$ for $i \geqq 1$, (recall that $\bar{a} \mid P = a \mid P$). Making the obvious identifications of $u_i \times T^{n-k}$ with T^{n-k}, and $x_i \times \boldsymbol{R}^{n-k}$ with $\boldsymbol{R}^{n-k}$ we see that

$$M_i(y) = \bar{C}x_i + \bar{B}y$$

for all $y \in \boldsymbol{R}^{n-k}$. Define

$$M_{it}(y) = t\bar{C}x_i + \bar{B}y .$$

Then $M_{i1} = M_i$ and M_{i0} has $\bar{B}$ as its matrix. Because $\bar{B}$ has integer entries and $t\bar{C}x_i$ does not depend on y, the homotopy M_{it} induces a homotopy g_{it} from $g_{i1} = g_i$ to the map induced by M_{i0}, which is b. Since b is homotopic to $\bar{b}$, each g_i is homotopic to $\bar{b}$. Since both g_i and $\bar{b}$ are smooth, we may find a smooth homotopy $h_i: T^{n-k} \times I \to T^{n-k}$ such that for some $\varepsilon > 0$, $h_i(v, t) = \bar{b}(v)$ for all $t < \varepsilon$, and $h_i(v, t) = g_i(v) = r(u_i, v)$ for all $t > 1 - \varepsilon$.

Pick coordinate charts (U_i, φ_i) about the points u_i such that $\{u_i\} = U_i \cap P$, $\varphi_i(U_i) = B_1(0) \subset \boldsymbol{R}^k$, and $\varphi_i(u_i) = 0$. Using the natural group structure on T^{n-k} we define

$$r_t: T^k \times T^{n-k} \longrightarrow T^{n-k} \qquad \text{for } t \in [0, 1]$$

by

$$r_t(u, v) = \begin{cases} r(u, v) + h_i(v, t\,\|\varphi_i(u)\| + 1 - t) - r(u_i, v) & \text{if } u \in U_i \\ r(u, v) & \text{if } u \notin \bigcup_{i=1}^{m} U_i \,. \end{cases}$$

Using r_t we obtain a homotopy

$$\bar{g}_t(u, v) = (\bar{a}(u), r_t(u, v))$$

from $\bar{g}_0 = \bar{g}$ to $\bar{g}_1$, where $\bar{g}_1(u, v) = (\bar{a}(u), r_1(u, v))$. Note that $r_1(u_i, v) = \bar{b}(v)$ for each $i = 0, 1, \cdots, m-1$. Proceed similarly with the other orbits in P and call the final map $\tilde{g}$.

The map $\tilde{g}$ will be smooth and homotopic $\bar{g}$ and hence homotopic to g. For all $(u, v) \in T^k \times T^{n-k}$, $\tilde{g}(u, v) \in (\bar{a}(u), \tilde{r}(u, v))$ for some map $\tilde{r}: T^k \times T^{n-k} \to T^{n-k}$ which satisfies $\tilde{r}(u, v) = \bar{b}(v)$ for all $u \in P$.

Now suppose $\tilde{g}^m(u, v) = (u, v)$. Then $\bar{a}^m(u) = u$ and so $u \in P$. Hence $\bar{a}^i(u) \in P$ for all i and so by an easy induction $\tilde{g}^i(u, v) = (\bar{a}^i(u), \bar{b}^i(v))$. Applying this with $i = m$ we see that $\bar{b}^m(v) = v$. Hence

$$\begin{aligned} &\#\{(u, v) \in T^k \times T^{n-k} \mid \tilde{g}^m(u, v) = (u, v)\} \\ &\leqq \#\{u \in T^k \mid \bar{a}^m(u) = u\} \times \#\{v \in T^{n-k} \mid b^m(v) = v\} \\ &\leqq \prod_{i=1}^{k} a_{im} \cdot \prod_{i=k+1}^{n} a_{im} = a_m \,. \end{aligned}$$

This completes the proof.

Proof of Theorem 1. The "if" direction follows from the Nielsen fixed point theorem and Theorem 2.

Next we prove the converse direction. Assume that $L(f^m)$, $m = 1, 2, \cdots$, are bounded. We may assume $n \geqq 1$. Let g be the map given by Theorem 3. If 1 is a characteristic root of $H_1(f): H_1(T^n) \to H_1(T^n)$, then g has no periodic points because $a_m(f) = 0$ for all $m \geqq 1$. So assume that 1 is not a characteristic root of $H_1(f)$. Now from Lemma 6 we have $|\lambda_i| \leqq 1$ for all i, where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of $H_1(f)$. Consequently, there exists a B such that

$a_{im} \leqq B$ for all $i = 1, \cdots, n$, and $m \geqq 1$. Thus $a_m(f) \leqq B^n$ for all $m \geqq 1$.

We will show that the number of periodic points of g is bounded by B^n. Suppose on the contrary that $S = \{x_i \mid 1 \leqq i \leqq B^n + 1\}$ is a set of $B^n + 1$ distinct periodic points such that x_i has period m_i. Set

$$m = \prod_{i=1}^{B^n+1} m_i .$$

Then $S \subset \{x \in T^n \mid g^m(x) = x\}$. But, by Theorem 3,

$$\# \{x \in T^n \mid g^m(x) = x\} \leqq a_m(f) \leqq B^n ,$$

a contradiction. This completes the proof.

REFERENCES

1. R. Brooks, R. Brown, J. Pak, and D. Taylor, *Nielsen numbers of maps of tori*, Proc. Amer. Math. Soc., **52** (1975), 398–400.

2. R. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, 1971.

3. M. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1976.

4. Hecke E. Leipzig, *Theorie der Algebraischen Zahlen*, Akademische Verlagsgesellschaft, 1923.

5. J. Milnor, *Lectures on the h-cobordism Theorem*, Princeton Math. Notes, Princeton, New Jersey, 1965.

Received February 15, 1978 and in revised form December 4, 1978.

INDIANA UNIVERSITY
BLOOMINGTON, IN 47401