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FINITE GROUPS WITH SMALL UNBALANCING 2-COMPONENTS

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In this paper we supply one link in a chain of results which will prove the following two conjectures:

$B(G)$**-Conjecture. If** H **is a 2-local subgroup of a finite group** G, **then** $[L(H), O(H)] \subseteqq O(G)$.

Unbalanced Group Conjecture. If G **is a finite group with** $O(C_G(t)) \nsubseteq O(G)$ **for some involution** $t \in G$, **then** $O(C_G(t))$ **acts nontrivially on** $L/Z^*(L)$ **where** L **is a 2-component of** G **with** $L/Z^*(L)$ **isomorphic to one of the following simple groups:**

(1) **A simple Chevalley group or twisted variation over a field of odd order;**

(2) **An alternating group of odd degree;**

(3) PSL(3, 4) **of** He, **the simple group of Held.**

In 1973 John Thompson, inspired by the Standard Component Theorem of Michael Aschbacher, formulated the $B(G)$-Conjecture. Daniel Gorenstein and John Walter had previously verified that the $B(G)$-Conjecture held in a wide variety of circumstances, but they had not formulated a general conjecture. The history of the $B(G)$-Conjecture is discussed in Chapter VI of Gorenstein's survey article, The classification of finite simple groups, I, II, which will appear as a monograph of the American Mathematical Society.

In 1974 converstions among Michael Aschbacher, David Goldschmidt, John Thompson, and John Walter directed attention toward the more general Unbalanced Group Conjecture. In § 4 we show how these various results fit together and how completion of some standard component problems will supply the missing link. Before discussing our work further we set up some notation.

For any finite group G, $O(G)$ is the largest normal subgroup of odd order, $O_2(G)$ is the largest normal 2-subgroup, $O_{2',2}(G)$ is the inverse image in G of $O_2(G/O(G))$, and $Z^*(G)$ is the inverse image in G of $Z(G/O(G))$. Syl$p\,(G)$ is the set of Sylow p-subgroups of G, and Inv$\,(G)$ is the set of involutions of G. The 2-rank of G is denoted by $m(G)$.

G is *quasisimple* if G is perfect and $G/Z(G)$ is simple. If G is simple, $\hat{G}$ stands for any quasisimple extension of G, (including G itself). A *component* of G is a subnormal quasisimple subgroup, and a 2-*component*, J, of G is a perfect subnormal subgroup with $J/O(J)$ quasisimple. $L(G)$ is the product of all 2-components of G, $F(G)$ is the fitting subgroup of G, $E(G)$ is the product of all components of

G, and $F^*(G) = E(G)F(G)$. The properties of components and 2-components are discussed in [2], [8], [13] and [26]. In particular each 2-component of G is normal in $L(G)$ by [13, § 2].

A 2-component J of $C_G(a)$, $a \in \mathrm{Inv}(g)$, is *standard* in G if

(1) J is quasisimple;

(2) $[J, J^g] \neq 1$ for all $g \in G$;

(3) If $g \in G$ with $|C_G(\langle J, J^g \rangle)|$ even, then $J = J^g$.

The standard component (or standard form) problem for J is to identify $\langle K^H \rangle$ whenever K is a standard component of H and K is isomorphic to J.

If a 2-group A acts on a group N, define $N_A = L(C_N(A))$. For cyclic groups write $N_{\langle a \rangle}$ as N_a. By [13, §§ 3 and 4] we know that $N_A \subseteq L(N)$. Further if A and B are 2-subgroups of G which normalize each other, and if J is a 2-component of $N_G(A)$ with $[J, B] \subseteq O(J)$, then $J_B = K_A$ where $K = \langle L^A \rangle$ for some 2-component L of $N_G(B)$. We refer to these results as *L-Balance* and we say that L *corresponds* to J. If $L = \langle L^A \rangle$ and J_B covers $L/O(L)$, then J also corresponds to L, and we will say in this case that J and L *correspond isomorphically*.

If a and b are commuting involutions of G and if J and K are 2-components of $C_G(a)$ and $C_G(b)$ respectively such that K corresponds to J (and necessarily $[J, b] \subseteq O(J)$), we will write $J \to K$. We define a relation $\twoheadrightarrow$ on the set of 2-components of centralizers of involutions of G by $L \twoheadrightarrow M$ if and only if $L = L_1 \to L_2 \to \cdots \to L_n = M$ for some sequence of 2-components. J is *maximal* with respect to G if and only if $J \twoheadrightarrow M$ implies $J/Z^*(J) \cong M/Z^*(M)$.

Finally an *unbalanced group* is one satisfying the hypothesis of what we shall refer to for brevity as the *U.G.*-Conjecture.

Now we return to our discussion of the context of this paper. A short argument using L-Balance together with the properties of the simple groups listed in the *U.G.*-Conjecture shows that the *U.G.*-Conjecture implies the $B(G)$-Conjecture. The advantage of the former is the inductive leverage provided by the following lemma, which incorporates results of Gorenstein and Walter [13], Gorenstein and Harada [12], and Aschbacher [1].

LEMMA 1.1. *Let G be an unbalanced group. Either G satisfies the Unbalanced Group Conjecture or G possesses a pair of commuting involutions (a, x), such that for some 2-component J of $C_G(a)$*

$$(*) \qquad [J, O(C_G(x)) \cap C_G(a)] = J = [J, x] .$$

By an *unbalancing triple* of G we mean a triple (a, x, J) with a, x, J as in Lemma 1.1 and satisfying (*). We refer to J as an *unbalancing 2-component* of G. G is a *minimal unbalanced group* if every proper section of G satisfies the $U.G$-Conjecture and G has an unbalancing triple. By Lemma 1.1 it suffices to prove the $U.G.$-Conjecture for minimal unbalanced groups.

We catalog the current work on the $U.G.$-Conjecture.

THEOREM 1.2 (*Aschbacher* [4]). *Let G be a finite group with $F^*(G)$ quasisimple. Suppose that $a \in \mathrm{Inv}(G)$, J is a 2-component of $C_G(a)$, $m(J) = 1$, and $a \in J$. Then $F^*(G)$ is isomorphic to a known group; in particular if G is unbalanced, $F^*(G)/Z(G)$ is isomorphic to a Chevalley group or twisted variation over a field of odd order.*

THEOREM 1.3 (*Aschbacher* [3], *Solomon* [25], [27], [28]). *Let G be a finite group with $F^*(G)$ quasisimple. Suppose that $a \in \mathrm{Inv}(G)$ and J is a 2-component of $C_G(a)$ with $J/Z^*(J) \cong A_n$ for some odd $n \geqq 9$. Then $F^*(G)/Z(G)$ is isomorphic to A_m for some odd m or to Lyons' simple group.*

THEOREM 1.4 (*Thompson, Burgoyne* [5], *Griess, Solomon* [14, *Theorem* 2.22]). *Let G be a minimal unbalanced group with $F^*(G)$ quasisimple. Suppose that $a \in \mathrm{Inv}(G)$ and J is an unbalancing 2-component of $C_G(a)$ with $J/Z^*(J)$ isomorphic to a simple Chevalley group or twisted variation over a finite field of odd order. Suppose further that $J/O(J) \not\cong L_2(q)$. Then G satisfies the conclusion of Theorem 1.2 or Theorem 1.3.*

THEOREM 1.5 (*Harris, Solomon* [16, 18]). *Let G be a finite group with $F^*(G)$ quasisimple. Suppose that $a \in \mathrm{Inv}(G)$ and J is a 2-component of $C_G(a)$ with a Sylow 2-subgroup isomorphic to D_8. Suppose that $C_G(J/O(J))$ has cyclic Sylow 2-subgroups. Then $F^*(G)$ is isomorphic to one of the following simple groups:*

(1) *A_8, $\mathrm{Sp}(4, 4)$, $L_5(2)$, $U_5(2)$ or HiS with $J/O(J) \cong A_6$;*
(2) *A_9 or He with $J/O(J) \cong A_7$;*
(3) *$L_3(4)$ or HJ with $J/O(J) \cong L_2(7)$;*
(4) *$L_2(q^2)$, $L_3(q)$ or $U_3(q)$ with $J/O(J) \cong L_2(q)$;*
(5) *$\mathrm{PS}p(4, q)$, $L_4(q)$ or $U_4(q)$ with $J/O(J) \cong L_2(q^2)$.*

THEOREM 1.6 (*Griess, Solomon* [14]). *Let G be a minimal unbalanced group with $F^*(G)$ quasisimple. Suppose that $a \in \mathrm{Inv}(G)$ and that J is an unbalancing 2-component of $C_G(a)$ with $J/O(J)$ isomorphic to He or to a covering group of $L_3(4)$. Suppose that G has no unbalancing triple (b, y, k) with $K/Z(K) \cong L_2(q)$ for $q \geqq 27$. Then $F^*(G)/Z(F^*(G))$ is isomorphic to He or $L_3(4)$.*

The next result assumes the solution of certain standard form problems. Specifically, the following hypothesis is needed.

Hypothesis S. Let H be a finite group satisfying

(1) $F^*(H)$ is simple.

(2) All sections of H satisfy the Unbalanced Group Conjecture.

(3) H has a standard subgroup, L, such that one of the following holds:

(a) $L/O(L) \cong \mathrm{PSL}(3, 4)$.

(b) $L/Z^*(L) \cong \mathrm{PSL}(4, 3)$, $\mathrm{PSU}(4, 3)$, $\mathrm{PSp}(4, 3)$, $\Omega_7(3)$, $\Omega^+(8, 3)$, $\Omega^-(8, 3)$.

Then $F^*(H)$ is a known simple group.

THEOREM 1.7 (*Foote* [7], *Harris* [17]). *Let G be a finite group with $F^*(G)$ simple. Suppose that the following conditions hold:*

(1) *Proper sections of G satisfy the Unbalanced Group Conjecture.*

(2) *There exists $t \in \mathrm{Inv}(G)$ and J a 2-component of $C_G(t)$ such that $J/O(J) \cong A_7$ or $L_2(q)$, q odd, and J is maximal in G.*

(3) *Proper sections of G satisfy Hypothesis* S.

Then either I or II holds:

I. *$J/O(J) \cong L_2(q)$ and $F^*(G)/F(G)$ is isomorphic to one of*

(a) *$L_2(q^2)$, t a field automorphism;*

(b) *$L_3(q)$, t a graph automorphism;*

(c) *$U_3(q)$, t a field automorphism;*

(d) *$L_4(q)$, t a graph automorphism;*

(e) *$L_4(p)$, t diagonal or graph, $q = p^2$;*

(f) *$U_4(q)$, t a graph automorphism;*

(g) *$U_4(p)$, t diagonal or graph, $q = p^2$;*

(h) $\mathrm{PSp}(4, q)$, *t inner or outer (2 classes);*

(i) $\mathrm{PSp}(4, p)$, *t outer, $q = p^2$;*

(j) $\mathrm{Re}^*(q)$, *t inner;*

(k) *$L_2(16)$, t a field automorphism, $q = 5$;*

(l) *$L_3(4)$, t field with $q = 7$ or t graph with $q = 5$;*

(m) *$U_3(4)$, t outer with $q = 5$;*

(n) *A_7, A_8, t outer with $q = 5, 9$ respectively;*

(o) *A_9, A_{10}, t inner with $q = 5, 9$ respectively;*

(p) *J_1, t inner with $q = 5$;*

(q) *HJ, t inner with $q = 5$ or outer with $q = 7$;*

(r) *J_3, t outer with $q = 17$;*

(s) *M_{12}, t inner or outer with $q = 5$;*

(t) *HiS, t inner with $q = 9$;*

(u) $\mathrm{Sp}(4, 4)$, *t outer, $q = 9$;*

(v) $\mathrm{GL}(5, 2)$, *t outer, $q = 9$;*

(w) $U_5(2)$, *t outer*, $q = 9$;
(x) $\Omega^-(8, p)$, *t outer*, $q = p^4$.
II. $J/O(J) \cong A_7$ *and* $F^*(G)/F(G)$ *is isomorphic to one of*
(y) A_9, *t outer*;
(z) A_{11}, *t inner*;
(zz) *He*, *t outer*.

THEOREM 1.8 (*Aschbacher* [2], *Seitz* [23], *Nah* [22], *Griess and Solomon* [14]). *Let G be a finite group satisfying the U.G.-Conjecture. Suppose that* $F^*(G)$ *is simple. Assume that J is a component of* $C_G(t)$, $t \in \mathrm{Inv}(G)$, *with* $J/Z(J) \cong L_3(4)$ *or* $J \cong He$ *and that either* $C_G(J)$ *has cyclic Sylow 2-subgroups or* $Z(J)$ *has even order. Let J be maximal in G. Then* $F^*(G)$ *is isomorphic to one of*
(a) $L_3(16)$.
(b) *He*.
(c) *O'NS, a sporadic simple group of O'Nan-type*.

In order to state our results we make some further definitions. A *maximal unbalancing triple* of G is an unbalancing triple (a, x, J) such that if $b \in \mathrm{Inv}(C_G(a))$, $[J, b] \subseteq O(J)$, and J corresponds to the 2-component L of $C_G(b)$, then

(1) If (b, y, L) is an unbalancing triple in G for some $y \in C_G(a) \cap N_G(J)$, then J corresponds isomorphically to L, and,

(2) If $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$ and $b \in Z(S)$, then $S \in \mathrm{Syl}_2(C_G(b) \cap N_G(L))$.

A *restricted* simple group is one isomorphic to $L_2(q)$, q odd, Suz, or one of the groups listed in the conclusions to Theorems 1.7 and 1.8. A group K is of *restricted type* if $K/Z^*(K)$ is a restricted simple group.

Theorem B below is our contribution to the proof of the Unbalanced Group Conjecture. In § 4 we show how Theorem B, all the work previously mentioned, and some additional results recorded below imply the validity of the Unbalanced Group Conjecture under the assumption of Hypothesis S. We formulate this observation as Theorem A.

THEOREM A. *Hypothesis* S *implies the Unbalanced Group Conjecture.*

THEOREM B. *Let G be a minimal unbalanced group satisfying the following conditions*:
(a) (a, x, J) *is a maximal unbalancing triple of G*;

(b) *$J/O(J) \cong L_2(q)$, q odd, or $J/O(J) \cong A_7$, or $J/Z^*(J) \cong L_3(4)$;*

(c) *The solution to the standard component problem for $L_2(q)$, q odd, A_7 and $L_3(4)$ in core-free proper sections of G is a central product of quasisimple groups of restricted type.*

Then one of the following conclusions holds:

(1) *G satisfies the conclusion of the Unbalanced Group Conjecture, or*

(2) *$J/O(J) \cong L_2(q)$ and there exists $b \in \mathrm{Inv}(G)$ and r an odd prime power with $L_b/O(L_b) \cong L_2(r^2)$ and $\langle b \rangle \in \mathrm{Syl}_2(C_G(L_b/O(L_b)))$, or*

(3) (a) *If $b \in \mathrm{Inv}(C_G(a))$ with $[J, b] \subseteq O(J)$, then J corresponds isomorphically to a 2-component K of $C_G(b)$; and*

(b) *If b is as in (a) and $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$ with $b \in Z(S)$, then $S \in \mathrm{Syl}_2(C_G(b) \cap N_G(K))$.*

COROLLARY C. *Let G be a minimal counterexample to the Unbalanced Group Conjecture with $F^*(G)$ simple, $a \in \mathrm{Inv}(G)$ and J a 2-component of $C_G(a)$ with $J/O(J) \cong A_7$. Suppose that there is a 4-subgroup E of $C_G(a) \cap N_G(J)$ with $\Delta = C_G(a) \cap (\bigcap_{e \in E^\#} O(C_G(e)))$ and $[J, \Delta] = J$. Then $F^*(G) \cong A_n$ for some odd $n \geqq 11$.*

We remark that Theorem B and Corollary C are used in the proof of Theorem 1.6. We use Theorem 1.6 only in § 4 in which we show how Theorem A follows from hypothesis (3) of Theorem 1.7 together with all the theorems in this section.

2. Properties of the restricted simple groups.

In this section we collect the properties of the restricted simple groups which we shall need in the proofs of Theorem B and Corollary C.

PROPOSITION 2.1. *Assume that H is a group such that*

(a) *$L \subseteq H \subseteq \mathrm{Aut}(L)$ with L simple of restricted type.*

(b) *$a \in \mathrm{Inv}(H)$, $J \cong L_2(q)$, A_7 or $\hat{L_3}(4)$, a component of $C_H(a)$, such that J is standard in H. Pick $S \in \mathrm{Syl}_2(C_H(a))$ and let $P = C_S(J)$, $D = S \cap J$. The following conditions hold:*

(1) *$P = \langle a \rangle$ except in the following cases:*

(i) *$J \cong A_n$, $H \cong A_{n+4}$ ($n \in \{5, 6, 7\}$), $P \cong E_4$.*

(ii) *$J \cong A_5$, $H \cong \mathrm{Aut}\, M_{12}$, $P \cong E_4$.*

(iii) *$J \cong L_2(q^2)$, $L \cong L_4(q)$ or $U_4(q)$, $C_L(J) \cong Z_{q+\varepsilon}$ with $\varepsilon = \pm 1$, $q + \varepsilon \equiv 0 \pmod 4$.*

(iv) *$J/Z(J) \cong L_3(4)$, $L \cong Suz$ or He, $P \cong E_4$.*

(v) *$J \cong A_5$, $L \cong HJ$, $P \cong E_4$.*

(2) *$P \cap Z(S) = \langle a \rangle$ except when*

(i) *$J \cong A_5$, $H \cong \mathrm{Aut}\, M_{12}$, or*

(ii) *$J \cong A_5$, $H \cong HJ$.*

PROPOSITION 2.2. *Same hypothesis as Proposition* 2.1. *Let* $z \in \operatorname{Inv} C_D(S)$. *Then* $a \sim_H az$ *except in the following cases*:

(1) $L \cong L_2(q^2)$ *and* H *does not contain a diagonal automorphism or a field-type automorphism* (*of order divisible by* 4) *acting nontrivially on* $D/[D, D]$.

(2) *Cases* (1)(i) *and* (1)(iii) *of Proposition* 2.1. *In these cases* $a \sim z$ *in* L. *Moreover if* $a_1 \in \operatorname{Inv} P$, *then* $a_1 \sim z$ *in* L.

(3) $J \cong A_n$, $H \cong S_{n+2}$ ($n \in \{5, 6, 7\}$). *Then* $a \sim \sigma$ *in* H *where* σ *acts as a transposition on* J.

(4) $J \cong L_2(q^2)$, $H \cong \operatorname{Aut} \operatorname{PSp}(4, q)$. *Then* $a \sim a\sigma$ *for some* $\sigma \in C_L(a)$ *inducing a field automorphism on* J.

(5) $J \cong Z_4 \cdot L_3(4)$, L of *O'Nan-Sims type*. *Then* $a \sim y$ *in* L *for every* $y \in \operatorname{Inv}(J)$.

Moreover, if case (1)(v) *of Proposition* 2.1 *holds, then* $a_1 \sim a_1 z$ *in* $C_L(z)$ *for all* $a_1 \in P^\sharp$.

We will be confronted with a slightly more general situation arising from the application of L-Balance to a group in which the $B(G)$-conjecture holds. If H is such a group and J is a component of $C_H(a)$, then $\langle J^{L(H)} \rangle$ will be quasisimple or a central product of two quasisimple groups permuted by $\langle a \rangle$. Let $a \in \operatorname{Inv}(H)$ and let J be a 2-component of $C_H(a)$ with $J/O(J) \cong L_2(q)$, A_7 or $L_3(4)$. Take z, S, P, D as above.

PROPOSITION 2.3. *Assume that* H *is a group such that*:

(a) *The* $B(G)$-*Conjecture holds in* H.

(b) $\bar{J}$ *is standard and not subnormal in* $\bar{H} = H/O_{2',2}(H)$.

(c) $L(H)/Z^*(L(H))$ *is a product of restricted groups*.

The following conclusions hold:

(1) $P = \langle a \rangle (P \cap O_{2',2}(H))$ *except as listed in* 2.1(1).

(2) $P \cap Z(S) = \langle a \rangle (P \cap O_{2',2}(H))$ *except as listed in* 2.1(2).

(3) $a \sim_H az$ *except in the following cases*:

(i) *The five cases listed in* 2.2.

(ii) $L/O(L) \cong \operatorname{SL}(2, q) * \operatorname{SL}(2, q)$ *with* $a \sim azy$, $y \in \operatorname{Inv} Z^*(L)$.

(iii) $\bar{J} \cong L_2(5)$ *or* $L_2(7)$, $L/Z^*(L) \cong L_3(4)$ *with* $a \sim azy$, $y \in \operatorname{Inv} Z^*(L)$.

(iv) $\bar{J} \cong L_2(q^2)$, $L/Z^*(L) \cong L_4(q)$, $U_4(q)$ *or* $\Omega^-(8, q^{1/2})$ *with* $a \sim azy$, $y \in \operatorname{Inv} Z^*(L)$.

(v) $\bar{J} \cong \hat{L_3}(4)$, $L/Z^*(L) \cong Suz$ *with* $a \sim azy$, $y \in \operatorname{Inv} Z^*(L)$.

(4) *If* $JO(H)/O(H) \cong \operatorname{SL}(2, q)$ *and* $a \not\sim az$ *in* H, *then* $L/Z^*(L)$ *is isomorphic to one of* $\operatorname{SL}(2, q) * \operatorname{SL}(2, q)$, $L_2(q^2)$, A_7, A_8, A_9, A_{10}, $L_3(4)$ $\operatorname{PSp}(4, \sqrt{q})$, $L_4(\sqrt{q})$, $U_4(\sqrt{q})$ *or* $\Omega^-(8, \sqrt[4]{q})$.

Propositions 2.1–2.3 incorporate all of the facts we need about restricted simple groups other than A_8, A_{10}, $\mathrm{PSp}(4, q)$, $L_4(q)$, $U_4(q)$ or $\Omega^-(8, q)$, which we treat more fully later.

Proposition 2.1 is easily checked for all of the listed groups. We remark that H may be taken to be $\mathrm{Aut}\, L$ except in the following cases:

(1) $J \cong A_n$, $L = H \cong A_{n+4}$. J is not maximal in S_{n+4}.

(2) $J \cong L_2(9)$, $L \cong HiS$. J is not maximal in $\mathrm{Aut}\, HiS$.

(3) $\mathrm{PSp}(4, q)$, $L_4(q)$, $U_4(q)$ discussed below.

Moreover the case $J \cong L_2(q)$, $L \cong \mathrm{PSp}(4, q)$ does not occur, since in this case J is not standard in L, even though J *is* maximal in L.

If $L/Z(L) \cong A_n$, then as $|Z(J)|$ is odd, $|Z(L)|$ is odd. The properties of A_n are well-known. If $L/Z(L) \cong L_{12}(16)$, $L_3(q)$ or $U_3(q)$ (q odd), $U_3(4)$, $\mathrm{Sp}(4, 4)$, $L_5(2)$, $U_5(2)$, $\mathrm{Re}(q)$, J_1, J_3, Held's group or a group of O'Nan type, then $|Z(L)|$ is odd. If $L \cong J_1$ or $\mathrm{Re}(q)$ or of O'Nan type, then $a \in L$ and L has one class of involutions. In the other cases La has one class of involutions. So Propositions 2.2 and 2.3 hold for all of these groups.

If $L/Z(L) \cong HiS$, we refer the reader to [15] for the desired properties of L.

If $L/Z(L) \cong L_3(4)$, then Proposition 2.3 only makes an assertion about fusion modulo $Z(L)$. As each coset of $\mathrm{Inn}\, L$ in $\mathrm{Aut}\, L$ has at most one class of involutions, this is clear.

We now treat M_{12}, HJ and Suz.

PROPOSITION 2.4. (a) *Suppose that* $L/Z(L) \cong M_{12}$. *Then* $J \cong A_5$ *and* $C_{\mathrm{Aut}\, L}(J) = P \cong E_4$ *with* $|P \cap \mathrm{Inn}\, L| = 2$. $N_{\mathrm{Aut}\, L}(J) = (P \times J)\langle b \rangle$ *with* $|C_P(b)| = 2$. *If* $a \in C_P(b)^\#$, *then* $a \sim az$ *in* $C_L(z)$.

(b) *Suppose that* $L/Z(L) \cong HJ$. *If* $a \notin \mathrm{Inn}\, L$, then $J \cong L_2(7)$, $C_{\mathrm{Aut}\, L}(J) = \langle a \rangle$ *and* $a \sim az$ *in* $\langle L, a \rangle$. *If* $a \in L$, *then* $J \cong A_5$ *and* $C_{\mathrm{Aut}\, L}(J) = P \cong E_4$. *If* $a_1 \in P^\#$, *then* $a \sim a_1$ *and* $a \sim az$ *in* $C_L(z)$.

(c) *Suppose that* $L/Z(L) \cong Suz$ *and* $J/Z(J) \cong L_3(4)$. *Then* $Z(L) \subseteq Z(J)$. *Also* $C_{\mathrm{Aut}\, L}(J) = P \cong E_4$ *and* $a_1 \sim a_1 z$ *in* L *for all* $a_1 \in P^\#$.

Proof. Let $\bar{L} = L/Z(L)$ and identify $\bar{L}$ with $\mathrm{Inn}\, L$.

(a) $\mathrm{Aut}\, L$ contains a 4-group, $\bar{P}$, with $C_{\mathrm{Aut}\, L}(\bar{P}) = \bar{P} \times \bar{J}$ and $N_{\mathrm{Aut}\, L}(\bar{P}) = (\bar{P} \times \bar{J})\langle \bar{b} \rangle$ with $\langle \bar{P}, \bar{b} \rangle \cong D_8$, $\langle \bar{J}, \bar{b} \rangle \cong S_5$. As $(\bar{P} \times \bar{D})\langle \bar{b} \rangle \notin \mathrm{Syl}_2(\mathrm{Aut}\, L)$ and $\bar{D}^\# = \bar{z}^{\mathrm{Aut}\, L} \cap (\bar{P} \times \bar{D})$, there exists $\bar{y} \in N_{\bar{L}}(\bar{P} \times \bar{D}) \cap N_{\bar{L}}(\bar{D})$ with $\bar{P} \cap \bar{P}^{\bar{y}} = 1$. Thus if $\langle \bar{a} \rangle = C_{\bar{P}}(\bar{b})$, then $\bar{a}^{\bar{y}} = \bar{z}$ and $\bar{z}^{\bar{y}} = \bar{z}$. If $\bar{a}_1 \in \bar{P} - \langle \bar{a} \rangle$, then $\bar{a}_1^{\bar{y}_1} = \bar{a}_1 \bar{z}_1$ for some $\bar{z}_1 \in \bar{D} - \langle \bar{z} \rangle$, $\bar{y}_1 \in N_{\bar{L}}(\bar{D})$. Thus $\bar{a}_1 \sim \bar{a}_1 \bar{z}$ in $N_{\bar{L}}(D)$. Suppose that $|Z(L)_1 = 2$. Now $\bar{b}$, $\bar{a}$ and $\bar{a}\bar{b}$ are noncentral involutions of $\bar{L}$. So $\langle a, b \rangle \cong Q_8$ and $a \sim az$ in $C_L(z)$. Also $\bar{a}_1 \sim \bar{a}\bar{a}_1$. So a_1 and aa_1 have the same order in L. Thus a_1 inverts a and $a_1 \sim a_1 z$ in $N_L(D)$.

(b) Suppose that $a \notin \mathrm{Inn}\, L$. Then La has only one class of involutions. Thus we may assume that $Z(L)\neq 1$. Now $C_{\bar{L}}(\bar{a})=\bar{J}\langle\bar{s}\rangle$ with $\bar{s}$ a noncentral involution. Thus s has order 4 in L. As $\bar{a}\sim\bar{a}\bar{s}$, a and as have the same order in L. So a inverts s. Thus $a\sim az$ in $\langle L, a\rangle$.

Now suppose that $a\in \operatorname{Inn} K$. Then $\bar{J}\cong A_5$ and $C_{\bar{L}}(\bar{J})=\bar{P}\cong E_4$. As $\bar{a}\sim\bar{a}_1$ for all $a_1\in\bar{P}^\#$, $\bar{P}\times\bar{J}=C_{\bar{L}}(\bar{a}_1)$ for all $\bar{a}_1\in\bar{P}^\#$. Thus there exists $\bar{r}\in C_{\bar{L}}(\bar{J})\cap N_{\bar{L}}(\bar{P})$ permuting the elements of $\bar{P}^\#$ and $\langle\bar{P},\bar{r}\rangle\times\bar{J}=N_{\bar{L}}(\bar{P})$. Let $\bar{E}\in\operatorname{Syl}_2(\bar{P}\times\bar{J})$. Then $N_{\bar{L}}(\bar{P})\cap N_{\bar{L}}(\bar{E})$ has three orbits on $\bar{E}^\#$ with lengths 3, 3, 9. As $\bar{a}^{\bar{L}}\cap\bar{D}=\varnothing$ and $\bar{E}\in\operatorname{Syl}_2(\bar{L})$, $|\bar{a}^{\bar{L}}\cap\bar{E}|=12$. Thus $N_{\bar{L}}(\bar{E})\cap C_{\bar{L}}(\bar{D})/\bar{E}\cong A_4$ and $N_{\bar{L}}(\bar{E})\cap C_{\bar{L}}(\bar{D})=O^2(N_{\bar{L}}(\bar{E})\cap C_{\bar{L}}(\bar{D}))$. Thus $\bar{a}_1\sim\bar{a}_1\bar{z}_1$ in $C_{\bar{L}}(\bar{D})\cap N_{\bar{L}}(\bar{E})$ for all $\bar{a}_1\in\bar{P}^\#$, $\bar{z}_1\in\bar{D}$. If $Z(L)\neq 1$, then $P\cong Q_8$ and $N_L(E)\cap C_L(D)$ is the full inverse image in L of $N_{\bar{L}}(\bar{E})\cap C_{\bar{L}}(\bar{D})$. Thus $a_1\sim a_1z_1$ in $C_L(D)$ for all $a_1\in P-Z(P)$, $z_1\in D$.

(c) Let $\hat{Suz}$ be the six-fold cover of Suz and let ρ be a 12-dimensional complex representation of $\hat{Suz}$. We wish to show that the inverse image, $\hat{J}$, of J in $\hat{Suz}$ is a six-fold cover of $L_3(4)$. Now $C_{Suz}(J)=P\cong E_4$ and $N_{Suz}(P)\cong(A_4\times L_3(4))\cdot \boldsymbol{Z}_2$.

Also $\hat{P}\cong Q_8$ and $Z(\hat{P})$ acts as $-I$. So $\hat{P}$ must act on a sum of six isomorphic 2-dimensional modules. Then $C_{\rho(\hat{Suz})}(\rho(\hat{P}))$ is isomorphic to a subgroup of $\mathrm{GL}(6, \boldsymbol{C})$. Then, by Lindsey [18], $\hat{J}$ is a six-fold cover of $L_3(4)$. Thus $Z(K)\subseteq Z(J)$. If $\bar{S}\in\operatorname{Syl}_2(N_{\bar{L}}(\bar{P}))$, then $\bar{S}\subset\bar{T}\in\operatorname{Syl}_2(\bar{L})$ and $\bar{P}\cap\bar{P}^{\bar{y}}=1$ for $\bar{y}\in N_{\bar{T}}(\bar{S})-\bar{S}$. As $a^{\bar{L}}\cap\bar{J}=\varnothing$ and $z^{\bar{J}}=\operatorname{Inv}\bar{J}$, $\bar{a}\sim\bar{a}\bar{z}$ in $\bar{L}$. As $\hat{P}\cong Q_8$ in $\hat{Suz}$, we see as before that $a\sim az$ in L.

PROPOSITION 2.5. *Let K be isomorphic to A_8 or to* $\mathrm{PSp}(4, q)$, *q odd. Let $N=S_8$ in the former case and let N be the extension of* $\mathrm{PSp}(4, q)$ *by a diagonal automorphism in the latter case,* i.e., $N\cong \mathrm{SO}(5, q)$ *in the latter case.*

(a) *If $f\in\operatorname{Aut}K-N$ of order 2, then $K\cong\mathrm{PSp}(4, q_1^2)$, f is a field automorphism and $L(C_K(f))\cong\mathrm{PSp}(4, q_1)$.*

(b) *N has four classes of involutions, two in K and two in $N-K$. N has exactly one 2-central class of involutions.*

(c) *If s is a 2-central involution of N, then $O^2(C_N(s))\cong\mathrm{SL}(2, q)*\mathrm{SL}(2, q)$ if $K\cong\mathrm{PSp}(4, q)$ and $C_N(s)$ is solvable if $K\cong A_8$.*

(d) *If $a\in K$ is a non-2-central involution, and $\varepsilon=\pm 1$ so that $4\mid q+\varepsilon$, then*

$$C_N(a)\cong\mathrm{PGL}(2, q)\times D_{2(q+\varepsilon)}$$

with $q=3$ when $K\cong A_8$.

Also $C_K(a)=(L\times F)\langle\alpha\rangle$ with $L\cong L_2(q)$, $\langle L, \alpha\rangle\cong\mathrm{PGL}(2, q)$, $F\cong D_{q+\varepsilon}$ and $\langle F, \alpha\rangle\cong D_{2(q+\varepsilon)}$.

If $z\in\operatorname{Inv}L$, then $a\sim z$ in K.

(e) *If $e\in\operatorname{Inv}(N-K)$, then $O^2(C_N(e))$ is isomorphic to $L_2(q)$ or $L_2(q^2)$, with $q=3$ when $K=A_8$. If $L(C_N(e))$ is a maximal component* *in N, then $L(C_N(e))\cong L_2(q^2)$. If $O^2(C_N(e))\cong L_2(q)$, then $C_K(\langle e, O^2(C_N(e))\rangle)\cong D_{q-\varepsilon}$ with ε as in* (d).

(f) *No proper covering of K has an involutory automorphism whose centralizer has an $L_2(q)$ component.*

Proof. The properties of S_8 are well-known. The conjugacy of all involutory field automorphism is a well-known consequence of Lang's theorem. Properties (b)-(e) may be found in [10], [11] or deduced directly by matrix calculations in $N\cong \mathrm{SO}(5, q)$. Elements of $\Omega(5, q)$ with exactly two eigenvalues -1 are known to lift to elements of order 4 in $\mathrm{Spin}(5, q)$. (See [24, Lemma 3.1].) If M is an $L_2(q)$ component in K, then the involutions of M have exactly two eigenvalues -1, whence $\hat{M}\cong \mathrm{SL}(2, q)$ in $\mathrm{Spin}(5, q)$.

PROPOSITION 2.6. *Same hypotheses as Proposition* 2.5. *Take $a\in \mathrm{Inv}\, K$ with a non-2-central and take $e\in \mathrm{Inv}(N-K)$ with $L=L(C_K(e))\cong L_2(q^2)$. Take $S\in \mathrm{Syl}_2(C_N(a))$, $S\subset R\in \mathrm{Syl}_2(N)$. Let $P=S\cap C_N(J)$, $D=S\cap J$, $S_0=S\cap K$, $P_0=P\cap K$, $R_0=R\cap K$.*

(a) *$[R_0, R_0]$ is nonabelian; $R=\Omega_1(R)$, $R_0=\Omega_1(R_0)$.*

(b) *When $K=\mathrm{PSp}(4, q)$, all elements of $\mathrm{Inv}(P_0-\langle a\rangle)$ are 2-central.*

(c) *$|R: S|=|R_0: S_0|=2$. For any $r\in \mathrm{Inv}(R-S)$, $a^r=z$, $z^r=a$.*

(d) *d_0 contains $E\cong E_{16}$ and all such E are conjugate in N. $N_N(E)/C_N(E)\cong S_5$ or $S_3\wr Z_2$, according as $K=\mathrm{PSp}(4, q)$ or A_8.*

(e) *Pick $Q\in \mathrm{Syl}_2(C_N(e))$ with $Q\subseteq T\in \mathrm{Syl}_2(N)$ and $z\in \mathrm{Inv}(C_L(Q))$. Then $C_K(e)=L\langle\tau\rangle$ with τ a non-2-central involution in K, τ acting as a field automorphism on L. Further $e^g=e^\tau$ for some $g\in N_T(Q)-Q$ with $g^2\in Q$. Also $z\tau$ is 2-central in K and $z\tau$, z, e, $ez\tau$ represent the N-classes of involutions.*

Proof. We refer the reader to [10], [11] and direct matrix calculation.

PROPOSITION 2.7. *Let $K=\mathrm{PSL}(4, q)$, q odd. Let N be a normal complement in* Aut K *to the cyclic group of field automorphisms.*

(a) *If $N\neq O^{2'}(\mathrm{Aut}\, K)$, then $(\mathrm{Aut}\, K)-N$ has two classes of involutions whose fixed points on K contain $\mathrm{PSL}(4, q^{1/2})$ and $\mathrm{PSU}(4, q^{1/2})$ respectively.*

(b) *If $q\equiv 3 \pmod 4$, then $\mathrm{Aut}\, K/\mathrm{Inn}\, K$ is abelian.*

(c) *N has six classes of involutions.*

(d) *If s is a 2-central involution in N, then $O^2(C_N(s))\cong \mathrm{SL}(2, q)*\mathrm{SL}(2, q)$.*

(e) *N has two classes of diagonal involutions with representatives a and a_1. $L(C_N(a))\cong L_2(q^2)$ and $L(C_N(a_1))\cong L_3(q)$. Also a is* *inner if and only if* $q \equiv 3 \pmod 4$. $C_{\langle K, a\rangle}(a) = (\langle t\rangle \times J)\langle\sigma\rangle$ *with* $\langle t\rangle \cong \boldsymbol{Z}_{q+1}$, $J \cong L_2(q^2)$, σ *inverting* $\langle t\rangle$ *and inducing a field automorphism on* J. *Also if* $z \in \operatorname{Inv} J$ *and* $a \in K$, *then* $a \sim z$ *in* K. *If* $a \notin K$, *then* $a \sim az$ *in* $\langle K, a\rangle$.

(f) *N has three classes of graph automorphisms of order 2 with representatives b, c, d.* $L(C_N(b)) \cong \mathrm{PSp}(4, q)$, $L(C_N(c)) \cong L_2(q^2)$ *and* $O^2(C_N(d)) \cong L_2(q) \times L_2(q)$. *Let M be a subgroup of N containing* $\langle K, a\rangle$ *such that* $L(C_H(a))$ *is maximal in* M. *Then* $b \notin M$ *and* $\langle t\rangle = C_M(J)$. *Let* M_1 *be a subgroup of N containing* $\langle K, c\rangle$ *such that* $L(C_K(c))$ *is maximal in* M_1. *Then* $b \notin M_1$ *and* $\langle c\rangle = C_{M_1}(L(C_K(c)))$. *Every involution of Kc is K-conjugate to c. In particular, if* $z \in \operatorname{Inv}(L(C_K(c)))$, *then* $c \sim cz$ *in* $\langle K, c\rangle$.

(g) *The full coverging group $\hat{K}$ of K does not admit an involutory automorphism whose centralizer has an $L_2(q)$ component. Also $Z(\hat{K})$ is cyclic of order 2 or 4.*

Proof. Most of these facts may be found in [6] and [20], [21]. The rest follow by direct matrix calculation. It is helpful to recall that $L_4(q) \cong P\Omega^+(6, q)$. The existence of precisely three classes of graph automorphisms may be found in [6]. Both b and d lie in the coset of $P\Omega^+(6, q)$ by a diagonal matrix with precisely one eigenvalue -1. It follows that all involutions in Kc are K-conjugate to c. Fact (g) follows as in Proposition 2.5 from the properties of Spin $(6, q)$.

PROPOSITION 2.8. (1) *Let* $K = \mathrm{PSU}(4, q)$, *q odd and* $N = \operatorname{Aut} K$. *The assertions of Proposition* 2.7 *remain true after replacing* (b) *by*:

(b′) *If* $q \equiv 1 \pmod 4$, *then* $\operatorname{Aut} K/\operatorname{Inn} K$ *is abelian.*

And replacing (e) *by*:

(e′) *N has two classes of diagonal involutions with representatives a and a_1.* $L(C_N(a)) \cong L_2(q^2)$ *and* $L(C_N(a_1)) \cong U_3(q)$. *Also a is inner if and only if* $q \equiv 1 \pmod 4$. $C_{\langle K, a\rangle}(a) = (\langle t\rangle \times J)\langle\sigma\rangle$ *with* $\langle t\rangle \cong \boldsymbol{Z}_{q-1}$, $J \cong L_2(q^2)$, σ *inverting* $\langle t\rangle$ *and inducing a field automorphism on* J. *If* $z \in \operatorname{Inv} J$ *and* $a \in K$, *then* $a \sim z$ *in* K. *If* $a \notin K$, *then* $a \sim az$ *in* $\langle K, a\rangle$.

(2) *Let* $K = A_{10}$, $N = S_{10}$. *N has one class of involutions with representative a satisfying* $L(C_N(a)) \cong A_6$. $C_N(a) \cong S_6 \times D_8$. $C_K(a) = (J \times P)\langle\sigma\rangle$ *with* $J \cong A_6$, $P \cong E_4$, $J\langle\sigma\rangle \cong S_6$, $P\langle\sigma\rangle \cong D_8$. *$N$ has one class of involutions with representative b satisfying* $L(C_N(b)) \cong A_8$. $C_K(b) \cong S_8$ *and* $b \notin K$. $\hat{A}_{10}$ *does not admit an involutory automorphism whose centralizer has a subnormal subgroup isomorphic to A_4 or A_6.*

Proof. (1) is handled like Proposition 2.7. The assertions about S_{10} are trivial. An involution r in an A_4 or A_6 subnormal in $C_G(s)$ for some $s \in \operatorname{Inv} S_{10}$ has the property that r is a product of two transpositions. Then r lifts to an element of order 4 in $\hat{A}_{10}$.

PROPOSITION 2.9. *Let* $K = A_{10}$, $L_4(q)$ *or* $U_4(q)$ *with notation as in* 2.7 *and* 2.8. *Suppose that* $a \in K$, i.e., $K = A_{10}$ *or* $K = L_4(q)$, $q \equiv 3 \pmod 4$ *or* $K = U_4(q)$, $q \equiv 1 \pmod 4$. *Let* $S \in \mathrm{Syl}_2(C_N(a))$, $S \subset R \in \mathrm{Syl}_2(N)$. *Let* $R_0 = R \cap K$, $D = S \cap J$, $P = C_S(J)$. *Then*

(1) $[R_0, R_0]$ *is nonabelian.*

(2) *There exists* $t \in N_{R_0}(S) - S$ *with* $t^2 \in S_0$ *and* $DD^t = D \times D^t = S_0$.

(3) $|D^t : D^t \cap P| = 2$ *and* $d \in D^t - P$ *acts as a field automorphism or transposition on* J.

(4) $(D^t \cap P)\langle b\rangle$ *is dihedral with center* $\langle a\rangle$.

Proof. Direct calculation. Note that $R_0 \cong D_{2^n} \wr Z_2$ with $2^n \,\|\, q^2-1$. Here $q = 3$ if $K = A_{10}$.

PROPOSITION 2.10. *Let* K *be a finite quasisimple group with* $\bar{K} = K/Z(K) \cong \Omega_8^-(q)$.

(1) $Z(K)$ *is cyclic of order* 2 *or* 4.

(2) *There is a unique class of involutory automorphisms with representative* a *such that* $J = L(C_K(a)) \cong L_2(q^4)$. *If* $b \in \mathrm{Inv}\,\mathrm{Aut}\,K$ *with* $C_K(b)$ *having a maximal component isomorphic to* $L_2(r)$ *for any* r, *then* $b \in a^K$.

(3) $\mathrm{Inv}(\bar{K}a) = a^K$. *In particular, if* $z \in \mathrm{Inv}(J)$, *then* $a \sim azy$ *in* $\langle K, a\rangle$ *for some* $y \in Z(K)$. *Also* $\langle a\rangle \in \mathrm{Syl}_2(C_{\mathrm{Aut}\,K}(J))$.

(4) *There is no involution* $\bar{\alpha}$ *in* $\bar{K}$ *with* $O^{2'}(C_{\bar{K}}(\bar{\alpha}))$ *of* 2*-rank* 1.

Proof. We refer the reader to [6].

PROPOSITION 2.11. *Let* G *be a finite group of sectional* 2*-rank* 4 *with* $F^*(G)$ *simple. Suppose that* G *has an involution* a *with* $L(C_G(a)) \cong L_2(q^2)$ *for some odd* $q \geqq 3$. *Suppose also that* $G \subseteq G_1$ *of index* 2 *and* $b \in \mathrm{Inv}\,C_{G_1}(a)$ *with either* $L(C_G(b)) \cong \mathrm{PSp}(4, q)$ *or* $q = 3$ *and* $L(C_G(b)) \cong A_8$. *Then* $F^*(G) \cong L_4(q)$, $U_4(q)$ *or* A_{10}.

Proof. We may check the list of conclusions to the Main Theorem of [12].

PROPOSITION 2.12. *Let* $K \cong \mathrm{PSp}(4, q)$, $L_4(q)$, $U_4(q)$ *or* $\Omega^-(8, q)$, q *odd. Suppose that* a *is an involutory automorphism of* K *with* $L(C_K(a)) = 1$. *Then* $K \cong \mathrm{PSp}(4, 3)$, $L_4(3)$ *or* $U_4(3)$. *Moreover* $C_K(a)$ *involves* A_4.

Proof. The information for $\mathrm{PSp}(4, q)$, $L_4(q)$ and $U_4(q)$ may be read off from Propositions 2.5, 2.7 and 2.8. For $\Omega^-(8, q)$ one may check the information in [6] or compute directly in $O^-(8, q)$.

3. **Preliminary results.** We are principally concerned here with properties of 2-components. We begin with a reduction of the Unbalanced Group Conjecture.

LEMMA 3.1. *Suppose that every proper section of G satisfies the Unbalanced Group Conjecture, then either G satisfies the Unbalanced Group Conjecture or $F^*(G)$ is simple.*

Proof. Suppose G does not satisfy the $U.G.$-Conjecture. Take $x \in \mathrm{Inv}(G)$ with $D = O(C_G(x)) \nsubseteq O(G)$. Minimality implies $O(G) = 1$, and by [26, Lemma 2.5] G has an (unbalanced) component L such that $L = [L, D] = [L, x]$. By minimality again $G = \langle L, D, x\rangle$ whence $L = F^*(G)$. It remains only to show $Z(L) = 1$. Suppose $Q = Z(L) \neq 1$. Q is a 2-group, so $C_Q(x) \leqq O_2(C_G(x))$ implies $[C_Q(x), D] = 1$. By Thompson's $A \times B$ lemma, $[Q, D] = 1$. Now $Z(G) = C_Q(x) \neq 1$, and one sees easily that $G/Z(G)$ is an unbalanced group. (Note that $x \notin Z(G)$ else $D \subseteqq O(G) = 1$.) Since the Unbalanced Group Conjecture holds for $G/Z(G)$, and $L/Z(G) = L(G/Z(G))$, we conclude that L is simple.

Next we wish to develop a particular property of 2-components. Suppose J is a 2-component of $C_G(a)$, $a \in \mathrm{Inv}(G)$, with $a \in Z(S)$, $S \in \mathrm{Syl}_2(N_G(J))$. Let $P = C_S(J)$; for any 2-group $B \subseteqq P$, B has a conjugate $B^x \subseteqq P$ such that $Q = N_S(B^x) \in \mathrm{Syl}_2(N_G(B^x) \cap N_G(J))$. If G satisfies the $B(G)$-Conjecture, then $J_{B^x} = J$ centralizes $O(C_G(B^x))$, and $Q \in \mathrm{Syl}_2(N_G(B^x) \cap N_G(J_{B^x}O(C_G(B^x))))$. Lemma 3.3 shows that this result holds if G is a minimal unbalanced group. The next lemma is used in the proof of Lemma 3.3.

LEMMA 3.2. *Suppose all 2-local subgroups of G satisfy the $B(G)$-Conjecture. Let J be a 2-component of $C_G(Q)$, Q a 2-subgroup of G, and let P and R be 2-subgroups with $P \subseteqq R \subseteqq C_G(Q) \cap C_G(J/O(J))$. Define $H = C_G(P)$, $\bar{H} = H/O(H)$. The following conditions hold:*

(i) $\overline{J_P} = \overline{J_R}$;

(ii) *If $Q \subseteqq R$, then* $\overline{J_P} = O^{2'}(\overline{J_R O(C_G(R))})$.

Proof. $[J, P] \subseteqq O(J)$ implies $J_P = O^{2'}(C_J(P))$. Likewise $J_R = O^{2'}(C_J(R))$; and it follows that $J_R \subseteqq J_P$ and J_R covers $J_P/O(J_P)$. As $\overline{J_P}$ is quasisimple by the $B(G)$-Conjecture, $\overline{J_R}$ covers $\overline{J_P}/Z(\overline{J_P})$. Since $\overline{J_P}$ is perfect, $\overline{J_R} = \overline{J_P}$ and (i) holds.

We prove (ii). By L-Balance $\overline{J_P} = \overline{J_R}$ is a component of $C_{\bar{H}}(\bar{Q})$. Let $F = O(C_G(R))$. As $PQ \subseteqq R$, $F \subseteqq C_H(Q)$. $[J_R, F] \subseteqq [C_G(R), F] \subseteqq F$

implies $[\overline{J_R}, \bar{F}] \subseteq \bar{F}$ in $\bar{H}$. Since $\bar{F}$ permutes the components of $C_{\bar{H}}(\bar{Q})$, $\bar{F}$ normalizes $\overline{J_R}$. Thus $O^{2'}(\overline{J_R F}) = \overline{J_R} = \overline{J_P}$.

LEMMA 3.3. *Let G be a group with all 2-local subgroups satisfying the $B(G)$-Conjecture. Suppose J is a 2-component of $C_G(a)$, $a \in \mathrm{Inv}(G)$, and $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$. Let $P = C_S(J/O(J))$, and assume*

$$(**) \qquad \begin{array}{l} \textit{For every } b \in \mathrm{Inv}(C_P(S)) , \\ S \in \mathrm{Syl}_2(C_G(b) \cap N_G(J_b O(C_G(b)))) . \end{array}$$

Then for every $B \subseteq P$ there exists $x \in G$ such that

(i) $x = x_1 \cdots x_t$ *and* $\langle (B^{x_1 \cdots x_{i-1}})^{\langle x_i \rangle} \rangle \subseteq P$, $1 \leq i \leq t$;

(ii) $N_S(B^{x_1 \cdots x_{i-1}})^{x_i} \subseteq S$, $1 \leq i \leq t$;

(iii) $J_{B^x} O(C_G(B^x)) = [J_B O(C_G(B))]^x$;

(iv) $N_S(B^x) \in \mathrm{Syl}_2(N_G(B^x) \cap N_G(J_{B^x} O(C_G(B^x))))$.

Proof. First we show that (i) implies (iii). By induction on t we may let $x = y x_t$ and suppose $J_{B^y} O(C_G(B^y)) = [J_B O(C_G(B))]^y$. Let $x_t = z$, $E = B^y$ and $T = \langle E^{\langle z \rangle} \rangle$; by (i) $T \subseteq P$. By Lemma 3.2 (i) $J_E O(C_G(E)) = J_T O(C_G(E))$ and likewise $J_{E^z} O(C_G(E^z)) = J_T O(C_G(E^z))$. Now

$$\begin{aligned} [J_B O(C_G(B))]^x &= [J_E O(C_G(E))]^z = [J_T O(C_G(E))]^z = J_{T^z} O(C_G(E^z)) \\ &= J_{E^z} O(C_G(E^z)) = J_{B^x} O(C_G(B^x)) . \end{aligned}$$

We will now use induction on $|P: B|$ to prove Lemma 3.3 (i), (ii), (iv). Let $X = J_B O(C_G(B))$ and suppose $Q = N_S(B) \notin \mathrm{Syl}_2(N_G(B) \cap N_G(X))$. Pick a 2-group Q_1 with $Q \trianglelefteq Q_1$, $Q \neq Q_1$, and $Q_1 \subseteq N_G(B) \cap N_G(X)$. If $B = P$, then $Q = S$ and Q_1 centralizes some $b \in \mathrm{Inv}(C_B(S))$. Q_1 normalizes $O^{2'}(X) O(C_G(b)) = J_b O(C_G(b))$ by Lemma 3.2 (ii) contradicting (**). Thus $B \neq P$. Let $T = Q \cap P$; we have $B \subset T$. By induction we may assume $N_S(T) \in \mathrm{Syl}_2(N_G(T) \cap N_G(J_T O(C_G(T))))$, Q_1 normalizes $T = C_Q(X/O(X))$ and $C_X(T)$. By Lemma 3.2 (i) $X = J_T O(C_G(B))$ whence $C_X(T) = J_T(O(C_G(B)) \cap C_G(T))$. It follows that Q_1 normalizes $J_T O(C_G(T)) = C_X(T) O(C_G(T))$. Thus $Q_1 \subseteq N_G(T) \cap N_G(J_T O(C_G(T)))$ and these exists $x \in N_G(T) \cap N_G(J_T O(C_G(T)))$ such that $Q_1^x \subseteq N_S(T)$. Clearly $\langle B^{\langle x \rangle} \rangle \subseteq T \subseteq P$ and $Q_1^x \subseteq N_S(B^x) \subseteq N_G(B^x) \cap N_G(J_{B^x} O(C_G(B^x)))$. As $|Q_1| > |Q|$, repeating our argument a finite number of times gives (iv).

LEMMA 3.4. *Let J be a 2-component of $C_G(a)$, $a \in \mathrm{Inv}(G)$. Suppose $m(J) \geqq 2$ and $b \in \mathrm{Inv}(C_G(a))$ with $[J, b] \subseteq O(J)$. Pick $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$ with $\langle a, b \rangle \trianglelefteq S$, and let $P = C_S(J/O(J))$. Let J correspond to the 2-component K of $C_G(b)$. Either J corresponds isomorphically to K or there exists $e \in \mathrm{Inv}(P)$ such that the following conditions hold*:

(1) $C_S(b) \subseteq C_S(e)$;

(2) *If J corresponds to the 2-component L of $C_G(e)$, then*

$|L: O(L)| > |J: O(J)|$ *or* $m(C_G(e) \cap C_G(L/O(L))) > m(C_G(a) \cap C_G(J(O(J))))$.

Proof. Suppose J does not correspond isomorphically to K. Pick $e \in \mathrm{Inv}(P)$ so that $C_S(b) \subseteqq C_S(e) = Q$ and Q is maximal with respect to inclusion among choices of e for which J does not correspond isomorphically to a 2-component of $C_G(e)$. By L-Balance either $L \neq L^a$ or $|L: O(L)| > |J: O(J)|$. We may assume $L \neq L^a$ and $|L: O(L)| = |J: O(J)|$. Define $M = C_G(e)$, $\bar{M} = M/O(M)$. $\bar{L} = \bar{J}_e$ implies $\bar{L}\bar{L}^{\bar{a}} = \bar{L} \times \bar{L}^{\bar{a}}$. By [13, Lemma 2.14], $\bar{J}_e = \{\bar{y}\bar{y}^{\bar{a}} \mid \bar{y} \in \bar{L}\}$ is isomorphic to $\bar{L}$. As $T = Q \cap P$ centralizes $J_e/O(J_e)$, T acts on $\bar{L}\bar{L}^{\bar{a}}$ and centralizes $\bar{J}_e$. It follows that $T = \langle a \rangle \times R$, $R = C_T\langle \bar{L}\bar{L}^{\bar{a}} \rangle$. Pick $F \in \mathrm{Syl}_2(L)$ so that $\langle F, Q \rangle$ is a 2-group. As $\bar{L} \cap \bar{L}^{\bar{a}} = 1$, $\bar{F}\bar{F}^{\bar{a}} = \bar{F} \times \bar{F}^{\bar{a}}$ and $C_{\bar{F}}(\bar{a}) = 1$ implies $\bar{F} \cap \bar{R} = 1$. It follows that $m(C_G(e) \cap C_G(\bar{L})) \geqq m(\bar{F}) + m(\bar{R}) \geqq 2 + m(\bar{R})$ as $m(\bar{J}) = m(\bar{L}) \geqq 2$. Since $m(\bar{T}) = 1 + m(\bar{R})$, $m(C_G(e) \cap C_G(\bar{L})) > m(T)$. If $m(T) = m(P)$ then (2) holds, so assume $m(T) < m(P)$ and pick $w \in N_P(Q) - Q$ with $w^2 \in T$. $T = C_P(e)$ implies $|T| \geqq 4$, and if $|T| = 4$, then P has maximal class and $m(P) = m(T)$. Thus $|T| \geqq 8$, $|R| \geqq 4$, and $R \cap R^w \neq 1$. As $R \trianglelefteq Q$, $R \cap R^w \trianglelefteq \langle Q, w \rangle$ and we can pick $z \in \mathrm{Inv}(R)$ with $\langle Q, w \rangle \subseteqq C_S(z)$. By choice of e, J corresponds isomorphically to a 2-component V of $C_G(z)$. Let $J_0 = J_{\langle b, z \rangle}$. It is easy to see that $\langle J_0^{L(C_G(z))} \rangle = V$ and $\langle J_0^{L(C_G(b))} \rangle = LL^a$. Likewise the normal closure of J_0 in $(LL^a)_z$ is $(LL^a)_z$ whence $(LL^a)_z \subseteqq L(C_G(z))$ implies $(LL^a)_z \subseteqq \langle J_0^{L(C_G(z))} \rangle = V$. But $|LL^a: O(LL^a)| > |J: O(J)| = |V: O(V)|$ and we see that $|T| \geqq 8$ forces $T = P$ and establishes (2).

LEMMA 3.5. *Suppose J is a 2-component of $C_G(a)$, $a \in \mathrm{Inv}(G)$, and $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$. Let $P = C_S(J/O(J))$. Assume that for every $b \in \mathrm{Inv}(P)$ J corresponds isomorphically to a 2-component L of $C_G(b)$ and that if $b \in \mathrm{Inv}(C_P(S))$, then $S \in \mathrm{Syl}_2(C_G(b) \cap N_G(L))$. Under these conditions J is maximal in G.*

Proof. We must show that if $J \twoheadrightarrow K$, then $J/Z^*(J) \cong K/Z^*(K)$. We are given a sequence of 2-components $J = L_1, \cdots, J_r = K$ such that for each pair L_i, L_{i+1} these are involutions a_i, b_{i+1} satisfying

(1) L_i is a 2-component of $C_G(a_i)$;
(2) L_{i+1} is a 2-component of $C_G(b_{i+1})$;
(3) $[a_i, b_{i+1}] = 1$;
(4) $L_i \rightarrow L_{i+1}$.

We will show that L_i corresponds isomorphically to L_{i+1}, $1 \leqq i \leqq r-1$, which will suffice for the proof of the lemma. Assume r is minimum such that for assertion fails. For any i with $2 \leqq i \leqq r-1$ and $x \in N_G(L_i)$ we may replace a_j by a_j^x, $i \leqq j \leqq r-1$, and b_j, L_j by b_j^x, L_j^x, respectively $i+1 \leqq j \leqq r$ to obtain another sequence of length

r for which our assertion fails. By (3) we can choose x such that $\langle a_{j-1}, b_j, a_j^x\rangle$ is a 2-group. Thus we may assume $\langle a_{i-1}, b_i, a_i\rangle$ is a 2-group for all i with $2 \leqq i \leqq r-1$.

Pick $x_i \in \mathrm{Inv}\,(Z(\langle a_{i-1}, b_i, a_i\rangle))$ and let L_{i-1} correspond to the 2-component K_i of $C_G(z_i)$. As $i < r$, each term in the sequence $L_1, \cdots, L_{i-1}, K_i$ corresponds isomorphically to the next term. A straightforward argument using L-Balance shows that K_i corresponds to L_i and L_{i+1}. Thus by replacing b_i, L_i by z_i, K_i respectively we may assume $b_i \in Z(\langle a_{i-1}, b_i, a_i\rangle)$.

We wish to apply Lemma 3.3. If $b \in \mathrm{Inv}\,(C_P(S))$ and J corresponds to the 2-component L of $C_G(b)$, then by hypothesis $S \in \mathrm{Syl}_2\,(C_G(b) \cap N_G(L))$. Thus S normalizes $J_bO(C_G(b)) = LO(C_G(b))$. As $L = L(J_bO(C_G(b)))$ is characteristic in $J_bO(C_G(b))$, it follows that condition (**) of Lemma 3.3 holds.

Pick $T \in \mathrm{Syl}_2\,(N_G(J))$ with $S \subseteqq T$. $S = C_T(a)$ implies $Z(T) \subseteqq Z(S)$, so for $z \in \mathrm{Inv}\,(Z(T))$ our hypothesis implies $S \in \mathrm{Syl}_2\,(C_G(z) \cap N_G(L))$ where J corresponds isomorphically to the 2-component L of $C_G(z)$. But T normalizes J_z whence T normalizes L, and it follows that $S = T$. In particular $\langle a_1, b_2\rangle^x \subseteqq P$ for some $x \in G$ and replacing our sequence of 2-components and involutions by their x-conjugates we may assume $\langle a_1, b_2\rangle \subseteqq P$. Clearly we may further assume $a_1 = a$. As $b_2 \in P$, our hypotheses force $r \geqq 3$.

Apply Lemma 3.3 to $\langle b_2\rangle$. These exists $x \in G$ such that for $e = (b_2)^x$ we have

(1) $e \in P$;

(2) $C_S(e) \subseteqq \mathrm{Syl}_2\,(C_G(e) \cap N_G(J_eO(C_G(e))))$;

(3) $J_eO(C_G(e)) = (J_{b_2}O(C_G(b_2)))^x$.

By (3) $L(J_eO(C_G(e))) = (L_2)^x$, a 2-component of $C_G(e)$. Replacing b_j by b_j^x, L_j by $(L_j)^x$, $2 \leqq x \leqq r$ and replacing a_j by a_j^x, $2 \leqq j \leqq r-1$, we may deduce from (2) that

$$Q = C_S(b_2) \in \mathrm{Syl}_2\,(C_G(b_2) \cap N_G(L_2))\ .$$

As $r \geqq 3$, we have $b_2 \in Z(\langle a, b_2, a_2\rangle)$. Whence $(a_2)^y \in Q$ for some $y \in C_G(b_2) \cap N_G(L_2)$. As before we may assume $a_2 \in Q$. $[a, a_2] = 1$ implies that we may replace b_2 by a_2 and assume $b_2 = a_2$. Since b_3 normalizes L_2 and centralizes a_2, $b_3^w \in Q$ for some $w \in C_G(a_2) \cap N_G(L_2)$. Again we may assume $b_3 \in Q$. Now by hypotheses J corresponds isomorphically to L_3. Thus $r \geqq 4$ and the sequence $J = L_1, L_3, \cdots, L_r = K$ contradicts our choice of r.

LEMMA 3.6. *Let G be a group in which all 2-local subgroups satisfy the $B(G)$-Conjecture. Suppose J is a 2-component of $C_G(a)$, $a \in \mathrm{Inv}\,(G)$, and $S \in \mathrm{Syl}_2\,(C_G(a) \cap N_G(J))$. Let $P = C_S(J/O(J))$. Assume the following conditions*:

(i) *If* $b \in \mathrm{Inv}(C_P(S))$, *then* $S \in \mathrm{Syl}_2(C_G(b)) \cap N_G(J_b O(C_G(b)))$.

(ii) *For some* $b \in \mathrm{Inv}(P)$, J *does not correspond isomorphically to any* 2-*component of* $C_G(b)$.

(iii) $m(J) \geqq 2$.

Under these conditions these exists a 2-*group* B, $1 \neq B \subseteqq P$ *for which the following conclusions hold where* $H = N_G(B)$, $\bar{H} = H/O_{2',2}(H)$, $V = \langle J^{L(H)} \rangle$.

(1) $\overline{J_B}$ *is a component of* $C_{\bar{H}}(\bar{a})$ *and is standard in* $\bar{H}$;

(2) $V = [V, a] = L(H)$;

(3) $B \in \mathrm{Syl}_2(C_H(V/O(V)))$ *and* $N_S(B) \in \mathrm{Syl}_2(N_G(B) \cap N_G(J_B O(C_G(B))))$;

(4) *If* $E \subseteqq P$ *with* $|E| > |B|$ *or with* $|E| = |B|$ *and* $|N_S(E)| > |N_S(B)|$, *then* J *corresponds isomorphically to a* 2-*component of* $N_G(E)$.

Proof. Pick $B_1 \subseteqq P$ maximal in the partial order indicated in (4) such that $H_1 = N_G(B_1)$ and $V_1 = \langle J_{B_1}^{L(H_1)} \rangle$ satisfy $V_1 = [V_1, a]$. By (ii) and L-Balance we have that $B_1 \neq 1$ and (4) holds. By Lemma 3.3 we can find $x \in G$ such that for $B_1^x = B$ we have

(a) $N_S(B) \in \mathrm{Syl}_2(N_G(B) \cap N_G(J_B O(C_G(B))))$.

(b) $N_S(B_1)^x \subseteqq S$.

(c) $J_B O(C_G(B)) = [J_{B_1} O(C_G(B_1))]^x$.

Let $H = N_G(B)$ and $V = \langle J_B^{L(H)} \rangle$. V_1^x is the normal closure of $(J_{B_1})^x$ in $L(H)$. By (c) $V_1^x \subseteqq \langle (J_B O(C_G(B)))^{L(H)} \rangle$. As V_1^x is a product of 2-components of $L(H)$, it follows that $V_1^x \subseteqq V$. In particular $|V_1: Z^*(V_1)| > |J: Z^*(J)|$ implies the same for V whence $V = [V, a]$. Let $Q = N_S(B)$. By (a) $Q \cap C_H(V/O(V)) \in \mathrm{Syl}_2(C_H(V)O(V))$, and maximality of $|B| = |B_1|$ implies $B = C_Q(V/O(V))$. Now (2) and (3) are immediate and (b) implies (4). Also by (3) $B \in \mathrm{Syl}_2(O_{2'2}(H))$.

J_B is a 2-component of $C_G(B) \cap C_G(a) = C_G(B\langle a \rangle)$. Consequently J_B is a 2-component of $N_G(B\langle a \rangle)$. From the structure of H, $\overline{N_H(B\langle a \rangle)} = N_{\bar{H}}(\overline{B\langle a \rangle}) = C_{\bar{H}}(\bar{a})$. Thus $\overline{J_B}$ is a 2-component of $C_{\bar{H}}(\bar{a})$. As H satisfies the $B(G)$-conjecture, $[J_B, O(J_B)] \subseteqq O(H)$ whence $\overline{J_B}$ is quasisimple and $\overline{J_B}$ is a component of $C_{\bar{H}}(\bar{a})$. We claim that if $\bar{t} \in \mathrm{Inv}(C_{\bar{H}}(\overline{J_B}))$, then $\overline{J_B}$ is a component of $C_{\bar{H}}(\bar{t})$. Let $Y = J_B O_{2',2}(H)$. As $J_B O(H) = O(Y)L(Y)$ is characteristic in Y, (3) implies $Q \in \mathrm{Syl}_2(N_H(Y))$. Accordingly $\bar{Q} \in \mathrm{Syl}_2(N_{\bar{H}}(\overline{J_B}))$ and we may assume $t \in C_Q(\overline{J_B})$ projects onto $\bar{t}$. We have $t^2 \in Q \cap O_{2',2}(H) = B$. Let $T = \langle t, B \rangle$; by maximality of $|B|$, $N_G(T)$ has a 2-component L with $J_T \subseteqq L$, $J_T O(L) = L$. Repeating the argument we used for $B\langle a \rangle$, we see that $\bar{L}$ is a component of $C_{\bar{H}}(\bar{t})$. Since $\bar{L}$ is quasisimple, $\bar{L} = \overline{J_T} O(\bar{L})$ implies $\bar{L} = \overline{J_T}$. Likewise $\overline{J_B} = \overline{J_T} = \bar{L}$ and our claim is proved.

Now by [8, Proposition 4.1] Theorem 5 of [2] applies to a maximal product of pairwise commuting $\bar{H}$-conjugates of $\bar{J}$. From (2) and hypothesis (iii) we have $[\bar{J}, \bar{J}^{\bar{h}}] \neq 1$ for any $\bar{h} \in \bar{H}$. Further if $\bar{t} \in C_{\bar{H}}(\bar{J}) \cap C_{\bar{H}}(\bar{J}^{\bar{h}})$, then $\bar{J}$ and $\bar{J}^{\bar{h}}$ are components of $C_{\bar{H}}(\bar{t})$, which forces $\bar{J} = \bar{J}^{\bar{h}}$ (else $[\bar{J}, \bar{J}^{\bar{h}}] = 1$) whence (1) holds.

LEMMA 3.7. *Let G be a group with commuting involutions a and x, and let $D = O(C_G(x)) \cap C_G(a)$. Either $D \subseteqq O(C_G(a))$ or there is a 2-component J of $C_G(a)$ such that $[J, D] = J = [J, x]$.*

Proof. See [28, Lemma 2.5].

LEMMA 3.8. *Let (a, x, J) be an unbalancing triple of G. If $b \in \mathrm{Inv}\,(C_G(a))$ with $[b, x] = 1$, $[b, J] \subseteqq O(J)$ and J corresponds to the 2-component L of $C_G(b)$, then (b, x, L) is an unbalancing triple of G.*

Proof. Let $F = O(C_G(x)) \cap C_G(a)$. F acts nontrivially on $J/O(J)$ and as $[J, b] \subseteqq O(J)$, $[F, b]$ centralizes $J/O(J)$. $[x, b] = 1$ implies that b acts on F, so $F = C_F(b)[F, b]$ and $E = C_F(b)$ acts nontrivially on $J/O(J)$. Hence E acts nontrivially on $J_b/O(J_b)$ and $LL^a/O(LL^a)$. Note that $LL^a = \langle J_b^{L(C_G(b))} \rangle$ implies that E normalizes LL^a. Likewise $\langle x \rangle$ acts nontrivially on $LL^a/O(LL^a)$. Now apply Lemma 3.7 to the group $LL^aE\langle a, x, b \rangle$ to obtain the desired conclusion.

LEMMA 3.9. *Let G be a group such that all 2-local subgroups satisfy the $B(G)$-Conjecture and such that for every unbalancing triple (a, x, J), $m(J) \geqq 2$. For any unbalancing triple (a, x, J) there exists a maximal unbalancing triple (b, y, L) with $J \twoheadrightarrow L$.*

Proof. It suffices to show that if (a, x, J) is not maximal, then there is an unbalancing triple (b, y, L) with $J \to L$ such that one of the following occurs:

(1) $|L: O(L)| > |J: O(J)|$;

(2) $|L: O(L)| \geqq |J: O(J)|$ and $m(C_G(b) \cap C_G(L/O(L))) > m(C_G(a) \cap C_G(J(O(J))))$:;

(3) $|L: O(L)| \geqq |J: O(J)|$ and $m(C_G(b) \cap C_G(L/O(L))) \geqq m(C_G(a) \cap C_G(J(O(J))))$ and $|C_G(b) \cap N_G(L)|_2 > |C_G(a) \cap N_G(J)|_2$.

Pick $b \in \mathrm{Inv}\,(C_G(a) \cap C_G(J/O(J)))$ so that condition (1) or (2) of the definition of maximal unbalancing triple fails. Pick $S \in \mathrm{Syl}_2\,(C_G(a) \cap N_G(J))$ with $Q = C_S(b) \in \mathrm{Syl}_2\,(C_G(\langle a, b \rangle) \cap N_G(J))$. Let $P = C_S(J/O(J))$.

If condition (1) fails, G has an unbalancing triple (b, y, L) such that J corresponds to L and $y \in C_G(a) \cap N_G(J)$. Thus $y \in C_G(\langle a, b \rangle) \cap N_G(J)$ and we may assume $y \in Q$. If condition (2) fails, then $b \in Z(S)$, and as we may assume by conjugation in $C_G(a) \cap N_G(J)$ that $x \in S$, Lemma 3.8 guarantees that (b, x, L) is an unbalancing triple where L is any 2-component of $C_G(b)$ to which J corresponds. In either case if J does not correspond isomorphically to L, apply Lemma 3.4 to find $b_1 \in \mathrm{Inv}(P)$ with $Q \subseteq C_S(b_1)$ such that J corresponds to the 2-component L_1 of $C_G(b_1)$ and either (1) or (2) above holds with b_1 and L_1 in place of b and L respectively. By Lemma 3.8 again either (b_1, y, L_1) or (b_1, x, L_1) is an unbalancing triple.

Thus we may assume that J corresponds isomorphically to L. Thus it is condition (2) which fails; $b \in Z(S)$ but $S \notin \mathrm{Syl}_2(C_G(b) \cap N_G(L))$. But again (b, x, L) is an unbalancing triple, and as J corresponds isomorphically to L, it is clear that (3) above holds.

Finally, the following result from [25] will be used. For any group G let $G^2 = \langle g^2 \mid g \in G\rangle$.

LEMMA 3.10. *Let G be a group with $P \in \mathrm{Syl}_2(G)$ and $Q \subseteq P$ weakly closed in P with respect to G. Suppose $x \in \mathrm{Inv}(Q \cap G^2) - N_G(Q)^2$. Then there exists a group $S \subseteq Q$ such that*

(1) *$C_Q(x) \subseteq S$.*

(2) *The transfer $V_{P \to S}(x) \notin \Phi(Q)$.*

4. Proof of Theorem A. As we noted after Lemma 1.1 it suffices to prove the Unbalanced Group Conjecture for a minimal unbalanced group G. Let (a, x, J) be an unbalancing triple in G. By Lemma 3.1 we may assume $F^*(G)$ is simple whence $Y = \langle x, O(C_G(x)) \cap C_G(a), J\rangle$ is a proper unbalanced subgroup of G as $Y \subseteq C_G(a) \subset G$. Applying the *U.G.*-Conjecture to Y, we see that $J/Z^*(J)$ is isomorphic to one of the simple groups listed in the conclusion of the *U.G.*-Conjecture. By Theorems 1.2–1.5 $J/O(J)$ is isomorphic to $L_2(q)$, q odd, A_7, $\hat{L_3}(4)$ or He.

By Lemma 3.9 we may assume that (a, x, J) is a maximal unbalancing triple in G. Assume Hypothesis S holds. By Theorem B, either we have the hypotheses of Theorem 1.7 satisfied by G or we have the hypotheses of Lemma 3.5 satisfied by J and G. In any case, by applying Theorem 1.7 we may assume that

(1) Either $J/Z^*(J) \cong L_3(4)$ or $J/O(J) \cong He$.

(2) J is maximal in G.

(3) If (b, y, K) is *any* unbalancing triple in G, then $K/Z^*(K)$ is isomorphic to $L_2(5)$, $L_2(7)$, $L_3(4)$ or He.

Now G satisfies the hypotheses of Theorem 1.6. Thus $F^*(G)$ is isomorphic either to $L_3(4)$ or to He. Thus the Unbalanced Group Theorem is proved.

5. The proof of Theorem B, Part 1. Throughout this section and the next, G will be a fixed counterexample to Theorem B and (a, x, J) will be a fixed maximal unbalancing triple of G with $J/O(J)$ isomorphic to A_7 or to $L_2(q)$, q odd, or $J/Z^*(J)$ isomorphic to $L_3(4)$.

Pick $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$ with $x \in S$ and define $D = S \cap J$, $P = C_S(J/O(J))$. Conclusion (3)(b) of Theorem B is part of the definition of maximal unbalancing triple, so we assume that (a, x, J) does not satisfy conclusion (3)(a). Choose $B \subseteq P$ to satisfy the conclusion of Lemma 3.6, and let $V = \langle J_B^{L(N_G(B))} \rangle$. From conclusion (1) of Lemma 3.6 and Hypothesis (c) of Theorem B we have that $V/O_{2',2}(V)$ and hence V itself are products of groups of restricted type. In fact by L-Balance, $V = KK^a$ where K is a 2-component of $N_G(B)$ and if $K \neq K^a$, then $K/Z^*(K) \cong J/Z^*(J)$. We prove Theorem B by considering the possibilities for K and showing that each possibility leads to a contradiction. By Lemma 3.1 we may suppose that $F^*(G)$ is simple.

LEMMA 5.1. $B \cap B^x = 1$.

Proof. If not, then as $x^2 = 1$, we may choose $b \in \mathrm{Inv}(C_B(x))$. By Lemma 3.8 and the definition of maximal unbalancing triple, J corresponds isomorphically to a 2-component L of $C_G(b)$. In particular $[L, a] \subseteq O(L)$. Now

$$J_B = O^{2'}(C_J(B)) \subseteq O^{2'}(C_J(b)) = J_b \subseteq L.$$

As $L(N_G(B)) = L(C_G(B))$ and $L(C_G(B)) \subseteq L(C_G(b))$ by L-Balance, we have

$$K = \langle J_B^{L(C_G(B))} \rangle \subseteq \langle J_b^{L(C_G(b))} \rangle = L .$$

From Lemma 3.6, $KK^a = [KK^a, a]$, whence $KK^a \subseteq [L, a] \subseteq O(L)$, which is impossible. We conclude that $B \cap B^x = 1$.

We now know that G satisfies the following hypothesis with G in place of H.

Hypothesis 5.2. H is a group such that

(1) All proper sections of H satisfy the *U.G.*-Conjecture;

(2) The solution to the standard component problem for $L_2(q)$, q odd, A_7 and $\hat{L_3}(4)$ in proper sections of H is a central product of groups of restricted type;

(3) There exists $a \in \mathrm{Inv}(H)$ and J a 2-component of $C_H(a)$ with $J/O(J) \cong L_2(q)$, q odd, $J/O(J) \cong A_7$, or $J/Z^*(J) \cong L_3(4)$;

(4) For any $S \in \mathrm{Syl}_2(C_H(a) \cap N_H(J))$ with $P = C_S(J/O(J))$, there exists $B \subseteq P$ with $B \neq 1$ such that for $W = N_H(B)$ and $\bar{W} = W/O_{2',2}(W)$, $\bar{J}_B$ is a component of $C_{\bar{W}}(\bar{a})$ and is standard in $\bar{W}$; and further

(5) For $KK^a = \langle J_B^{L(W)} \rangle$, we have $KK^a = [KK^a, a] = L(W)$;

(6) $B \in \mathrm{Syl}_2(C_W(KK^a/O(KK^a)))$;

(7) If $E \subseteq P$ with either $|E| > |B|$ or $|E| = |B|$ and $|N_S(E)| > |N_S(B)|$, then J corresponds isomorphically to a 2-component of $N_G(E)$;

(8) There exists $x \in S$ with $B \cap B^x = 1$ and $x^2 \in N_P(B)$;

(9) $N_S(B)$ contains a Sylow 2-subgroup of $L(W) \cap N_H(J_B O(W))$.

By the hypothesis of Theorem B, G satisfies conditions (1)-(3) of Hypothesis 5.2. Hypothesis (a) of Theorem B, Lemma 3.6 and the assumption that G fails conclusion (1) of Theorem B give (4)-(7) and (9). Lemma 5.1 implies (8).

The advantage of Hypothesis 5.2 is that it is inherited by certain sections of H. Thus we may argue in certain sections of G which do not have an unbalancing triple.

LEMMA 5.3. *Let H satisfy Hypothesis 5.2 and let $a^g \in S - P$ for some $g \in N_H(B)$. The following conditions hold:*

(1) $B = C_P(a^g) \cap C_P(J^g/O(J^g))$;

(2) $a^g \notin C_{\langle D, a\rangle}(x)$.

Proof. Let $a^g = e$. Since $g \in N_H(B)$, clearly $B \subseteq C_P(e) \cap C_P(J^g/O(J^g))$. Suppose that

$$B \subset E \subseteq C_P(e) \cap C_P(J^g/O(J^g))$$

with $|E: B| = 2$. By 5.2(7) J corresponds isomorphically to a 2-component L of $C_H(E)$. As $[e, E] = 1 = [e, a]$, e normalizes J_E and so $L^e = L$. Likewise $e \in S - P$ implies $L = [L, e]$. Let $M = J^g$ correspond to the 2-component N of $N_G(E)$. Our conditions imply $N \neq L \neq N^e$, else $N = N^e = L$ and $M/O(M) \cong J/O(J) \cong L/O(L)$ would force e to centralize $L/O(L)$. Thus $[M_E, J_E] \subseteq O(N)$, and defining $Y = \langle M_E, J_E\rangle$, we have

$$Y/O(Y) = M_E O(Y)/O(Y) \times J_E O(Y)/O(Y) .$$

Further, $O(Y) \subseteq O(L(N_H(E))) = O(L(C_H(E)))$.

Let $V = N_H(B)$ and $\bar{V} = V/O(V)$. Since $|E: B| = 2$, $E \subseteq V$. Since $C_{\bar{V}}(\bar{E}) = \overline{C_V(E)} = \overline{C_H(E)}$, $\overline{L(C_H(E))} = L(C_{\bar{V}}(\bar{E}))$. By 5.2(1) $\bar{V}$ satisfies the $B(G)$-Conjecture, whence

$$[\overline{L(C_H(E))}, O(\overline{L(C_H(E))})] = 1 .$$

Consequently $[\bar{Y}, O(\bar{Y})] = 1$ and $\bar{Y}$ is the central product of $\bar{J}_E$ and $\bar{M}_E$. By Lemma 3.2, $\bar{J}_E = \bar{J}_B$ and $\bar{M}_E = \bar{M}_B = \bar{M}_{B^g} = (\bar{J}_B)^{\bar{g}}$. Thus $[\bar{J}_B, \bar{J}_B^{\bar{g}}] = 1$ contradicting 5.2(4). This proves that there is no such E and (1) holds.

Suppose $e \in C_{\langle D, a\rangle}(x)$. If x normalizes $M = J^g$, then x normalizes B by (1). But $B \cap B^x = 1$ by 5.2(8), so M^x is a 2-component of $C_H(e)$ distinct from M. Thus $L(C_H(a))$ is a product of 2-components $JL_1 \cdots L_t$ with $t \geqq 1$ and we may take L_1 to be a $C_H(a)$-conjugate of J. Further e centralizes $L_i/O(L_i)$, $1 \leqq i \leqq t$. Let $P_i = P \cap L_i \in \mathrm{Syl}_2(L_i)$.

Suppose $(L_i)^x = L_j$; then x normalizes $P_i P_j$, so $P_i J_j \nsubseteq B$. As $e \in$

$\langle D, a\rangle$, $[P_iP_j, e] = 1$, so by (1) P_iP_j must act nontrivially on $M/O(M)$. By L-Balance, $(L_i)_e$ or $(L_j)_e$ is a 2-component of $L(C_{MM^a}(a))$. As $M/O(M)$ is known by 5.2(3), $L(C_{MM^a}(a))$ has at most one component, and it follows that $t = 1$ or $t = 2$ and $L_2 = (L_1)^x$.

In either case, the 2-components of $C_H(a)$ are all conjugate in $C_H(a)$ and hence isomorphic. We may assume that P_1 acts nontrivially on $M/O(M)$. By L-Balance, $(L_1)_e \subseteq MM^a$. If $M = M^a$, then L_1 corresponds isomorphically to M, whence $a \in B$ by (1). But this is not the case. If $M \neq M^a$, let $y = a^{g^{-1}}$. Clearly $[y, a] = [a, e]^{g^{-1}} = 1$ and $[J, J^y] \subseteq O(L(C_G(a)))$. But now since $y \in N_H(B)$, we may argue as in the proof of (1) with $\langle B, y\rangle$ in the role of E and reach a contradiction to 5.2(4).

Lemma 5.3(2) puts a severe restriction on the fusion of a in $N_G(B)$. Using this fact and the properties of K listed in Proposition 2.3, we can immediately rule out many possibilities for K.

LEMMA 5.4. *Let H satisfy Hypothesis 5.2. The possibilities for $J/O(J)$ and $KK^a/O(KK^a)$ are as follows:*

	$J/O(J)$		$KK^a/O(KK^a)$
(1)	A_n, $n = 5, 6, 7$	(1)	A_{n+2}, A_{n+4}, $\hat{A}_n * \hat{A}_n$
(2)	$L_2(5)$, $L_2(7)$	(2)	*A proper covering group of* $L_3(4)$
(3)	$\hat{L_3}(4)$	(3)	*A proper covering group of Suz*
(4)	$L_2(q^2)$, *q odd*	(4)	*A proper covering group of* $L_4(q)$, $U_4(q)$ *or* $\Omega^-(8, q^{1/2})$
(5)	$L_2(q)$, *q odd*, $q > 3$	(5)	$L_2(q^2)$ *with no diagonal automorphism*
(6)	$L_2(q)$, *q odd* $q > 3$	(6)	*a central product* $\mathrm{SL}(2, q) * \mathrm{SL}(2, q)$
(7)	$L_2(q^2)$, *q odd* .	(7)	$\mathrm{PSp}(4, q)$.

In the remainder of this section we shall develop some more general lemmas and use these to eliminate cases (1)-(4) on the above list with the exception of the case $J/O(J) \cong A_6$, $K/O(K) \cong A_8$. Cases (5)-(7) cause the greatest difficulties and are deferred to the next sections.

We fix the notation $Q = N_S(B)$, $T = Q \cap P$.

LEMMA 5.5. *Let H satisfy Hypothesis 5.2. Then $KK^a/O(KK^a) \not\cong \hat{Suz}$ and $Z(J/O(J)) = 1$.*

Proof. By Lemma 5.4, the first assertion implies the second. Suppose $KK^a/O(KK^a) \cong \hat{Suz}$. Then $K = K^a$ and by Lemma 5.4, $|Z^*(K)|$ is even. By Proposition 2.4(c), $Z^*(L(C_K(a)))$ has even order and lies in $Z^*(K)$. From L-balance, $Z^*(J)$ has even order. As B centralizes

$J/O(J)$, $Q \cap J \in \mathrm{Syl}_2(J)$ and $Q \cap Z^*(J) \in \mathrm{Syl}_2(Z^*(J))$. Thus $Q \cap Z^*(J) = S \cap Z^*(J)$ is normalized by x. But $Q \cap Z^*(J) \subseteqq Z^*(J_B) \subseteqq Z^*(K)$, whence $Q \cap Z^*(J) \subseteqq B$ by 5.2(6), contradicting 5.2(8).

LEMMA 5.6. *Let H satisfy Hypothesis* 5.2. *The following conditions hold:*;

(1) $C_P(S) = \langle a \rangle$

(2) $J \trianglelefteq C_G(a)$ *and* $S \in \mathrm{Syl}_2(C_G(a))$;

(3) *If L is a 2-component of $C_G(a)$ distinct from J, then* $Z(L/O(L)) \neq 1$.

Proof. From 5.2(9) we see that Q contains a Sylow 2-subgroup of $L(W) \cap N_H(J_B O(N_H(B)))$. As $J_B O(N_H(B))$ is characteristic in $J_B O_{2',2}(N_H(B))$, $Q \cap L(W)$ projects onto a Sylow 2-subgroup of $N_{\overline{L(W)}}(\bar{J}_B)$ in the notation of 5.2(4). From 5.2(6) we have

$$B = C_Q(KK^a/O(KK^a)) = C_Q(KK^a/O_{2',2}(KK^a)) .$$

It follows now from Proposition 2.3(2) that

$$C_P(S) \subseteqq C_T(Q) \subseteqq \langle a, B \rangle .$$

Now (1) follows from Lemma 5.1. If L is as in (3), then $\Omega_1(C_P(S)) \cap \langle L^S \rangle \neq \langle 1 \rangle$ implies that $a \in \langle L^S \rangle$, whence $Z(\langle L^S \rangle / O(\langle L^S \rangle)) \neq 1$ and (3) holds. Then (3) and Lemma 5.5 imply (2).

LEMMA 5.7. *Let H satisfy Hypothesis* 5.2.

(1) *If $g \in N_H(B)$ and $a^g \in S - P$, then $C_S(a^g) \subseteqq Q$.*

(2) *If $g \in H$ and $a^g \in \langle J, a \rangle - \langle a \rangle$, then $P \cap P^g = 1$.*

Proof. Let $L = a^g$ and $E = C_S(e)$. By Lemma 5.6(2), $J^g \trianglelefteq C_G(e)$; so E normalizes J^g. By Lemma 5.3(1), $E \subseteqq N_S(B) = Q$. This proves (1).

Suppose $F = P \cap P^g \neq 1$ in (2). As $F \subseteqq P$, J_F covers $J/O(J)$ and $J_F = O^{2'}(C_J(F))$. As $e \in C_{\langle J,a \rangle}(F)$, $e \in \langle J_F, a \rangle$; likewise $D \in \mathrm{Syl}_2(J_F)$. By Lemmas 5.4 and 5.5, J has one class of involutions. Hence by replacing g by gh for some $h \in J_F$ we may assume that $e \in \langle C_D(S), a \rangle - \langle a \rangle$.

By Lemma 5.6(2), $S \in \mathrm{Syl}_2(C_G(a))$. Likewise as $S \subseteqq C_G(e)$, $S \in \mathrm{Syl}_2(C_G(e))$ and S acts on J^g. Thus $U = \langle F^S \rangle$ centralizes $J/O(J)$ and $J^g/O(J^g)$. Consequently $U \subseteqq P \cap C_S(J^g/O(J^g))$ and $U \trianglelefteq S$ implies $U \cap Z(S) \neq 1$. From Lemma 5.6(1) applied to $C_H(a)$ and to $C_H(e)$ we obtain $\langle a \rangle = \Omega_1(C_U(S)) = \langle e \rangle$, a contradiction. We conclude that $P \cap P^g = 1$.

At this point we know little about the structure of P. When $P \subseteqq Q = N_S(B)$, B^x normalizes B and as $B^x \cap B = 1$, we have $BB^x = B \times B^x$. Thus B is isomorphic to a subgroup of T/B; in particular, if $|T{:}\,B| = 2$, then $|T| = 4$ and P has maximal class. In the case where $P \not\subseteq Q$, we attempt to recover this advantage in a section of G. We define an element $x_0 \in S$ as follows:

$$\begin{aligned} &x_0 = x \quad \text{if} \quad B \trianglelefteq P \\ &x_0 \in P - N_P(B) \quad \text{otherwise} \\ &\text{with} \quad x_0^2 \in N_P(B)\ . \end{aligned}$$

We let $B_0 = B \cap B^{x_0}$, $S_0 = N_S(B_0)$, $P_0 = S_0 \cap P$, $J_0 = J_{B_0}$, $G_0 = N_G(B_0)$ and $\bar{G}_0 = G_0/B_0$. Note $[D, B_0] = 1$ implies $D = S_0 \cap J_{B_0}$. $\bar{G}_0$ will be the appropriate section of G.

LEMMA 5.8. *$\bar{G}_0$ satisfies Hypothesis 5.2 with H replaced by $\bar{G}_0$, a by $\bar{a}$, J by $\bar{J}_0$, B by $\bar{B}$, S by $\bar{S}_0$, K by $\bar{K}_0$ and x by $\bar{x}_0$.*

Proof. We consider conditions (1)-(9) in turn; 5.2(1) and 5.2(2) are immediate. As $a \in B_0$, $\bar{a} \notin B_0$, $\bar{a} \in \text{Inv}(\bar{G}_0)$. J_0 is a 2-component of $C_G(\langle B_0, a\rangle)$ and hence of $N_G(\langle B_0, a\rangle)$. It follows that $\bar{J}_0$ is a 2-component of $\overline{N_G(\langle B_0, a\rangle) \cap G_0} = C_{\bar{G}_0}(\bar{a})$. Clearly $J/Z^*(J) \cong \bar{J}_0/Z^*(\bar{J}_0)$, so 5.2(3) holds. Since $KK^a = L(N_G(B))$, $[KK^a, B] = 1$. Thus $KK^a = L(N_{G_0}(B))$. As $N_{\bar{G}_0}(\bar{B}) = \overline{N_{G_0}(B)}$, it is straightforward to check 5.2(4)-5.2(6) and 5.2(9). To check 5.2(7) suppose $\bar{E} \subseteq \bar{P}_0$ with $|\bar{E}| > |\bar{B}|$ or $|\bar{E}| = |\bar{B}|$ and $|N_{\bar{S}_0}(\bar{E})| > |\bar{Q}|$. Letting E be the inverse image of E in P_0, we can apply 5.2(7) to obtain that J corresponds isomorphically to a 2-component L of $N_G(E)$. As $[L, B_0] \subseteq [L, E] = 1$, L is a 2-component of $N_{G_0}(E)$ whence $\bar{L}$ is a 2-component of $N_{\bar{G}_0}(\bar{E})$. As $\bar{J}_E = \bar{L}_a$ lies in $C_{\bar{G}_0}(\langle\bar{a}, \bar{E}\rangle)$ and covers $\bar{J}_0/O(\bar{J}_0)$ and $\bar{L}/O(\bar{L})$, it is clear from L-Balance that $\bar{J}_0$ corresponds isomorphically to $\bar{L}$. Finally 5.2(8) is immediate from our choice of x_0 and B_0 and Lemma 5.8 is proved.

It is easy to see that if H satisfies Hypothesis 5.2, then so does $H/O(H)$. By Lemma 5.8, $\bar{G}_0/O(\bar{G}_0)$ satisfies Hypothesis 5.2 and, by our choice of x_0 and B_0, $\bar{G}_0/O(\bar{G}_0)$ satisfies the following hypothesis.

Hypothesis 5.9. H is a group such that in the notation of Hypothesis 5.2

(1) H satisfies Hypothesis 5.2;

(2) $O(H) = 1$;

(3) Either $x \in N_P(Q)$ or $P \subseteq Q$, $x^2 = 1$ and $O(C_J(x)) \not\subseteq O(J)$.

LEMMA 5.10. *Let H be a group satisfying Hypothesis* 5.9. *Then $F^*(H)$ is simple.*

Proof. Let $L = \langle J^{L(H)}\rangle$; by L-Balance, $L = L_1L_1^a$ where L_1 is a 2-component of $L(H)$. As $O(H) = 1$, L_1 is a component. From 5.2(5),

$$KK^a = \langle J_B^{KK^a} \rangle \subseteq L ,$$

as $KK^a \subseteq L(H)$ by L-Balance. If $L \neq L_1$, then $[KK^a, a] = KK^a$ forces $KK^a = L$ and $B \subseteq C_H(L)$. However we claim $C_H(L) = 1$, whether or not $L = L_1$. As $O(H) = 1$, it suffices to show $|C_H(L)|$ is odd. Pick $U \in \mathrm{Syl}_2(H)$ with $S \subseteq U$. If $B \trianglelefteq V \subseteq U$ with $[KK^a, V] \subseteq O(KK^a)$, then $V \subseteq N_H(B)$ implies $V = B$ by 5.2(6). Thus $B = C_U(KK^a/O(KK^a))$, whence $C_U(L) \subseteq B$. By 5.2(8), $C_U(L) \cap C_U(L)^x = 1$. As x normalizes J, $L = L^x$ and we have $C_U(L) = 1$. Because L is subnormal in H, our claim is proved. But now $L = L_1$ and $C_H(L) = 1$ imply that the lemma is valid.

By the argument of the preceding proof we have

LEMMA 5.11. *If H satisfies Hypothesis 5.2, then*

$$B \in \mathrm{Syl}_2(C_H(KK^a/O(KK^a))) .$$

LEMMA 5.12. *Let H satisfy Hypothesis 5.9. The following conditions hold:*

(1) $\langle B, B^x \rangle = B \times B^x$;

(2) *B is isomorphic to a subgroup of T/B;*

(3) *If $|T : B| = 2$, then P is dihedral or semidihedral and $T = \langle a, b \rangle$ where $B = \langle b \rangle$ has order 2. Further $[S, S] \subseteq D \times P$, $[S, S]$ is a direct product of two cyclic groups and $\langle a \rangle \subseteq \Omega_1([S, S]) \subseteq \langle z, a \rangle$ where $z \in \mathrm{Inv}(C_D(S))$.*

Proof. By 5.9(3), $\langle B, B^x \rangle = B \times B^x$. As $B^x \subseteq T = Q \cap P$, (2) holds and implies the first part of (3). By Lemmas 5.4 and 5.5, $D \in \mathrm{Syl}_2(J)$ is dihedral. Pick cyclic groups $\langle d \rangle$ and $\langle p \rangle$ to be of maximum order in D and P respectively and normal in S. As $J/O(J) \cong L_2(q)$ or A_7, the structure of $\mathrm{Aut}(J/O(J))$ forces $[S, S] \subseteq \langle d \rangle P$. As $S/C_S(\langle p \rangle)$ is abelian,

$$[S, S] \subseteq \langle d \rangle P \cap C_S(\langle p \rangle) \subseteq \langle d \rangle C_P(\langle p \rangle) .$$

If $|P| \geqq 8$, $C_P(\langle p \rangle) = \langle p \rangle$ and we are done, so assume $P = T = \langle a, b \rangle$. As $T \trianglelefteq S$, $[b, x] = a$ and $S = \langle x \rangle Q$ where $|S : Q| = 2$. (Recall $Q = N_S(B)$.) Consider $\bar{S} = S/D$; it suffices to show $[\bar{S}, \bar{S}] \subseteq \langle \bar{a} \rangle$. From knowledge of $J/O(J)$, $\bar{S}/\bar{T}$ is isomorphic to a subgroup of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_{2^m}$. In particular $[\bar{S}, \bar{S}] \subseteq \bar{T}$. By 5.2(8), $x^2 \in T$, whence $x^2 \in C_T(x) = \langle a \rangle$. Thus $\bar{x}\bar{T}$ is an involution and $S = \langle x \rangle Q$ implies

$$\bar{S}/\bar{T} = \langle \bar{x} \rangle \bar{T}/\bar{T} \times \bar{Q}/\bar{T} .$$

In particular $\bar{Q}/\bar{T}$ is cyclic, forcing $\bar{Q}$ to be abelian. As $\bar{x}$ acts as an automorphism of order 2 on $\bar{Q}$, $\bar{x}$ inverts $[\bar{Q}, \bar{x}]$. As $\bar{x}$ does not invert

$\bar{b}$ or $\bar{b}\bar{a}$, $[\bar{Q}, \bar{x}] \subseteq \langle \bar{a} \rangle$, and the desired conclusion follows.

Now we return to consideration of the possibilities for K.

LEMMA 5.13. *If $K = K^a$, then $K/O(K)$ is not isomorphic to A_7, A_9 or A_{11}.*

Proof. Suppose the contrary. By Theorem 1.1 of [25], G has no 2-subgroup, B_1, and 2-component, K_1, of $C_G(B_1)$ with $K_1/O(K_1) \cong A_{2n+1}$ for any $n \geqq 9$. In particular, $K/O(K) \cong A_7$. Recall that $T = Q \cap P = N_P(B)$. As $|T: B| = 2$ by Proposition 2.3(1), $|B| = 2$ by 5.12(2). By Lemmas 5.10 and 5.11, Theorem 1.5 is applicable to $\bar{G}_0/O(\bar{G}_0)$ and yields $L(G_0)/O_{2'2}(L(G_0)) \cong A_9$ or He. As $K = K^a$, we have $J/O(J) \cong A_5$ by Lemma 5.4. Since He does not admit an action of a with $L(C_{He}(a))$ having a component of type A_5, $L(G_0)/O_{2',2}(L(G_0)) \cong A_9$. Hence $L(G_0)/O(L(G_0))$ is isomorphic to a covering group of A_9. As $O(L(G_0))L(C_{G_0}(a))/O(L(G_0)) \cong A_5$, we have $L(G_0)/O(L(G_0)) \cong A_9$, contradicting Theorem 1.1 of [25].

LEMMA 5.14. *The case $K/Z^*(K) \cong L_3(4)$ does not occur in G.*

Proof. Assume the contrary. We have $J/O(J) \cong L_2(5)$ or $L_2(7)$ and by Proposition 2.3(1), $|T: B| = 2$. By Lemma 5.8 and Hypothesis 5.9, Lemma 5.12 applies to $\bar{G}_0$, so it suffices to show that the assumption that there is a group H satisfying the hypotheses of Lemma 5.12 with $K/Z^*(K) \cong L_3(4)$ leads to a contradiction. Assume the notation of Lemma 5.12 and Hypotheses 5.2 and 5.9.

By Lemma 5.4, $Z(K/O(K)) \neq 1$ and so $Z^*(K) = BO(K)$ by Lemma 5.11. As $|B| = 2$, $m(K) = 5 > 4 = m(S)$ from Lemma 5.12. Thus $S < U \in \mathrm{Syl}_2(H)$ and we can find $u \in N_U(S) - S$ with $u^2 \in S$. As $a^u \neq a$ and $\langle a \rangle = P \cap Z(S)$ by Lemma 5.5, $P^u \cap P = 1$ and P acts faithfully on $J/O(J)$. Since P has maximal class, P is dihedral of order $\leqq 16$. In particular, $J = L(C_G(a))$.

By the structure of $L_3(4)$, there exists $g \in K$ with $a^g \in azB$. By Lemma 5.7, $a^g = azb$ where $\langle b \rangle = B$. Since $b^x = ab$, $b^{xg} = az$. Let $Y = C_H(\langle a, z \rangle)$ and $K^{xg} = L(C_G(az))$. Note that $S \in \mathrm{Syl}_2(Y)$. If a acts as an inner automorphism on K^{xg}, then $Y \cap K^{xg} = C_{K^{xg}}(a)$ has 2-rank 5, contrary to $m(S) = 4$. Thus a is outer on K^{xg} whence $a \notin [S, S]$, contrary to Lemma 5.12(3).

LEMMA 5.15. *The case $K/O(K) \cong A_{10}$ does not occur in G.*

Proof. In this case $T/B \cong E_4$ by Proposition 2.1(1), and as in Lemma 5.14 it suffices to show that no H exists satisfying Hypothesis 5.9 with $K/O(K) \cong A_{10}$.

Suppose such an H exists. By Lemma 5.12(2) B is elementary abelian of order 2 or 4. By Hypothesis 5.9(3) either $x \in N_P(Q)$ and $B^x \subseteq T = N_P(B)$ or $B \trianglelefteq P$, $B^x \trianglelefteq P$ and $P = T$. In either case $B \trianglelefteq T$, $B^x \trianglelefteq T$ and $B \cap B^x = 1$ imply $T \cong E_8$ or E_{16}. Also x normalizes T in both cases.

We claim a is not fused to any element of B, for if so, then for some $w \in H$, $a \in B^w \subseteq S$. Let $L = L(C_H(B^w)) = K^w$. $L/O(L) \cong A_{10}$ and by Lemma 5.11, $B^w \in \mathrm{Syl}_2(C_H(L/O(L)))$. But then as J and L are distinct components, $D = S \cap J \subseteq B^w$, whence $|D| \leq 4$, not the case.

Now consider $N = N_H(T)$ and let $A = T \cap K \cong E_4$. Then $T = A \times B$ with $N \cap K$ inducing an S_3 on A and centralizing B. If $a \in A$, then $a \sim z$ in $K \subseteq C_H(B)$, contrary to Lemma 5.7. Thus $a \in T - A$. Further, x acts on T and $B \cap B^x = 1$.

Suppose first that $A = A^x$; we see that x normalizes $\langle A, a\rangle = \langle A, b\rangle$ for some $b \in B$. But then all elements of $\langle A, a\rangle - A$ are conjugate under $\langle N \cap K, x\rangle$, contrary to $a \not\sim b$.

If $|A \cap A^x| = 2$, then $AA^x \cong E_8$ and $|C_{AA^x}(x)| = 4$. If $T = AA^x$, then $a \in C_T(x) \subseteq AA^x$, while if $T \cong E_{16}$, then $|B| = 4$ and $T = B \times B^x$ implies $|C_T(x)| = 4$ and again $a \in AA^x$. But $AA^x = A \times \langle b\rangle$ for some $b \in B$ and $\langle N \cap K, x\rangle$ acts as $\mathrm{GL}(3, 2)$ on AA^x contrary to $a \not\sim b$.

Finally suppose $|B| = 4$ and $T = A \times A^x$. If $A^x = B$, then $N_H(B)$ contains a 2-element acting nontrivially on B. By Hypotheses 5.2(4) $N_H(B)$ does not contain an element acting as an outer automorphism on $K/O(K)$, so by Lemma 5.11, KB contains a Sylow 2-subgroup of $N_H(B)$. Thus no 2-element acts nontrivially on B, and $|A^x \cap B| \leq 2$. If $A^x \cap B = 2$, then $\langle N \cap K, x\rangle$ acts as S_5 on T and every element of T is fused to B contrary to $a \not\sim B$. Similarly if $A^x \cap B = 1$, then all involutions of $T - B$ are fused by $\langle N \cap K, x\rangle$ and $B \cap B^x \neq 1$, again a contradiction.

LEMMA 5.16. *The cases* $K/Z^*(K) \cong L_4(q)$, $K/Z^*(K) \cong U_4(q)$ *and* $K/Z^*(K) \cong \Omega^-(8, q^{1/2})$ *do not occur in* G.

Proof. Suppose one of these cases occurs. By Lemma 5.4, $J/O(J) \cong L_2(q^2)$.

Consider first the possibility that a acts as an outer automorphism on $K/O(K)$. By Proposition 2.3(3), $a \sim azZ^*(K)$ in $N_G(B)$, so Lemma 5.3 implies $O(K) \subset Z^*(K)$. As a is outer on $K/O(K)$, we have $|T: B| = 2$. Passing to the section $\tilde{G}_0$ of Lemma 5.8, it suffices to show that no group H exists satisfying Hypothesis 5.9 with $K/Z^*(K) \cong L_4(q)$ or $U_4(q)$ or $\Omega^-(8, q^{1/2})$, a outer on $K/Z^*(K)$ and $|T: B| = 2$.

Suppose H exists. By Propositions 2.7, 2.8 and 2.10 and Lemmas 5.11 and 5.12, $Z^*(K) = O(K)B$ with $B = \langle b\rangle$ cyclic. Further, as

$|T: B| = 2$, B has order 2, P has maximal class and $a \sim azb$ in $C_H(b)$. Thus $b \sim ab \sim az$ in H. Consider $Y = C_H(\langle a, az\rangle)$. Clearly $S \in \mathrm{Syl}_2(Y)$ and S acts on $L = L(C_H(az)) \cong K$. Now let $M = O^{2'}(C_L(a)) = O^{2'}(C_{L(C_H(a))}(az))$. As $J \trianglelefteq C_H(a)$, $C_L(a)$ acts on J and a Sylow 2-subgroup of $O^{2'}(C_L(a))$ must centralize $J/O(J)$. Thus a Sylow 2-subgroup of $\langle M, a\rangle$ is isomorphic to a subgroup of P, a group of maximal class. Let $R \in \mathrm{Syl}_2(M)$. As $[R, a] = 1$ and $\langle R, a\rangle$ is of maximal class, $a \in Z(R)$. Thus $a \in Z^*(M)$ and $M/O(M) \cong \mathrm{SL}(2, r)$, r odd, or $\hat{A}_7$. By Propositions 2.7, 2.8 and 2.10 there is no involutory automorphism α of L with $\alpha \in O^{2'}(C_L(\alpha)) \cong \mathrm{SL}(2, r)$ or $\hat{A}_7$, a contradiction.

We have proved that a does not act as an outer automorphism on $K/O(K)$. Thus $K/Z^*(K)$ is isomorphic to $L_4(q)$ with $q \equiv 3 \pmod 4$ or to $U_4(q)$ with $q \equiv 1 \pmod 4$. By Hypotheses 5.2(6) and the structure of K, $T \cap K$ is cyclic of order at least 4. Again it suffices to show that no H exists satisfying Hypothesis 5.9 with K and a as above.

Suppose such an H exists. As $J/O(J) \cong L_2(q^2)$, $Z^*(K) = O(K)$. If $a \in K$, then $a \sim z$ in K by Proposition 2.2, contrary to Lemma 5.3. Thus $a \in KB - B$. By Proposition 2.1, T/B is cyclic; so by Lemma 5.12, B is cyclic. We have $KB = K \times B$, $T = (T \cap K) \times B$ and $\langle ab\rangle = \Omega_1(T \cap K)$. As $\langle x\rangle$ normalizes $T = N_P(B)$ by 5.9(3), $b^x = ab$ implies $B \cong T \cap K$.

From the structure of $K/O(K)$ we know $T \cap K \cong \boldsymbol{Z}_{2^s}$ where $2^s \| q + \varepsilon$ with $\varepsilon = -1$ when $K/O(K) \cong L_4(q)$ and $\varepsilon = 1$ when $K/O(K) \cong U_4(q)$. If $S \notin \mathrm{Syl}_2(H)$, then we could find $U \in \mathrm{Syl}_2(H)$ with $S \subset U$ and $u \in N_U(S) - S$. As before P^u would act faithfully on $J/O(J)$. But $T \cong \boldsymbol{Z}_{2^s} \times \boldsymbol{Z}_{2^s}$ cannot act faithfully on $L_2(q^2)$.

Thus $S \in \mathrm{Syl}_2(H)$. By Proposition 2.2(2), $(ab)^g = z$. As $b^x = ab$, $b^{xg} = z$. Consider the action of S on $L = K^{xg} \trianglelefteq C_H(z)$. By Propositions 2.7 and 2.8 and the congruences on q, the outer automorphism group of $L/O(L)$ is abelian. As $a \in [B, x] \subseteq [S, S]$, a is inner on $L/O(L)$. Let $L_0 = O^{2'}(C_L(a))$. By Propositions 2.7 and 2.8, $L_0/O(L_0)$ is isomorphic to $\mathrm{SL}(2, q) * \mathrm{SL}(2, q)$ or to $L_2(q^2)$. As $z \in J$, we argue as before that either $[J, L_0] \subseteq O(C_G(a))$ or L_0 is solvable and if $R_0 \in \mathrm{Syl}_2(C_G(L_0))$, then $[J, R_0] \subseteq O(C_G(a))$. Thus either $[D, L_0] \subseteq O(L_0)$ or $[D, R_0] = 1$ for some $R_0 \in \mathrm{Syl}_2(C_G(L_0))$. Since $B^{xg} = C_S(L/O(L))$ is cyclic with $\langle z\rangle = \Omega_1(B^{xg})$, $\langle a, e\rangle$ acts faithfully on $L/O(L)$ for any $e \in \mathrm{Inv}(D - \langle z\rangle)$. From $[L_0, e] \subseteq O(L)$, we deduce by Propositions 2.7 and 2.8 that $L(C_{L/O(L)}(e)) \cong \mathrm{PSp}(4, q)$. But by 5.2(4), J_B projects to a standard component in $K/O(K)$, whence $N_H(B)$ does not contain an involution f with $L(C_{K/O(K)}(f)) \cong \mathrm{PSp}(4, q)$. This contradicts the existence of e, completing the proof of Lemma 5.16.

We conclude this section by collecting the results of Lemmas 5.4, 5.5 and 5.13–5.16.

LEMMA 5.17. *The possibilities for $J/O(J)$ and $KK^a/O(KK^a)$ are as follows*:

	$J/O(J)$		$KK^a/O(KK^a)$
(1)	A_7 *or* $L_2(q)$, *q odd*, $q > 3$	(1)	*A central product* $\hat{A}_7 * \hat{A}_7$ *or* $\mathrm{SL}(2, q) * \mathrm{SL}(2, q)$
(2)	$L_2(q)$, *q odd*, $q > 3$	(2)	$L_2(q^2)$ *with no diagonal automorphism*
(3)	$A_6 = L_2(9)$	(3)	A_8
(4)	$L_2(q^2)$, *q odd*.	(4)	$\mathrm{PSp}(4, q)$.

An immediate corollary of Lemmas 5.11 and 5.17 and Proposition 2.3(1) is the following.

COROLLARY 5.18. $N_P(B) = B \times \langle a \rangle$.

6. The choice of B_0. In this section, we do some technical refinement of the choice of B_0 which is useful in the remaining cases. We first pick x_0 as follows:

$$x_0 = x \qquad \text{if } B \trianglelefteq P$$
$$x_0 \in (Z_i(S) \cap P) - N_P(B) \text{ with } i \text{ minimum and } x_0^2 \in N_P(B) \qquad \text{otherwise}.$$

Assume henceforth that $B \ntrianglelefteq P$. By Corollary 5.18, we have $N_P(B) = B \times \langle a \rangle$ and $\langle a \rangle = Z(S) \cap P$. Thus $i \geqq 2$ and if $i = 2$, then $[B, x_0] = \langle a \rangle$. Suppose that $i > 2$. Then $Z_2(S) \cap P \subseteqq B \times \langle a \rangle$. As $[Z_2(S) \cap P, x] \subseteqq \langle a \rangle$ and $B^x \cap B = 1$, we have $Z_2(S) \cap P = \langle a, b \rangle$ with $b^2 = 1, b^x = ab$. Repeating this argument we see that $Z_3(S) \cap (\langle B, a \rangle) = \langle a, b \rangle$. Thus $x_0 \in Z_3(S) \cap P$, $x_0^2 \in \langle a, b \rangle$ and $[B, x_0] \subseteqq \langle a, b \rangle$.

DEFINITION. If $i = 2$, let $b = 1$. If $i = 3$, let b be the element of $B^\# \cap Z_2(S)$ described above.

LEMMA 6.1. *One of the following holds*:

(1) $x_0 \in Z_2(S) \cap P$ *and* $[B, x_0] = \langle a \rangle$.

(2) $x_0 \in Z_3(S) \cap P$, $Z_2(S) \cap P = \langle a, b \rangle$, $b \in B_0$, $[B, x_0] \subseteqq \langle a, b \rangle$ *and* $|S : C_S(b)| = 2$.

Proof. It remains only to show that $b \in B_0$. If not, then $b^{x_0} = ab$ and $D = \langle b, x_0 \rangle$ is dihedral of order 8. As $D \trianglelefteq S$ and $\langle a, b \rangle \trianglelefteq S$, every element of S induces an inner automorphism on D. Thus S is a central product $D * C_S(D)$. If $P \cap C_S(D) \neq \langle a \rangle$, then as $P \trianglelefteq S$, $Z_2(S) \cap P \cap C_S(D) \nsubseteq \langle a \rangle$, contrary to fact. Thus $P = D$ and $x_0 \in Z_2(S) \cap P$.

We would like to have

$$(*)\qquad N_S(B_0) \in \mathrm{Syl}_2\,(N_G(B_0) \cap C_G(\langle a\rangle[B, x_0]))\ .$$

NOTATION. Set $E = \langle a\rangle[B, x_0]$. Note that $E = \langle a\rangle$ or $E = \langle a, b\rangle$.

Assume that (*) does not hold and let $T \in \mathrm{Syl}_2\,(N_G(B_0) \cap C_G(E))$ with $N_S(B_0) \subsetneqq T$. Note $[S, b] \subseteqq \langle a\rangle$, so $[N_S(B_0), b] \subseteqq \langle a\rangle \cap B_0 = 1$. Pick $g \in C_G(E)$ so that $T^g \subseteqq C_S(E) \in \mathrm{Syl}_2\,(C_G(E))$. Replace B by B^g, x_0 by x_0^g, B_0 by B_0^g. Since $J \trianglelefteq C_G(a)$, $J_E \trianglelefteq C_G(E)$ and $C_G(J_E/O(J_E)) \cap C_G(E) \trianglelefteq C_G(E)$. Thus, as $g \in C_G(E)$, $B^g \subseteqq C_S(J_E/O(J_E)) \cap C_S(E) = C_P(E)$. It follows that B^g satisfies the conclusions of Lemma 3.6. In particular, $N_S(B) \subsetneqq N_S(B_0)$, so $|N_S(B^g)| = |N_S(B)|$. We have proved the following.

PROPOSITION 6.2. *We may choose B, x_0 and B_0 in such a way that the following hold*:

(1) *Hypothesis* 5.2 *holds for G.*

(2) *Hypothesis* 5.9 *holds for $\bar{G}_0$.*

(3) *$N_S(B_0) \in \mathrm{Syl}_2\,(N_G(B_0) \cap C_G(E))$, where $E = \langle a\rangle[B, x_0]$.*

(4) *Either $[B, x_0] = \langle a\rangle$ or $[B, x_0] = \langle a, b\rangle$ with $b \in C_B(x_0)$, $\langle a, b\rangle = Z_2(S) \cap P$.*

7. The cases $K/O(K) \cong L_2(q^2)$ and $K \neq K^a$. In this section we obtain the following reduction of Theorem B.

LEMMA 7.1. *The cases $K/O(K) \cong L_2(q^2)$ and $K \neq K^a$ do not occur in G.*

Combining this with Lemma 5.17, we have the following immediate corollary.

COROLLARY 7.2. *$J/O(J) \cong L_2(q^2)$ and either $K/O(K) \cong \mathrm{PSp}\,(4, q)$ or $q = 3$ and $K/O(K) \cong A_8$.*

We continue the notation and hypotheses of § 5. Moreover we assume throughout this section that eithen $K/O(K) \cong L_2(q^2)$ or $K \neq K^a$. We prove Lemma 7.1 via a sequence of reductions.

$$(1)\qquad |B| > 2\ .$$

Proof. Assume that $|B| = 2$. If $K/O(K) \cong L_2(q^2)$, then by Lemma 5.11, one of the conclusions of Theorem B holds. If $K \neq K^a$, then $m(K) = 1$ by Lemma 5.4 and $F^*(G)$ is known by Theorem 1.3. Again Theorem B holds.

(2) $B_0 \neq 1$. $\bar{B} = \langle \bar{b}_1 \rangle$ has order 2. $F^*(\bar{G}_0)/O(F^*(\bar{G}_0))$ is isomorphic to $\mathrm{PSp}(4, q)$.

Proof. We have $|T: B| = 2$; so $B \cap B^x = 1$ implies $T \subset P$. Thus $\bar{G}_0$ is proper, $B_0 \neq 1$ and $\bar{B} = \langle \bar{b}_1 \rangle$ has order 2. If $K/O(K) \cong L_2(q^2)$, Lemma 5.11 and the conditions of Lemma 5.8 imply that $\bar{K}$ is a standard component in $\bar{G}_0$. Now $|P_0: B_0| \geqq 8$ and by Lemma 5.12(3), $\bar{P}_0$ has maximal class. By Lemma 5.8, $\bar{P}_0$ contains a Sylow 2-subgroup of $C_{F^*(\bar{G}_0)}(\bar{a}) \cap C_{F^*(\bar{G}_0)}(\bar{J}_0/O(\bar{J}_0))$. By the hypotheses of Theorem B, if $K/O(K) \cong L_2(q^2)$, then $F^*(\bar{G}_0)/O(F^*(\bar{G}_0))$ is simple of restricted type and we conclude

$$F^*(\bar{G}_0)/O(F^*(\bar{G}_0)) \cong \mathrm{PSp}(4, q) .$$

If $K \neq K^a$, then $m(K) = 1$ and Theorem 1.3 lead to the same conclusion.

(3) We may assume that $K \neq K^a$.

Proof. Assume $K/O(K) \cong L_2(q^2)$. We shall find $B_1 \subseteqq P_0$ satisfying the conclusions of Lemma 3.6 and with $\langle (J_{B_1})^{L(C_G(B_1))} \rangle = K_1 K_1^a$ with K_1 of 2-rank 1.

By Lemma 5.12(3) and the structure of Aut PSp (4, q), it is clear that

$$(*) \qquad P_0 \in \mathrm{Syl}_2\,(C_{\bar{G}_0}(\bar{a}) \cap C_{\bar{G}_0}(\bar{J}_0/O(\bar{J}_0))) .$$

Further our knowledge of PSp (4, q) tells us that $\bar{P}_0$ contains an involution $\bar{b}_2$ which is 2-central in $\bar{G}_0$. Pick $b_2 \in P_0$ so that b_2 projects to $\bar{b}_2$ and define $B_2 = \langle b_2, B \rangle$. Clearly

$$(**) \quad [J, B_2] \subseteqq O(J) , \quad \text{and} \quad L(C_G(B_2))/O(L(C_G(B_2)) \cong \mathrm{SL}(2, q) * \mathrm{SL}(2, q) .$$

Let $Q_2 = N_{S_0}(B_2)$. As $\langle \bar{b}_1, \bar{b}_2 \rangle \subseteqq \bar{P}_0$ and $\bar{P}_0$ is of maximal class (with $\langle \bar{a} \rangle = Z(\bar{P}_0)$), $|\bar{S}_0: C_{\bar{S}_0}(\bar{b}_2)| = |\bar{S}_0: C_{\bar{S}_0}(\bar{b}_1)|$, whence $|Q_1| = |Q|$. We claim

$$(***) \qquad Q_2 \in \mathrm{Syl}_2(N_G(B_2) \cap N_G(J_{B_2} O(C_G(B_2)))) .$$

If not, apply Lemma 3.3 to find a conjugate, B_3, of B_2 with $B_3 \subseteqq P$ and $|N_S(B_3)| > |Q_2| = |Q|$. By Lemma 3.6(4) applied to B, J_{B_3} must correspond isomorphically to a 2-component of $N_G(B_3)$, not the case. Thus (***) holds and as $Q_2 \subseteqq G_0$, the structure of PSp (4, q) implies

$$(****) \qquad B_2 \in \mathrm{Syl}_2\,(C_G(L(C_G(B_2)))/O(L(C_G(B_2)))) .$$

Finally the conclusions of Lemma 3.6 follow easily from (**), (***) and (****).

NOTATION. Let $L = L(C_G(B_0))$. Note that by Proposition 6.2,

$S \cap L \in \mathrm{Syl}_2(C_L(a))$.

$$(4) \qquad N_G(S) \subseteq C_G(a) .$$

Proof. Suppose that $N_G(S) \not\subseteq C_G(a)$. We have $C_P(S) = \langle a \rangle$ by Lemma 5.5 and $C_L(a)$ contains an involution acting on $L(C_{L/O(L)}(a))$ and hence on $J/O(J)$ as an outer diagonal automorphism. Accordingly, $Z(S) = \langle z, a \rangle$ and $a^u \in \{z, az\}$, for some $u \in N_G(S)$. Then $(P \cap Z(S)) \cap (P^u \cap Z(S)) = \langle a \rangle \cap \langle a^u \rangle = 1$. So $P \cap P^u = 1$ and P^u acts faithfully on $J/O(J)$. Consequently P is isomorphic to a subgroup of $\mathrm{Aut}(L_2(q))$. As $N_S(B_0)$ contains a Sylow 2-subgroup of $C_L(a)$ we may see in L that $N_P(B_0) \supseteq D_1 \times B_0$ with $D_1 \cong D$. The structure of $\mathrm{Aut}(L_2(q))$ forces $|B_0| = 2$ and $D \subseteq PP^u$.

Let $F = J_e(S)$. As $PP^u = P \times P^u$, $m(S) = 6$ and $F \subseteq PP^u$. Likewise $D \times D^u \subseteq F$. Let $B_0 = \langle b \rangle$. Since b^u acts on $J/O(J)$ as a field automorphism, $|D| \geqq 8$. We have $F = D \times \langle b^u \rangle \times D^u \times \langle b \rangle$. If a is fused in G to az, then we may take $a^u = az$, whence $P^u \cap D = 1$, P^u centralizes D and by the structure of $\mathrm{Aut}(L_2(q))$, $|P^u| \leqq 4$. As $|P| \geqq |D||B_0| \geqq 16$, we see that a is not G-conjugate to az. Thus $a^u = z$ and $u^2 \in S$.

Pick $U \in \mathrm{Syl}_2(G)$ with $\langle S, u \rangle \subseteq U$ and $N_U(F) \in \mathrm{Syl}_2(N_G(F))$. Our conditions imply $\langle a, z \rangle = \Omega_1([F, F])$, whence $|N_U(F){:}\, S| = 2$. It follows easily that $F = J_e(N_U(F))$ and $|U{:}\, S| = 2$. Consequently fusion in $Z(F) = \langle a, z, b, b^u \rangle$ is controlled in $N_G(F)$. We have $a^u = z$, $b^x = ab$ and $b^{ux} = b^u z$. The last equation holds because first $[b^u, x] \in zP$ by properties of $\mathrm{Aut}(L_2(q))$ and second, $[b^u, x] \in P^u \cap Z(F) = \langle z, b^u \rangle$. So far we have the following fusion information:

(A) $a^G \cap Z(F) = \{a, z\}$.

(B) $(az)^G \cap Z(F) = \{az\}$.

(C) $b \sim b^u \sim b^u z \sim ab$.

(D) $baz \sim b^u az \sim b^u a \sim bz$.

(E) $bb^u \sim bb^u az$ and $bb^u a \sim bb^u z$.

Pick $R \in \mathrm{Syl}_2(C_G(b))$ with $C_S(b) \subseteq R$. Since $a \in [F, F]$ and $F \subseteq C_S(b) \subseteq R$, $a \in R' \subseteq L$ and $Z(R) = \langle az, b \rangle$. $Z(U) = \langle az \rangle$, so b is not 2-central in G and $R \subset V \in \mathrm{Syl}_2(G)$. Let $v \in N_V(R) - R$. As $F \trianglelefteq V$, x centralizes $\langle az \rangle = [F, F] \cap Z(R)$ and we have $b^v = baz$. Thus $C_b = \{b^G\} \cap Z(F)$ contains at least 8 involutions. If $|C_b| = 8$, then $|U{:}\, C_U(b)| = 8$. But b^u acts on $J/O(J)$ as a field automorphism, and $[S, b^u] \subseteq P^u$ implies $|S{:}\, C_S(b^u)| = 2$. Thus $|U{:}\, C_U(b)| = |U{:}\, C_U(b^u)| = 4$ and $|C_b| > 8$. Our conditions imply $|C_b| = 10$ or 12. As $N_G(F)$ normalizes the series $1 \subseteq [F, F] \subseteq Z(F)$, no element of order 5 can act nontrivially on $Z(F)$, whence $|C_b| = 12$ and

$$(\mathrm{F}) \qquad b^G \cap Z(F) = Z(F) - \langle a, z \rangle .$$

Let $N = N_G(F)$ and $\bar{N} = N/C_N(Z(F))$. N normalizes the series $1 \subsetneqq \langle az \rangle \subsetneqq \langle a, z \rangle \subsetneqq Z(F)$ and we have $\bar{S} = O_2(\bar{N})$, $\bar{N}/\bar{S} \cong S_3$. Suppose $\lambda \in N$ with $|\bar{\lambda}| = 3$ and $|\lambda| = 3^m$ for some $m \geqq 1$. Clearly λ centralizes a, and it follows that λ acts on J and normalizes $D = F \cap J$ and $F \cap P = C_F(J/O(J))$. But then λ normalizes $\langle a, b \rangle = Z(F) \cap P$, whence $|b^N| = 4$, a contradiction.

(5) $\qquad Z_2(S) \cap P = \langle a, b \rangle\ , \quad x_0 \in Z_3(S) \cap P\ , \quad b \in B_0\ .$

Proof. If not, then by § 6, $x_0 \in Z_2(S)$ and $[x_0, B] = \langle a \rangle$, $[x_0, B_0] \subseteq B_0 \cap \langle a \rangle = 1$. Then $B = (B \cap KK^a) \times B_0$ with $B \cap KK^a \subseteq L$. But then $\langle a \rangle = [B \cap KK^a, x_0] \subseteq L$ and $a \sim z$ in L. Then $a \sim z$ in $N_G(S)$, not the case.

(6) $\qquad ab \in L \quad \text{and} \quad ab \sim z \text{ in } C_G(B_0)\ .$

Proof. $[B \cap KK^a, x_0] \subseteq L \cap \langle a, b \rangle$. As $b \in B_0$, $b \in L$. The argument in (5) shows $a \notin L$. Thus $ab \in L$ and $ab \sim z$ in L.

NOTATION. $U = C_S(ab)$, $N = N_G(Z(U))$, $C = C_N(Z(U))$.

(7) $\qquad Z(U) = \langle z, a, b \rangle\ .$

Proof. We have $|S \colon U| = 2$ and $N_S(B_0)$ normalizes $B_0 \cap U = \langle b \rangle$, so $N_S(B_0) \subseteq U$. $N_S(B_0)$ contains a 2-element acting as an outer diagonal automorphism on $J/O(J)$, so $Z(U) \subseteq \langle z \rangle P$ where $\langle z \rangle = C_S(D)$. Also $Z(S) = C_{Z(U)}(x) = \langle z, a \rangle$, forcing $Z(U) = \langle z, a, b \rangle$.

(8) $N/C \cong S_3$. The G-classes of $Z(U)$ are:

$$a \sim zb \sim azb\ ; \quad b \sim ab \sim z\ ; \quad az\ .$$

Proof. First, by the structure of $\mathrm{PSp}(4, q)$, $\langle ab, z \rangle$ is normal in a Sylow 2-subgroup; R, of L. (Note that $L/O(L) \cong \mathrm{PSp}(4, q)$ not $\mathrm{Sp}(4, q)$ because $KK^a/O(KK^a) \cong \mathrm{SL}(2, q)^*\mathrm{SL}(2, q)$.) Further, for some $r \in \mathrm{Inv}(R)$, $(ab)^r = z$, whence r normalizes $\langle ab, z \rangle$ and centralizes $b \in B_0$. Thus $r \in N$ and $(ab)^r = z$. Now $S \subseteq N$ and $|S \colon S \cap C| = |S \colon U| = 2$, so $|N \colon C|_2 = 2$. Further $N_G(S) \subseteq C_G(a)$ and the structure of $C_G(a)$ imply that none of the involutions in $Z(S)$ are G-conjugate. Thus $N = \langle r, x, C \rangle$, $N/C \cong S_3$ and the G-classes of $Z(U)$ are as described.

(9) $Z(U)$ has at most two G-classes of involutions.

Proof. As $C \subseteq C_G(\langle a, z \rangle)$, C acts on J and centralizes z. It follows that $C/C_C(J/O(J))$ has a normal 2-complement. Further if X is the largest subgroup of $C_C(J/O(J))$ normal in N, then $C_C(J/O(J)) \cap Z(U) = \langle a, b \rangle$ forces $X \cap Z(U) = 1$, whence $X \subseteq O(C)$. We conclude that $C =$

$UO(C)$ and U acts faithfully on a direct product of 3 copies of $J/O(J)$. Roughly speaking each pair of copies generates a $\mathrm{PSp}(4, q)$.

Let $N_1 = N_G(U)$. Clearly N_1 covers $N/O(C)$ and $x \in N_1$. Further $rw = g \in N_1$ for some $w \in O(C)$. As g centralizes $B_0O(C)/O(C)$ and normalizes U, $\langle g, r\rangle$ centralizes B_0 and acts on L. Also g^2 centralizes U and we can find a conjugate, y, of g such that $\langle g, y\rangle$ acts on U as a dihedral group of order 6. In particular, $\langle g, y\rangle$ covers N/C and we may assume that $[y, a] = 1$.

It follows from $y \in C_G(a)$ that $D = D^y$ whence $|N_1: N_{N_1}(D)| = 3$. As $D \cap Z(U) = \langle z\rangle$ and $D^g \cap Z(U) = \langle ab\rangle$, DD^g is a direct product. Let $P_1 = DD^g \cap P$. As $DD^g \subseteq L$, $b \notin DD^g$, whence $DD^g \cap D^{gy} = 1$ and $E = DD^gD^{gy} = D \times D^g \times D^{gy}$. Looking in $\mathrm{PSp}(4, q)$ we see that $DD^g = D \times P_1$ with $\langle ab\rangle = P_1 \cap Z(U)$. Since $P_1 \trianglelefteq U$ and $(ab)^y = b$, $P_1P_1^y = P_1 \times P_1^y \subseteq P \cap E$, as y normalizes $P \cap U$. Thus $|P \cap U| \geqq |P_1|^2 = |D|^2$ and as $DP = D \times P$, we have $E = D \times (P \cap E)$. We claim $B_1 = (P \cap E) \cap (P \cap E)^g \subseteq B_0$. Indeed, since $[B_0, g] = 1$, $B_0 \subseteq (P \cap U) \cap (P \cap U)^g$. Suppose $B_0 \trianglelefteq V \subseteq (P \cap U) \cap (P \cap U)^g$. Then V centralizes $J/O(J)$ and $J^g/O(J^g)$. Consequently V centralizes $J_{B_0}/O(J_{B_0})$ and $J^g_{B_0}/O(J^g_{B_0})$. But

$$\langle J_{B_0}O(L)/O(L)\ ,\ \ J^g_{B_0}O(L)/O(L)\rangle = L/O(L)\ .$$

It is immediate from Lemma 5.11 that $B_0 \in \mathrm{Syl}_2(C_G(L/O(L)))$ and we conclude that $B_0 = V$, whence $(P \cap U) \cap (P \cap U)^g \subseteq B_0$ and our claim is valid.

Now $|B: B_1| \leqq |D|^2$ and $D \times D^g \subseteq L$ force $E = D \times D^g \times B_1$ with $D \cong E/DD^g \cong B_1$. Then $DD^g \subseteq L$ implies $B_1 = E \cap B_0$. As $B_1 \trianglelefteq U$ and $[g, B_1] = 1$, we have $|N_1: N_{N_1}(B_1)| = 1$ or 3. As $B \cap Z(U) = \langle b\rangle$, $B_1B_1^y = B_1 \times B_1^y$ and $|N_1: N_{N_1}(B_1)| = 3$. Now $B_1B_1^y \subseteq P$, so $E = D \times (E \cap P) = D \times B_1 \times B_1^y$. Also $P_1 \subseteq B_1B_1^y$ and $DD^g = P_1P_1^g$ imply $D \subseteq \langle B_1^{\langle y, g\rangle}\rangle$. We conclude

$$E = B_1 \times B_1^y \times B_1^{yg}\ .$$

Pick a foursgroup A_1 in B_1. Since $[B_1, g] = 1$, we have

$$A = \langle A_1^{\langle y, g\rangle}\rangle = A_1 \times A_1^y \times A_1^{yg} \cong E_{64}\ .$$

Consequently $A \cap L = A \cap DD^g \cong E_{16}$ and, looking in $\mathrm{PSp}(4, q)$, we see that $N_L(A \cap DD^g) = N_L(A)$ acts as A_5 or S_5 on $A \cap DD^g$ and centralizes A_1. More precisely we have:

(a) $A \cap DD^g \cap P = DD^g \cap (A \cap P) = DD^g \cap A_1A_1^y = A_1^g$.

(b) Every involution in DD^g is $N_L(A)$-fused into A_1^y.

(c) A_1^y contains two L-classes of involutions.

Repeating this argument with $A_1^yA_1^{yg}$ in place of $A_1A_1^y$, we see that all the involutions in A are $N_L(A)$-fused into A_1^{yg}, whence A intersects at most 2 G-classes of involutions. As $Z(U) \subseteqq A$, we are done.

Since (8) and (9) contradict each other, we have proved Lemma 7.1.

8. The case $K/O(K) \cong \mathrm{PSp}(4, q)$ or A_8. In this section we handle the remaining possibilities for K and thus complete the proof of Theorem B. In Lemma 8.1, we determine the structure of $\bar{G}_0$ in the case when $|P| \geqq 8$. In particular, this determines G if $|P| \geqq 8$ and $|B| = 2$, since $G = \bar{G}_0$ if $|B| = 2$. In Lemma 8.2, we use this information to completely eliminate the case $|P| \geqq 8$ and in Lemma 8.3 we treat the case $|P| = 4$.

LEMMA 8.1. *Suppose that* $|P| \geqq 8$. *Then* $F^*(\bar{G}_0/O(\bar{G}_0)) \cong A_{10}$, $L_4(q)$ *or* $U_4(q)$.

Proof. For simplicity of notation, we shall write $H = \bar{G}_0$ and use the notation of Hypothesis 5.9. Our first task will be to determine a Sylow 2-subgroup of H. By Lemma 5.12, $T \cong E_4$ and P has maximal class and $[S, S]$ is abelian. Looking in K, we see that $S \notin \mathrm{Syl}_2(H)$. Since $J/O(J) \cong A_6$ or $L_2(q^2)$, we must have $Z^*(K) = O(K)$. Let $F = J_e(S)$ and pick $U \in \mathrm{Syl}_2(H)$ with $S \subset U$ and $N_U(F) \in \mathrm{Syl}_2(N_H(F))$.

(1) $$|U: S| = 2 .$$

Proof. Let $B = \langle b \rangle$. As we have noted in similar circumstances above, $Q \in \mathrm{Syl}_2(C_H(\langle a, b \rangle))$. Looking in K we see that $Q_1 = Q \cap K\langle a, b \rangle = D \times \langle \tau \rangle \times \langle a \rangle \times \langle b \rangle$, where τ is a non-2-central involution of K and τ acts as a transposition or field automorphism on $J/O(J)$. For later use we record the existence of $g \in N_K(Q)$ with $a^g = a\tau$, $(a\tau)^g = a$.

Returning to S, we have $m(S) = 5$ and $F = D \times \langle \tau \rangle (F \cap P)$. Since $\langle a, b \rangle \subseteqq C_P(\tau)$, either τ or τb centralizes a cyclic subgroup of order at least 4 in P. Thus $|F \cap P| \geqq 8$, and as $F \cap P \trianglelefteq P$, $|P: F \cap P| \leqq 2$. Consequently $\Omega_1([F, F]) = \Omega_1([S, S]) = \langle a, z \rangle$. (Note that $|D| \geqq 8$.) From $\langle a, z \rangle \trianglelefteq N_U(F)$, we deduce $|N_U(F): S| = 2$. We claim $F = J_e(N_U(F))$. If not we could find $A \cong E_{32}$ with $x \in A$, $v \in N_U(F) - S$. Since $P \cap P^v = 1$, $A \cap P = 1$ and $A \cap S$ acts faithfully on $J/O(J)$. But this is impossible because $m(A \cap S) = 4$. Thus $F = J_e(N_U(F))$, whence $|U: S| = 2$.

(2) $Z(F) = \langle a, z, \tau b \rangle$. We can divide $Z(Q)^\#$ into the following sets of H-conjugates:

$$C_1 = a^H \cap Z(Q): \quad z \overset{v}{\sim} a \overset{g}{\sim} a\tau \overset{x}{\sim} \tau$$

$$C_2 = (az)^H \cap Z(Q): \quad az \overset{g}{\sim} az\tau \overset{x}{\sim} z\tau$$

$$C_3: \quad a\tau b \overset{g}{\sim} ab \overset{x}{\sim} b$$

C_4: $a\tau bz \overset{g}{\sim} abz \overset{x}{\sim} bz \sim \tau b$
C_5: τbz.

Proof. Picking $v \in U - S$, we have $a^v = z$ or $a^v = az$. In the latter case, $P^v \cap P = 1 = P^v \cap D$, whence P^v acts faithfully on $J/O(J)$ and centralizes D, forcing $|P^v| \leqq 4$. Thus $a^v = z$. As P^v still acts faithfully on $J/O(J)$ and has maximal class, $|P^v| = |P| \leqq 2|D|$. Further az is 2-central in H.

Let $\langle p \rangle$ be the cyclic subgroup of index 2 in P and consider how the fours group $\langle \tau, b \rangle$ acts on $\langle p \rangle$. If $\langle \tau, b \rangle$ acts unfaithfully, then $F = D \times J_e(\langle \tau \rangle P) = D \times \langle \tau \rangle \Omega_1(P)$, and $Z(F) = \langle z, \sigma, a \rangle$ with $\sigma \in \{\tau, \tau b\}$. If $\langle \tau, b \rangle$ acts faithfully, $F = D \times \langle \tau, b \rangle \langle p^2 \rangle$ and again $Z(F) = \langle z, \sigma, a \rangle$ with $\sigma \in \{\tau, \tau b\}$. As $N_H(F)$ controls fusion in $Z(F)$ and $Z(F) \cap [F, F] = \langle z, a \rangle$, we see

$$(*) \qquad a^H \cap Z(F) = \{a, b\} \ ; \quad (az)^H \cap Z(F) = \{az\} \ .$$

Consider the action of $\langle x, g \rangle$ on $Z(Q) = \langle z, \tau, a, b \rangle$. As $\langle z \rangle = \Omega_1([Q, Q])$, we have

$$z^g = z \ , \quad b^g = b \ , \quad a^g = a\tau \ , \quad (a\tau)^g = a \ .$$

In particular, $a\tau \sim a$ implies $a\tau \notin Z(F)$, whence $\sigma = \tau b$. Since τ normalizes $N_P(\langle a, b \rangle) = \langle a, b, x \rangle$ and $\langle a, b, x \rangle \cong D_8$, it follows that $x \in F$ and $[x, \sigma] = 1$. We have

$$z^x = z \ , \quad b^x = ba \ , \quad a^x = a \ , \quad \tau^x = \tau a \ .$$

Since $\sigma = \tau b \in Z(F)$, $a\tau b \in Z(F)$ and we can pick $R \in \mathrm{Syl}_2(C_H(a\tau b))$ with $F \subset R$. As $a\tau b \sim b$ and $\langle b \rangle \in \mathrm{Syl}_2(C_H(K/O(K)))$, we know that $Z(R) \cong E_4$. Thus $\langle az \rangle = C_F(R)$ and $a \sim z$ in R. Consequently

$$\tau b = a(a\tau b) \sim za\tau b$$

in R. This gives all of the desired fusion. Since C_3 through C_5 have representatives in $Z(F)$, we have $a^H \cap Z(Q) = C_1$, $(az)^H \cap Z(Q) = C_2$.

$$(3) \qquad C_3 \not\sim C_4 \ .$$

Proof. Suppose that $C_3 \sim C_4$; then $b \sim bz$. Let $Y = C_H(\langle a, bz \rangle)$. We will show that $Y/O(Y)$ is a 2-group, contradicting any possible action of a on $L(C_H(bz)) \cong K$.

If $L(C_H(a)) \neq J$, then as P has maximal class, Lemma 5.6 implies that $L(C_H(a))$ contains a single additional 2-component, L, with $L/O(L)$ isomorphic to $\hat{A}_7$ or $\mathrm{SL}(2, q_1)$ for some odd prime q_1. Further $a \in \mathrm{Syl}_2(Z^*(L))$ and P is semidihedral with $\langle a \rangle = C_P(L/O(L))$. Now $T = N_P(\langle b \rangle) \cong E_4$ implies b acts as an outer diagonal automorphism on

$L/O(L)$. We have

$$C_{JL/O(JL)}(bz) = C_{JO(JL)/O(JL)}(z) \times C_{LO(JL)/O(JL)}(b)$$

is a product of two dihedral groups. As $JLO(C_H(a))/O(C_H(a))$ is self-centralizing in $C_H(a)/O(C_H(a))$, it follows from the structure of Aut $L_2(q)$ that $Y/O(Y)$ is a 2-group.

If $L(C_H(a)) = J$, P contains a Sylow 2-subgroup of $C_H(a) \cap C_H(J/O(J))$ and $C_H(a) \cap C_H(J/O(J))$ is 2-constrained. Clearly $a \in O_{2',2}(C_H(a))$ and as $b \not\sim a$, we obtain

$$C_H(a) \cap C_H(J/O(J))/O(C_H(a)) \cong P \quad \text{or} \quad S_4$$

with $b \notin O_{2',2}(C_H(a))$ in the latter case. Thus

$$C_Y(J/O(J))/Y \cap O(C_H(a)) \cong E_4 \quad \text{or} \quad S_3 .$$

Again $a \not\sim b$ precludes the possibility that a 3-element acts on the E_4. Again we achieve $Y/O(Y)$ a 2-group.

(4) $$C_4 = (\tau b)^H \cap Z(Q) .$$

Proof. As $N_G(F)$ stabilizes the series $1 \subseteq \langle ab \rangle \subseteq \langle a, z \rangle \subseteq \langle a, z, \tau b \rangle$, $N_G(F)$ acts on $Z(F)$ as a 2-group. Two elements of the coset $\langle a, z \rangle \tau b$ are H-conjugate and two are not. The only further fusion possible is $\tau ba \sim \tau bz$.

(5) $$P \cong D \quad \text{and} \quad P \subseteq F .$$

Proof. We have $F = D \times (P \cap F) \times \langle \sigma \rangle$ and as z is fused to a in $N_G(F)$, we have $D \cong P \cap F$. It suffices to show $|P| \leqq |D|$. Assume the contrary. As P^v acts faithfully on $J/O(J)$ and P has maximal class, $|P| = 2|D|$. Further $|P: C_P(\sigma)| \leqq 2$; so $[P, \sigma] \neq 1$ implies $\sigma \sim \sigma a$, i.e., $\tau b \sim \tau ba$, not the case. Thus P and $\langle P, P^v \rangle$ centralize $Z(F) = \langle a, z, \sigma \rangle$. As σ acts as a field automorphism on $J/O(J)$, $\langle P, P^v \rangle$ acts as inner or field automorphisms, but not as diagonal automorphisms. Thus $\langle P, P^v \rangle = P \times P^v \subseteq D \times \langle \sigma \rangle \times P$ forces $|P| \leqq |D|$.

(6) We may assume that $b^H \cap Z(Q) = C_3 \cup C_5$.

Proof. Suppose not. Then τba is 2-central in $N_H(F)$, so b is 2-central in H. We know that a acts as an outer automorphism on K and that the K-classes of involutions in $K\langle a \rangle$ are represented by a, $az\tau$, z, $z\tau$. Thus every involution in $K\langle a \rangle$ is H-fused into $C_1 \cup C_2$. In particular, $b \not\sim K\langle a \rangle$. If a 2-element η acts on K as a field automorphism, then $m(F) = m(H) = 5$ implies $\Omega_1(\langle \eta \rangle) = \langle b \rangle$, else $m(C_H(b)) = 6$. Pick η with $|\langle \eta \rangle|$ as large as possible. If no 2-element acts as a field automorphism, set $\eta = b$. We see that $K\langle a \rangle\langle \eta \rangle$ contains a Sylow

2-subgroup of $C_H(b)$ and of H. As $a \sim z \in K \subseteq O^2(H)$ and $b \not\sim K\langle a\rangle$, transfer gives $\langle K, a, \eta\rangle \cap O^2(H) = \langle K, a\rangle$. As $K\langle a\rangle$ has sectional 2-rank 4, we have by Lemma 5.10 and Proposition 2.11,

$$F^*(H) \cong L_4(q)\ , \quad U_4(q) \quad \text{or} \quad A_{10}\ ,$$

and we are done in this case.

(7) There exists $y \in U$ with $(\tau bz)^y = \tau ba$, with $C_{Z(F)}(y) = \langle a, z\rangle$ and with y acting nontrivially on both $D/[D, D]$ and $P/[P, P]$. $P - \langle a\rangle = P \cap b^H$.

Proof. As $\langle \tau bz\tau, ba\rangle \subseteq Z(F)$ and U covers $N_H(Z(F))/C_H(Z(F))$, there is an element $y \in U$ such that $(\tau bz)^y = \tau ba$. At this point the action of $N_H(Z(F))$ on $Z(F)$ is completely determined. In particular $U/C_U(Z(F)) \cong E_4$ and we can choose y so that either $C_{Z(F)}(y) = \langle a, z\rangle$ or so that $C_{Z(F)}(y) = \langle a\tau, az\rangle$. We choose it so that $C_{Z(F)}(y) = \langle a, z\rangle$.

Now y acts on J and normalizes D. Further since τbz acts as a field automorphism on $J/O(J)$ and $(\tau bz)^y = \tau ba$, the structure of $\mathrm{Aut}\,(J/O(J))$ forces y to act nontrivially on $D/[D, D]$.

Cleary y normalizes P. As $a\tau b \in Z(F)$, $N_H(F)$ contains a Sylow 2-subgroup of $C_H(a\tau b)$. Since $N_H(Z(F))$ acts as a 2-group on $Z(F)$, any Sylow 2-subgroup of $N_H(Z(F))$ contains a Sylow 2-subgroup of $C_H(a\tau b)$. In particular $R = C_U(a\tau b) \in \mathrm{Syl}_2\,(C_H(a\tau b))$. As we have seen, we can choose $v \in R$ such that $a^v = z$. Note that $U = SR$ with $S \cap R = C_U(Z(F))$ and $|U: S| = |U: R| = 2$. Now $D^v \in \mathrm{Syl}_2\,(J^v)$, so the reasoning in the previous paragraph shows that y acts nontrivially on $D^v/[D^v, D^v]$. Let $E = D\langle az, a\tau b\rangle$. We have $D^vE = F$ and $D^v \cap E = \langle a\rangle$. Since y normalizes D^v, y acts nontrivially on the commutator quotient of F/E. But also $PE = F$ and $P \cap E = \langle a\rangle = [P, P]$, so our claim is valid.

(8) $\qquad K/O(K) \cong \mathrm{PSp}\,(4, q) \qquad$ for some odd $q \geqq 3$.

Proof. Suppose $K/O(K) \cong A_8$. Then $L/O(L) \cong A_8$ where $L = L(C_H(a\tau b))$. Now $R \cap L\langle a\tau b\rangle$ contains $A \cong E_{32}$ with

$$N_L(A)/C_L(A) \cong S_3 \wr Z_2\ .$$

Further $N_L(A)$ fuses every involution in $C_{A\cap L}(C_J(a\tau b)/O(C_J(a\tau b)))$ to a. Thus every involution in $C_A(J/O(J)) - \langle a\tau b\rangle$ is fused to a or τb. But $C_A(J/O(J)) = A \cap P \cong E_4$, and so must contain two involutions fused to b, contrary to $a \not\sim b \not\sim \tau b$.

(9) $L(C_H(\langle az, a\rangle)) = 1$. If $q = 3$, $C_H(\langle az, a\rangle) = C_X(az)S$, where $X = O(C_H(a))$.

Proof. As $P \in \mathrm{Syl}_2(C_H(J/O(J)))$ and P is dihedral with $\mathrm{Inv}\, P = (P \cap b^G) \cup \{a\}$, $C_H(J/O(J))$ has a normal 2-complement. Thus by the structure of $\mathrm{Aut}\, L_2(q^2)$,

$$O^{2'}(C_H(a)/O(C_H(a))) = SJO(C_H(a))/O(C_H(a)) .$$

Moreover if $q=3$, $C_H(a)/O(C_H(a)) = SJO(C_H(a))/O(C_H(a))$. As $C_{J/O(J)}(az)$ is dihedral and is of order 8 if $q = 3$, the claims follow.

NOTATION. Set $F = C_H(az)$, $\bar{F} = F/\langle O(F)\langle az\rangle\rangle$, $L = L(C_H(\langle az, a\tau b\rangle))$.

(10) $$q = 3 .$$

Proof. Suppose that $q > 3$. As $C_F(a\tau b)$ covers a subgroup of index 2 of $C_{\bar{F}}(\overline{a\tau b})$, $\bar{L}$ is subnormal in $C_{\bar{F}}(\overline{a\tau b})$ and we conclude that $L(C_{\bar{F}}(\overline{a\tau b})) = \bar{L} \cong L_2(q) \times L_2(q)$. Suppose that $\bar{L}$ is not subnormal in $\bar{F}$. Let $\bar{J}_0$ be a summand of $\bar{L}$ which is not subnormal in $\bar{F}$. As $\bar{a}$ normalizes $\bar{J}_0$, $\bar{a}$ normalizes a Sylow 2-subgroup, $\bar{P}_1$, of $C_{\bar{F}}(\bar{J}_0)$. Let $\bar{E}$ be a maximal $\langle \bar{a}\rangle$-invariant subgroup of $\bar{P}_1$ satisfying:

$$\bar{J}_0 \text{ is not subnormal in } N_{\bar{F}}(\bar{E}) .$$

As $\bar{F}$ satisfies the $B(G)$-Conjecture, it follows easily that $\bar{J}_0$ projects to a standard component of $N_{\bar{F}}(\bar{E})/\bar{E}$. Thus $\bar{K}_0 = \langle \bar{J}_0 L(N_{\bar{F}}(\bar{E}))\rangle$ is a central product of restricted quasi-simple groups. Let $\bar{e}$ be an involution of order 4 in F, $\overline{a\tau b} \not\sim \overline{a\tau b e}$. By Proposition 2.3(4), the possibilities for $\bar{K}_0$ are:

$$\mathrm{SL}(2, q) * \mathrm{SL}(2, q),\ L_2(q^2),\ A_7,\ A_8,\ A_9,\ A_{10}\ ,$$
$$L_3\hat{}(4),\ \mathrm{PSp}(4, \sqrt{q}),\ L_4\hat{}(\sqrt{q}),\ U_4\hat{}(\sqrt{q}),\ \Omega^-(8, \sqrt[4]{q}) .$$

Now $\bar{a}$ normalizes $\bar{E}$, hence $\bar{K}_0$. As $L(C_{\bar{K}_0}(\bar{a})) = 1$, a quasisimple component of $\bar{K}_0$ must be isomorphic to $\mathrm{SL}(2, q) * \mathrm{SL}(2, q)$, $L_2(q^2)$, A_7, A_8, A_9, A_{10}, $L_3(4)$, $\mathrm{PSp}(4, 3)$, $L_4(3)$ or $U_4(3)$ by Proposition 2.12. As $az \in Z(K_0)$, $\bar{K}_0 \not\cong \mathrm{A}_n$ for $n \geqq 8$ by Theorem 1.1 of [22].

If $\bar{K}_0 \cong L_3(4)$, then as e has order 4 in F, $K_0/O(K_0)$ is a 16-fold covering group of $L_3(4)$. As $\bar{F}$ satisfies the $B(G)$-Conjecture, this is impossible by [22]. Suppose that $K_0/O(K_0)$ has a component isomorphic to $\mathrm{SL}(2, q)$, $\mathrm{SL}(2, q^2)$ or A_7. If this component is subnormal in $F/O(F)$, then it is intrinsic and H is known by the main theorem of [4]. This we may repeat our earlier argument to produce a component K_1 with $\bar{K}_1$ not isomorphic to $\mathrm{SL}(2, q)$, $L_2(q^2)$ or A_7. But then we must have $\bar{K}_1 \cong \mathrm{PSp}(4, 3)$, $L_4(3)$ or $U_4(3)$. Then necessarily $q = 9$ and $\bar{J} \cong L_2(81)$. It follows that $C_H(\langle a, az\rangle)$ is 2-nilpotent, but the fixed point subgroup of every involutory automorphism of $\mathrm{PSp}(4, 3)$,

$L_4(3)$ or $U_4(3)$ involves A_4 by Proposition 2.12, a contradiction.

Thus $\bar{L}$ is subnormal in $\bar{F}$ and, again, H is known by the main theorem of [4].

(11) $$q \neq 3 .$$

Proof. Let $R = C_U(a\tau b)$. Since $q = 3$, $K\langle a\rangle\langle b\rangle$ contains a Sylow 2-subgroup of $C_H(b)$. Thus $R \subseteqq L\langle r\rangle\langle a\tau b\rangle$ for some $r \in \mathrm{Inv}\,(R)$ with $r \sim a$. We see that $R = \Omega_1(R)$ and we can pick $v \in \mathrm{Inv}\,((R \cap L) - F)$ with $F \cap L\langle r\rangle = F_1 \times (F_1)^v$, $F_1 \cong D$. The coset vF contains $2\,|\,D\,|$ involutions and $|\,R\colon C_R(v)\,| = |\,D\,|$. If $C_R(v) = C_U(v)$, then $|\,U\colon C_U(v)\,| = 2\,|\,D\,|$ and all involutions in vF are conjugate. By inspection $v \sim_L a$ or az and, correspondingly $va\tau b \sim_L \tau b$ or τbz. As $a \not\sim \tau b$ and $az \not\sim \tau bz$, we conclude that $v \not\sim va\tau b$. Thus $C_R(v) \subset C_U(v)$. It follows that $U = FC_U(v)$ and $S = FC_S(v)$. As $F = P \times P^v \times \langle a\tau b\rangle$, we have $S = PC_S(v)$, $P \cap C_S(v) = 1$.

Recall that there exists $y \in U$ with $\langle a, z\rangle \subseteqq C_U(y)$ and $(a\tau b)^y = z\tau b$. As $q = 3$, we must have S/P isomorphic to a full Sylow 2-subgroup of $\mathrm{Aut}\,(J/O(J))$. It is now clear that we can choose y so that $y \in \mathrm{Inv}\,(C_S(v))$ and y acts as a diagonal automorphism on $J/O(J)$.

Now $U = PC_U(v)$ and as $v \notin \Phi(U)$, $C_U(v) = R_1 \times \langle v\rangle$ with R_1 isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}\, L_2(9)$. Let $P_1 = C_{PP^v}(v)$. Then $P_1 \cong D_8$ and $P_1 \leqq C_U(v)$. As

$$C_U(v)/P_1 = \langle P_1 a\tau b,\ P_1 y,\ P_1 v\rangle$$

with $[a\tau b, y] \in P_1$ and $v \notin C_U(v)'$, we have $C_U(v)/P_1 \cong E_8 \cong U/PP^v$. In particular, $a\tau b \notin \Phi(U)$. We have seen that $N_H(F) = O(N_H(F))U$ and F is weakly closed in U. Thus as $a\tau b \in Z(F)$ and $a\tau b \notin N_H(F)'N_H(F)^2$, we have $a\tau b \notin H'$ by Lemma 3.10. As $a\tau \sim z \in H$, $b \notin H'$. Thus $P \cap H' \cong Z_4$ and J is maximal in H. Thus H is of restricted type and our conclusion follows by inspection of the possibilities.

As (9) and (10) exhaust all cases, we are done.

LEMMA 8.2. $|\,P\,| = 4$.

Proof. Suppose that $|\,P\,| \geqq 8$. By Lemma 8.1, $|\,B\,| \geqq 4$ and, as $B \cap B^x = 1$, $|\,P\,| \geqq 16$. Further, $|\,N_P(B)\colon B\,| = 2$, so B is not normal in P.

We let x_0, B_0 be as in § 6. As $A \ntrianglelefteq P$, Proposition 8.2 holds for B, x_0 and B_0. Let $L = L(C_G(B_0) \trianglelefteq G_0 = N_G(B_0)$. As $\bar{a} \in Z(\langle \bar{x}_0, \bar{B}\rangle)$, $\bar{a}$ acts as an inner automorphism on $\bar{L}$. Thus, by Lemma 8.1, we have

$$\begin{aligned} L/O(L) \cong\ & L_4(q) && q \equiv 3 \pmod 4\,, \text{ or} \\ & U_4(q) && q \equiv 1 \pmod 4\,, \text{ or} \\ & A_{10}\,. \end{aligned}$$

By Proposition 6.3(3),

$$N_S(B_0) \in \mathrm{Syl}_2\,(N_G(B_0) \cap C_G(\langle a\rangle[B, x_0]))\ ,$$

where $[B, x_0] \subseteq \langle a, b\rangle$. It follows from the structure of L as described in Proposition 2.9 that there exists an element t with the following properties:

(a) $t \in N_L(S_0) - S_0$ and $t^2 \in S_0$.

(b) $DD^t = D \times D^t = L \cap S_0$.

(c) D^t acts on $J/O(J)$ as a field automorphism.

(d) $D^t \cap P = C_{D^t}(J/O(J))$ is cyclic of index 2 in D^t.

(e) $(D^t \cap P)B/B_0$ is dihedral and $a \in (\Omega_1(D^t \cap P))B_0 - B_0$.

We assume for the next few steps that $N_G(S) \subseteq C_G(a)$.

(1) x_0 acts nontrivially on D^t.

Proof. Clearly $x_0 \in P_0 = S_0 \cap P$, so x_0 acts on $D^t \trianglelefteq S_0$. Further P_0/B_0 has maximal class by Lemma 5.12 and $\langle x_0, B\rangle/B_0 \cong D_8$. As $D^t \cap P$ is cyclic of order at least 4 and projects isomorphically onto P_0/B_0, either x_0 inverts $D^t \cap P$ or $x_0 \in (D^t \cap P)B_0$. In the latter case $x_0 \notin \langle a, B_0\rangle$ implies that x_0 acts nontrivially on D^t.

(2) Let $\langle a_0\rangle = Z(D^t)$. Then $b \in B_0 \cap Z_2(S)$ and $a_0 = ab$.

Proof. Now $\langle a_0\rangle \subseteq [D^t, x_0] \subseteq [S, x_0] \cap P$. If $x_0 \in Z_2(S)$, then as $\langle a\rangle = Z(S) \cap P$, $a_0 = a$, whence a is fused to z in $N_G(S)$, contrary to $N_G(S) \subseteq C_G(a)$. Thus $x_0 \in Z_3(S)$ and $a_0 \in [S, x_0] \cap P = \langle a, b\rangle$. As $b \in B_0$, $a_0 = ab$.

(3) Let $A = \langle a, z, b\rangle$, $N = N_G\langle A\rangle$, $C = C_N(a)$. Then $N/C \cong S_3$ and $z \sim az \sim b$, $a \sim bz \sim baz$ in N; $az \in Z(N)$.

Proof. Let $U = C_S(a_0)$. Clearly $|S: U| = 2$. Now t interchanges z and ab and fixes b. Since $\langle a, z\rangle \subseteq Z(S)$, $A \subseteq Z(U)$ and $U = C_S(A)$. Picking $y \in S - U$, we have $a^y = a$, $z^y = z$ and as $\langle a, b\rangle \trianglelefteq S$, $b^y = ab$. Thus $\langle y, t\rangle$ acts on A as S_3. Since $N_G(S) \subseteq C_G(a)$, $N_G(S)$ centralizes $\langle a, z\rangle$. So $a \not\sim z \not\sim az$ in G. The conclusion follows.

We may copy the argument in the proof of Lemma 7.1 to conclude that C acts on J and centralizes z. Thus $C/C_C(J/O(J))$ has a normal 2-complement.

(4) $$C = UO(C)\ .$$

Proof. Let X be the largest subgroup of $C_C(J/O(J))$ which is normal in N. Clearly C/X has a normal 2-complement. We claim $|X|$ is odd. If not, then $1 \neq X \cap U \trianglelefteq U$ implies $E = X \cap \Omega_1(Z(U)) \neq 1$.

As X centralizes $J/O(J)$, $X \cap U \subseteq P$ so that $E \subseteq P$. S acts as an involution on E and $C_E(S) \subseteq Z(S) \cap P$ implies $C_E(S) = \langle a \rangle$. Thus $|E| \leqq 4$. But $X \trianglelefteq N$ implies $E \trianglelefteq N$ whence $|E| \geqq |\langle a^N \rangle| = 8$, a contradiction.

(5) Let $N_1 = N_G(U)$, $E = \langle D^{N_1} \rangle$. Then $E = D \times D^t \times D^{ty}$. Every involution in DD^t is conjugate to z or a.

Proof. As N_1 covers $N/O(C)$, $N_1/C_{N_1}(A) \cong S_3$. As $D \trianglelefteq S$, D has 3 N_1-conjugates with centers $\langle z \rangle$, $\langle ab \rangle$, $\langle b \rangle$ and as each conjugate is normal in U, E is a direct sum of the conjugates of D. Since $t \in N$, one of the N_1-conjugates of D projects onto $D^t O(C)/O(C)$. But $D^t \subseteq S_0 \subseteq U$ and U projects isomorphically onto $UO(C)/O(C)$, so D^t is one of the N_1-conjugates of D. Since y normalizes U, we have $E = D \times D^t \times D^{ty}$. Since $DD^t \subseteq L$, $\mathrm{Inv}(DD^t) \subseteq z^L \cup (abz)^L$. As $abz \sim a$ in N, our claim is valid.

(6) $$az^G \cap DD^t \neq \varnothing .$$

Proof. Pick $v \in N_1$ so that $\langle v \rangle$ covers $O(N_1/C_{N_1}(A)) \cong \boldsymbol{Z}_3$ and so that $|v|$ is a power of 3. From the structure of N, $v^3 \in N_1 \cap O(C)$ forces $v^3 \in C_N(U)$. Since D^t acts on $J/O(J)$ as a field automorphism of order 2, we can pick $E_0 = \langle z, e_1 \rangle \subseteq D$ so that

$$|E_0 \cap L(C_J(D^t))| = 2 .$$

Letting $\{e_1, e_2, e_3\}$ be the $\langle v \rangle$-conjugates of e_1, we see that $\langle v \rangle$ normalizes $F = \langle z, e_1, ab, e_2, b, e_3 \rangle \cong E_{64}$, and we may assume that with respect to the basis $\{z, e_1, ab, e_2, b, e_3\}$, v is represented by

$$M_v = \left(\begin{array}{cc|cc|cc} & & & & 1 & 0 \\ & & & & 0 & 1 \\ \hline 1 & 0 & & & & \\ 0 & 1 & & & & \\ \hline & & 1 & 0 & & \\ & & 0 & 1 & & \end{array}\right),$$

where omitted entries are zero. F acts on J as a group of order 8 and $N_J(F)$ acts as S_3 on F. We can pick $r \in N_D(F) - F$ so that r acts as

$$M_r = \left(\begin{array}{cc|cc|cc} 1 & 1 & & & & \\ 0 & 1 & & & & \\ \hline & & 1 & 0 & & \\ & & 0 & 1 & & \\ \hline & & & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right)$$

and we can pick $w \in F$ so that $|w|$ is a power of 3 and $\langle r, w\rangle$ acts on E_0 as S_3. Clearly w centralizes $F \cap P \supseteqq \langle ab, b\rangle$ and by our choice of E_0, $[w, e_2] \neq 1$. From the structure of Aut $L_2(q^2)$, w must centralize e_2z. Likewise w centralizes e_3 or e_3z. We have $F = E_0 \times C_F(w)$ with $C_F(w)$ equal to $\langle ab, b, e_2z, e_3\rangle$ or $\langle ab, b, e_2z, e_3z\rangle$. In either case the action of w on E_0 determines its action on F, and we may assume

$$M_w = \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 1 & 0 & \varepsilon \\ 1 & 1 & 0 & 1 & 0 & \varepsilon \\ \hline & & 1 & 0 & & \\ & & 0 & 1 & & \\ \hline & & & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right)$$

where $\varepsilon = 0$ or 1. In any event

$$M_w(M_vM_w)^2(az) = e_2z \in DD^t .$$

But now (5) and (6) yield a contradiction, as $z \not\sim az \not\sim a$. Thus $N_G(S) \not\subseteq C_G(a)$. $N_G(S)$ acts on $\langle a, z\rangle = [S, S] \cap Z(S)$. If $a^u = az$ for $u \in N_G(S)$, then $P^u \cap P = 1 = P^u \cap D$ implies $[P^u, D] = 1$, whence P^u acts faithfully on $J/O(J)$ as a fours group, contrary to $|P| > 8$. Thus we have the following.

(7) Let $U \in \mathrm{Syl}_2(G)$ with $S \subset U$. Then $|N_U(S){:}\, U| = 2$ and if $u \in N_U(S) - S$, then $a^u = z$ and P^u acts faithfully on $J/O(J)$.

(8) $$B_0 = \langle b\rangle \quad \text{with } b^2 = 1 .$$

Proof. Recall the existence of $t \in N_L(N_S(B_0))$ with $D \times D^t = N_S(B_0) \cap L$ and $D^t \cap P$ cyclic of index 2 in D^t. $[(D^t \cap P) \times B_0]^u$ acts faithfully on $J/O(J)$. The claim is immediate from the structure of Aut $(L_2(q^2))$.

(9) b^u acts on $J/O(J)$ as a field automorphism and $b \underset{S}{\sim} ba$.

Proof. The first assertion is clear. As $b \notin Z(S) \cap P$, $b^u \notin Z(S) \cap P^u$, whence some element, y, of S acts as a diagonal outer automorphism on $J/O(J)$. Considering the action of y on P^u we have

$$b^u \underset{S}{\sim} b^uz$$

and conjugating y by u gives

$$b \underset{S}{\sim} ba .$$

(10) $$Z(J_e(S)) = \langle z, b^u, a, b\rangle \quad \text{and} \quad |U{:}\, S| = 2 .$$

Proof. Since P^u is faithful in $J/O(J)$, $m(P^u)=m(P)\leqq 3$. $[D, D^u]=1$ implies D^u must act on $J/O(J)$ as a subgroup of $\langle z, b^u\rangle$. As $\langle z, b^u\rangle^u = \langle a, b\rangle \subseteqq P$, $\langle z, b^u\rangle \subseteqq P^u$, whence

$$D^u \subseteqq \langle z, b^u\rangle P \subseteqq PP^u .$$

Now $P \cap Z(S) = \langle a\rangle$ and $P^u \cap Z(S) = \langle z\rangle$, so $P \cap P^u = 1$ and $PP^u = P \times P^u$. Further, $D\langle b^u\rangle \cap P = 1$, as $D\langle b^u\rangle$ acts faithfully on $J/O(J)$. Thus P^u contains a subgroup isomorphic to $D\langle b^u\rangle$, whence $m(P^u) = 3 = m(P)$ and $m(S) = 6$. Clearly $J_e(S) = J_e(P) \times J_e(P^u) \cong D \times \langle b^u\rangle \times D \times \langle b^u\rangle$. In particular $Z(J_e(S)) \cong E_{16}$. Also exactly as in the proof of Lemma 7.1, $|U: S| = 2$.

Set $F = J_e(S)$, $A = Z(F)$. Fusion in A is controlled by $N_G(F)$ and $\langle a, z\rangle = [F, F] \cap A$ is invariant under $N_G(F)$. Thus

$$(az)^G \cap A = \{az\}$$
$$a^G \cap A = \{a, z\} .$$

The following fusion occurs in U:

$$ab \sim b \sim b^u \sim b^u z , \quad zb \sim azb \sim azb^u \sim ab^u .$$

(11) $a \sim z$ in $N_G(F) \cap C_G(b)$. Thus $ab \sim bz$.

Proof. $C_G(b) = G_0$ and we know that $F^*(\bar{G}_0/O(\bar{G}_0))$ is simple, where $\bar{G}_0 = G_0/\langle b\rangle$. As $F \subseteqq G_0$, $\bar{a} \in [\bar{F}, \bar{F}] \subseteqq \bar{L}$. Thus $a \in L$ or $ab \in L$. In the latter case $a \sim b$, contrary to $a^G \cap A = \{a, z\}$. Thus $a \in L$ and $a \sim_L z$. Thus $a \sim z$ in $N_{G_0}(F)$.

Now $8 \leqq |b^G \cap A| \leqq 12$. $C_A(u) = \langle az, bb^u\rangle$ and, as bb^u acts on $J/O(J)$ as a field automorphism, $bb^u \notin Z(S)$. Thus $C_A(U) = \langle az\rangle$. Consequently any involution in A which is 2-central in G is fused to az, and so b is not 2-central in G. Since $\langle b, a\rangle = A \cap P \trianglelefteq S$ and $|U: S| = 2$, $|U: C_U(b)| \leqq 2\,|S: C_S(b)| = 4$. Thus $|b^G \cap A| = 10$ or 12. Clearly no 5-element acts on F, so $|b^G \cap A| = 12$ and

$$b^G \cap A = A - \langle a, z\rangle .$$

Consequently there exists $\lambda \in N_G(F)$ of order 3^a acting nontrivially on A. As λ centralizes $\{a, b\}$, λ must normalize $P \cap A = \langle a, b\rangle$, whence λ centralizes $\langle a, z, b\rangle$, a contradiction.

This completes the proof of Lemma 8.2.

LEMMA 8.3. $|P| \neq 4$.

Proof. Assume that $|P| = 4$ and let $P = \langle a, b\rangle$. It is clear that x, the unbalancing involution, acts nontrivially on P with $[b, x] = a$. Thus x acts as an outer diagonal automorphism on $J/O(J)$. Since $Q=$

$N_S(B) \in \mathrm{Syl}_2(C_G(a) \cap N_G(B))$, we know from the structure of PSp (4, q) and A_8 described in Proposition 2.6 that there exists $g \in N_K(Q) - Q$ such that $g^2 \in Q$ and $a^g = a\tau$ where τ is a non-2-central involution of K acting on $J/O(J)$ as a field automorphism and τz is 2-central in K. In particular, $\Omega_1(S) = DP\langle x, \tau\rangle$.

(1) $$b \in F^*(G) .$$

Proof. By the minimality of G, $G = \langle F^*(G), x\rangle$ with $F^*(G)$ simple. Thus if $b \notin F^*(G)$, then $F^*(G)$ is a balanced group. Then J is simple and $\langle a\rangle = C_{F^*(G)}(J)$. Thus J is a standard component in $F^*(G)$, whence G satisfies one of the conclusions of Theorem B.

(2) $$q = 3 .$$

Proof. Suppose $q \geqq 5$. We have $b^{xg} = (ba)^g = ba\tau$. Consider $H = C_G(\langle a, ba\tau\rangle)$. Since $|P| = 4$ and $a \not\sim b$, $C_{C_G(a)}(J/O(J))$ is 2-constrained; so

$$C_{C_G(a)}(J/O(J)) = O(C_G(a))P .$$

It follows that $L(C_J(ba\tau))P$ covers $L(H)O_{2',2}(H)/H \cap O(C_G(a))$. In particular

(a) $O(H) \subseteqq O(C_G(a))$, and

(b) $P(H \cap O(C_G(a)) = C_H(L(H)/O(L(H)))$.

Now look at the action of a on $L = L(C_G(ba\tau))$. Let $\bar{L} = L/O(L) =$ PSp (4, q). As $L(H)/O(L(H)) \cong L_2(q)$, $q \geqq 5$, we know by L-Balance and structure of Aut (PSp (4, q)) that $C_{\bar{L}}(a)$ contains $\overline{L(H)} \times Z_{(q+\varepsilon)/2}$ where $\varepsilon = \pm 1$. If $q+\varepsilon$ is not a power of 2, then $H \cap O(C_G(a))$ covers the odd part of the $Z_{(q+\varepsilon)/2}$ whence $(ba\tau, a, L)$ is an unbalancing triple, contrary to the hypothesis of Theorem B. Thus $q+\varepsilon$ is a power of 2 and, as the $Z_{(q+\varepsilon)/2}$ must be isomorphic to a subgroup of P, we have $q+\varepsilon = 2$ or 4, whence $q = 3$ or 5. Further, we know by the action of a on K that there is a conjugate, e of a acting on L as an outer diagonal automorphism on $\bar{L}$. If a acts as an inner automorphism on $\bar{L}$, then $C_{\langle L, e\rangle/O(L)}(a)$ contains a cyclic group of order 4. Since P is elementary, we conclude that a acts as an outer automorphism. In this case $q + \varepsilon = 2$ and $q = 3$.

(3) $S = DP\langle x, \tau\rangle$ and $Q = DP\langle \tau\rangle \cong D_8 \times E_8$. $Z(Q) = \langle z, a, b, \tau\rangle$ and we have

(i) $a \overset{g}{\sim} a\tau \overset{x}{\sim} a\tau z \overset{g}{\sim} az$

(ii) $b \overset{x}{\sim} ab \overset{g}{\sim} ab\tau \overset{x}{\sim} \tau bz$

(iii) $\tau b \overset{x}{\sim} \tau zab \overset{g}{\sim} abz \overset{x}{\sim} bz$

(iv) $\tau z \overset{x}{\sim} \tau \underset{K}{\sim} z$.

Proof. Everything but the fusion is clear. Recall that in $N_K(Q) - Q$ we have an element g with $g^2 \in Q$ and $a^g = a\tau$. Further $b^g = b$, $\tau^g = \tau$ and $z^g = z$. We know $b^x = ba$, $a^x = a$, $z^x = z$ and from the structure of $\mathrm{Aut}(L_2(q^2))$, $\tau^x \in \tau z P$. Since $|x| = 2$, $\tau^x \in \{\tau z, \tau z a\}$. In the latter case $(\tau z)^{xg} = (\tau a)^g = a$. But $|S| = 2^7$, while $K\langle a, b\rangle$ contains a Sylow 2-subgroup of $C_G(b)$ of order 2^8 and τz is 2-central in $K\langle a, b\rangle$. Thus $a \not\sim_G \tau z$ and $\tau^x = \tau z$.

Likewise $a \not\sim_G b$. Further τz lies in the commutator subgroup of a Sylow 2-subgroup of $C_G(\langle \tau z, b\rangle)$, while $b \notin [C_G(b), C_G(b)]$. Thus $b \not\sim_G \tau z$.

We now see that $\langle x, g\rangle$ acts as a dihedral group of order 8 on $Z(Q)$ with the indicated fusion. Further $z \sim \tau$ in K.

(4) No two of a, b, bz, τz are fused in G.

Proof. From the remarks in (3), it suffices to show that bz is not fused to any of the other involutions. Pick $R \in \mathrm{Syl}_2(C_G(b))$ with $\langle Q, g\rangle \subsetneqq R$. $R_0 = C_R(bz) \in \mathrm{Syl}_2(C_G(\langle b, bz\rangle))$ with $R_0 \cong D_8 \times D_8 \times \boldsymbol{Z}_2$. If $bz \sim_G a$, then $|R_0| = |S|$ implies $R_0 \in \mathrm{Syl}_2(C_G(bz))$. But $a \in [S, S]$ while $bz \notin [R_0, R_0]$, so $a \not\sim_G bz$.

Suppose $b \sim_G bz$ and consider $H = C_G(\langle a, bz\rangle)$. As in (2), we see that $H/H \cap O(C_G(a))$ is a 2-group. On the other hand, a cannot act on $L = L(C_G(bz))$ in such a way that $C_L(a) = O_{2',2}(C_L(a))$. Thus $b \not\sim_G bz$.

Finally, suppose $bz \sim_G z\tau$ and pick $T \in \mathrm{Syl}_2(C_G(bz))$ with $R_0 \subsetneqq T$. We know $z\tau \sim_G z$ and $z \in Z(\langle Q, x, g\rangle)$. Thus $|T| \geqq 2^9$. In particular, $R_0 \subset T$. Let $v \in N_T(R_0)$. Since v normalizes $\mho^1(R_0) = \langle z, \tau\rangle$ and $b \sim_G bz \sim b\tau$ (and $b \neq b^v$), we have $b^v = bz\tau$. Hence

$$|N_T(R_0): R_0| = |N_T(R_0): N_T(R_0) \cap C_T(b)| = 2$$

and $R_0 = J_e(N_T(R_0))$. But now $T = N_T(R_0)$, whence $|T| \leqq 2^8$, a contradiction.

(5) Let E_0 be the unique E_{16} in $R \cap K$. Let $E = \langle E_0, b\rangle$, $N = N_G(E)$, $\bar{N} = N/C_N(E)$. Then $\bar{N} \cong S_6$.

Proof. $\overline{N \cap K\langle a\rangle} \cong S_5$ or $S_3 \wr \boldsymbol{Z}_2$ according as $K/O(K) \cong \mathrm{PSp}(4, 3)$ or A_8. Further $|R| = 2^8$ implies b is not 2-central in G, so we may pick $U \in \mathrm{Syl}_2(G)$ with $R \subset U$ and $N_U(E) \in \mathrm{Syl}_2(N)$. Pick $v \in N_U(R) - R$ with $v^2 \in R$. As $Z(R) = \langle z\tau, b\rangle$, $b^v = bz\tau$. The $\overline{N \cap K\langle a\rangle}$-classes of involutions are represented by z and $z\tau$ in E_0 and b, bz, $bz\tau$ in $E - E_0$. We see that E_0 is strongly closed in E with respect to G. Further $bz \not\sim_G bz\tau$. Letting C_{bz} denote the $\overline{N \cap K\langle a\rangle}$-class of bz, etc. we see that C_{bz} and $C_{bz\tau} \cup \{b\}$ are N-classes.

Suppose $K/O(K) \cong \mathrm{PSp}(4, 3)$. Then $|C_{z\tau}| = |C_{z\tau b}| = 5$ and $|C_z| = |C_{bz}| = 10$. Further $\langle C_{bz\tau}, b\rangle = E$ so that, considering the action of $\bar{N}$ on $C_{bz\tau} \cup \{b\}$, we see that $\bar{N}$ acts as a subgroup of S_6. As $\overline{N \cap K\langle a\rangle} \cong S_5$ and $\overline{N \cap K\langle a\rangle} \subset \bar{N}$, it follows that $\bar{N} \cong S_6$.

Next suppose $K/O(K) \cong A_8$. Then $|C_{z\tau}| = |C_{z\tau b}| = |C_{z\tau b}| = 9$ and $|C_z| = |C_{bz}| = 6$. Now $\langle C_{bz}\rangle = E$ and considering the action of $\bar{N}$ on C_{bz} we again obtain $\bar{N} \cong S_6$.

(6!) Define $C = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ to be the elements of the $\bar{N}$-class of size 6 in $E - E_0$. Let $W = N_U(E)$. Then $C_W(E) = E$ and if $w \in \mathrm{Inv}(N)$ acts as a transposition on C, then $W \sim_G a, b$ or bz.

Proof. Since $\bar{N}$ acts as S_6 on C, it is clear that as an S_6-module, E is the quotient of the permutation module by its 1-dimensional fixed subspace. In particular, all involutions of E_0 are fused in $\bar{N}$.

One shows by direct calculation that $|[E_0, a]| = 2$. Indeed $|[E_0, v]| = 2$ for any $v \in \mathrm{Inv}(R - (R \cap K)E)$. As $[a, b] = 1$, $|[E, a]| = 2$ and a acts on C as a transposition. Let $W = N_U(E)$. Since $C_R(E) = E$ and $C_W(E) \subseteq C_G(b)$, we have $E = C_W(E)$.

Let w be any involution of N acting as a transposition on C. Clearly w is N-fused to $aC_W(E) = aE$. The $K\langle a\rangle$-classes of Ka are represented by a and $az\tau$. Thus any involution in aE is $K\langle a\rangle$-conjugate to $a, ab, az\tau, az\tau b$ or b. The claim follows.

(7) $$W \neq U\,.$$

Proof. Assume $W = U$ and let N_0 be the inverse image in G of $\bar{N}'$. Since $a \in N - N_0$ and $a \in [S, S]$, it follows that a has a G-conjugate in N_0. $\bar{N}_0$ contains just one class of involutions and from the structure of $\overline{N \cap K\langle a\rangle}$ it follows that $\overline{N_0 \cap K}$ contains involutions. Thus $\overline{R \cap K}$ contains involutions, whence $\bar{a}$ is conjugate in $\bar{N}$ to $\bar{e} \in \overline{R \cap K}$. It follows that $a \sim e \in (R \cap K)E$. But every involution in $(R \cap K)E$ is $K\langle a\rangle$-fused to $z, zb, z\tau, z\tau b$, or b none of which is G-conjugate to a. Thus $W \neq U$.

(8) Let $y \in N_U(W) - W$. Then we may assume that
- (i) $\bar{E}^y = \langle(12), (34), (56)\rangle$
- (ii) $\bar{E}_0^y = \langle(12)(34), (12)(56)\rangle$
- (iii) $E \cap E^y = \langle e_1 + e_2, e_3 + e_4\rangle$
- (iv) $E_0E_0^y$ is isomorphic to a Sylow 2-subgroup of $L_3(4)$.

Proof. We may assume that $\bar{W}$ acts on C as $\langle(12)(34), (13)(24), (12), (56)\rangle$. Every involution in E_0 is fused to z, whence $\bar{E}_0^y$ contains no transpositions. Suppose $|E \cap E^y| = 16$. Then $e^y \in E^y - E$ acts on

E as a transposition and every involution of Ee^y lies in E^y. Then we may assume that $e^y \sim a$, not the case as $e \sim z$, b or bz. As $\bar{W} \cong D_8 \times Z_2$, $\bar{E}^y$ is elementary of order at most 8 and $\bar{E}^y \trianglelefteq \bar{W}$. One now verifies the claim in a straightforward way.

(9) $$|N_U(W): W| = 2 .$$

Proof. Let $u \in N_U(W) - (W \cup Wy)$. Since y was arbitrary in (8), $\bar{E}^y = \bar{E}^u$ and $E \cap E^y = E \cap E^u = E \cap E^{uy^{-1}}$. Thus $|E^u \cap E^y| = |E \cap E^u| = 4$. Since all involutions of $E_0E_0^y$ lie in $E \cup E^y$, this yields $|E^u \cap E_0E_0^y| = 4$. We know from the fusion of involutions that $E_0 \trianglelefteq N$. Thus $E_0E_0^y$ is normal of index 4 in EE^y. Our conditions imply $EE^y = EE^u$, a contradiction.

(10) EE^y is weakly closed in $N_U(W)$ with respect to G.

Proof. Let $F = E^h$ be a G-conjugate of E with $F \subseteq N_U(W)$ and $F_0 = E_0^h$. Suppose $F \not\subseteq W$. Then $F \cap EE^y$ lies in X, the inverse image in EE^y of $C_{EE^y/E \cap E^y}(f)$, where $f \in F - W$. Now $|X: E \cap E^y| = 8$ and $|X: X \cap E_0E_0^y| = 2$. As all involutions of $X \cap E_0E_0^y$ lie in $E_0 \cap E_0^y = E \cap E^y$, we conclude $|X \cap F| \leqq 8$. As $|N_U(W): EE^y| = 2^4$ and $|F| = 2^5$, we must have $|X \cap F| = 8$, whence $E \cap E^y \subseteq F$. But also F covers $N_U(W)/EE^y$ and $EE^y = C_W(E \cap E^y) \subset W$, a contradiction. Thus $F \subseteq W$.

We assume $F \not\subseteq EE^y$. Then $\bar{F}$ is elementary and $\bar{F} \not\subseteq \overline{F^y}$, so $\bar{F}$ contains at least one of

$$(13)(24) \quad (13)(24)(56) \quad (14)(23) \quad (14)(23)(56) .$$

The last two possibilities are handled in the same way as the first two, so assume $\bar{F}$ contains (13)(24) or (13)(24)(56). Consequently

$$C_E(F) \subseteq C_E(\langle(13)(24)\rangle) = \langle e_1 + e_3, e_2 + e_4, e_5\rangle ,$$

or

$$C_E(F) \subseteq C_E(\langle(13)(24)(56)\rangle) = \langle e_1 + e_3, e_2 + e_4, e_1 + e_2 + e_5\rangle .$$

In either case $E \cap F \subseteq C_E(F)$ implies

$$|E \cap F| \leqq |C_E(F)| \leqq 8 \quad \text{and} \quad |\bar{F}| \geqq 4 .$$

If $|\bar{F}| = 8$, then $\bar{F} = \langle(12)(34), (13)(24), (56)\rangle$ and $|C_E(\bar{F})| = 2$, forcing $|F| \leqq 16$, not the case. Thus $E \cap F = C_E(\bar{F})$. As $\langle e_1 + e_3, e_2 + e_4, e_1 + e_2 + e_5\rangle$ is not centralized by a fours group in $\bar{W}$, we must have $C_E(\bar{F}) = \langle e_1 + e_3, e_2 + e_4, e_5\rangle$ and $\bar{F} = \langle(13), (24)\rangle$.

We know that E_0 is strongly closed in E whence $E \cap F_0 \subseteq E_0$. Thus

$$|E \cap F_0| \leqq |E_0 \cap C_E(\bar{F})| = 4 ,$$

whence $\bar{F}_0 = \bar{F}$. But then some element of F_0 acts as a transposition on E and so some element of E_0 is G-fused to a, b, bz, not the case.

(11) $$U = N_U(W) .$$

Proof. By (10) it suffices to show $N_U(W) = N_U(EE^y)$. Assume not and pick $w \in (N_U(EE^y) \cap N_U(N_U(W))) - N_U(W)$, with $w^2 \in N_U(W)$. E_0^w contains no element acting on E as a transposition, whence $E_0^w \subseteqq EE_0^y$. Likewise $E_0^{wy} \subseteqq EE_0^y$, whence $E_0^w \subseteqq E^{y^{-1}}E_0 = E^yE_0$. Thus $E_0^w \subseteqq EE_0^y \cap E_0^yE_0 = E_0E_0^y$. Replacing w by wy if necessary we may assume $E_0^w = E_0$. As $\bar{W}$ acts faithfully on E_0, we have

$$E^w \subseteqq C_{EE^y}(E_0) = E ,$$

contrary to our choice of w.

(12) $$b \notin G^2 .$$

Proof. Let $M = N_G(EE^y)$. For any $w \in M$ the preceding argument has shown $E_0^w \subseteqq E_0E_0^y$. Since the same argument shows $E_0^{yw} \subseteqq E_0E_0^y$, we conclude $E_0E_0^y \trianglelefteq M$. Let $\bar{M} = M/E_0E_0^y$. Since $E \cap E^y = Z(EE^y) \trianglelefteq U$ and W acts nontrivially on $E \cap E^y$, we have

(a) $U = WU_0$

(b) $U_0 = C_U(E \cap E^y)$

(c) $W \cap U_0 = EE^y$.

In particular, any element of $U_0 - EE^y$ permutes E and E^y, whence $\bar{U}_0 \cong D_8$. Further $E \trianglelefteq W$, $E^y \trianglelefteq W$ imply that $\bar{W}$ is abelian, and it follows that $\bar{U} \cong D_8 \times \boldsymbol{Z}_2$ or $D_8 * \boldsymbol{Z}_4$. Let $e \in \operatorname{Inv}(E - E_0)$. We have in either case $\bar{e} \notin \bar{U}^2$. Also $C_{\bar{M}}(\bar{E}\bar{E}^y)/\bar{E}\bar{E}^y$ has a Sylow 2-subgroup of order two, whence $C_{\bar{M}}(\bar{E}\bar{E}^y)$ has a normal 2-complement. Once we show $\bar{M}/C_{\bar{M}}(\bar{E}\bar{E}^y) \cong \boldsymbol{Z}_2$, we have $\bar{M} = \bar{U}O(\bar{M})$ and $e \notin M^2$. The only other possibility is $\bar{M}/C_{\bar{M}}(\bar{E}\bar{E}^y) \cong S_3$, in which case $\lambda \in M$ of order 3^m with λ acting nontrivially on $EE^y/E_0E_0^y$. But the structure of $E_0E_0^y$ implies that λ normalizes E_0 and E_0^y, hence normalizes $E = C_{EE^y}(E_0)$ and E^y. But then λ acts trivially on $EE^y/E_0E_0^y$.

Now suppose $e \in G^2$ and apply Lemma 3.10. There exists $V \subset EE^y$ such that

(1) $C_{EE^y}(e) \subseteqq V$ and

(2) The transfer $V_{EE^y \to V}(e) \notin \Phi(EE^y)$.

Choosing $e = e_5 \in E - E_0$, we see that $|EE^y : C_{EE^y}(e_5)| = 2$ and $\Phi(C_{EE^y}(e_5)) = E \cap E^y = \Phi(EE^y)$. Thus $V = C_{EE^y}(e_5)$ and $|EE^y : V| = 2$. Hence $V_{EE^y \to V}(e_5) \in$

$\Phi(EE^y) = \Phi(V)$, contrary to Lemma 3.10. Thus $e_5 \notin G^2$.

Now e_5 is $N_G(E)$-fused to b or bz. As $z \in G^2$, we conclude in either case that $b \notin G^2$. But by (1), $b \in F^*(G) = G^2$, a contradiction.

This completes the proof of Lemma 8.3. As Lemmas 8.2 and 8.3 are mutually contradictory, the proof of Theorem B is complete.

9. **The proof of Corollary C.** In this section G will be a fixed counterexample to Corollary C and (a, x, J) will be a fixed unbalancing triple satisfying

(1) $J/O(J) \cong A_7$.

(2) $x \in E$, a fours subgroup of $C_G(a) \cap N_G(J)$ with $\Delta = C_G(a) \cap \left(\bigcap_{e \in E^\#} O(C_G(e))\right)$ and $[J, \Delta] = J$.

(3) $|C_G(a) \cap N_G(J)|_2$ is maximal subject to (1) and (2).

Suppose that (a, x, J) is a maximal unbalancing triple in G. Then either conclusion (1) or (3) of Theorem B must hold. If (1) holds, then as G is not 2-balanced, $F^*(G) \cong A_n$ for some odd $n \geqq 11$. If (3) holds, then by Lemma 3.5, J is maximal in G. Now by Theorem 1.5 and the main theorems of [7], [16] and [18], $F^*(G)$ is isomorphic to A_9, A_{11} or He. Again, as G is not 2-balanced, $F^*(G) \cong A_{11}$.

Thus we may assume that (a, x, J) is not a maximal unbalancing triple in G. Thus one of the two conditions in the definition of maximal unbalancing triple must fail to hold. Let $S \in \mathrm{Syl}_2(C_G(a) \cap N_G(J))$ and let $P = C_S(J/O(J))$, $D = S \cap J$. Let $b \in \mathrm{Inv}\, C_P(S)$ and let K be a 2-component of $\langle J^{L(C_G(b))} \rangle$. The proof of Lemma 3.8 shows that if $\Delta_1 = C_\Delta(b)$, then $[K, \Delta_1] = K = [K, E]$. Thus $N_G(K)$ is not 2-balanced and, as G is a minimal counterexample to the Unbalanced Group Conjecture, $K/Z^*(K) \cong A_n$ for some odd $n \geqq 7$. By [25] and [28], $K/O(K) \cong A_7$. Thus (b, x, K) has the same properties as (a, x, J) with $|C_G(b) \cap N_G(K)|_2 \geqq |S|$. It follows from the choice of (a, x, J) that $S \in \mathrm{Syl}_2(C_G(b) \cap N_G(K))$ for all $b \in \mathrm{Inv}\, C_P(S)$. Thus condition 1 in the definition of a maximal unbalancing triple fails to hold.

Now pick $B \subseteq C_P(x)$ to satisfy the conclusions of Lemma 3.6 and let K be a component of $\langle J_B^{L(C_G(B))} \rangle$. Then $N_G(K)$ is unbalanced and as above $K/Z^*(K) \not\cong A_n$ for any n. On the other hand J_B projects onto a standard component of $K/Z^*(K)$. Thus $K/O(K) \cong He$. As in § 5, $N_P(B) = \langle a \rangle \times B$ and $B \cap B^e = \langle 1 \rangle$ for some $e \in E$, since Aut He is 2-balanced. Thus $\langle a \rangle = C_P(S)$. Also $a \sim az$ in $K\langle a \rangle$. Hence $a \sim az$ in $N_G(S)$. Let $n \in N_G(S)$ with $a^n = az$. Then $P \cap P^n = \langle 1 \rangle = D \cap P^n$. Thus P^n acts faithfully on $J/O(J)$ and centralizes D. So P^n is isomorphic to $Z_2 \times Z_2$. But $E \subseteq D \times P$ and E does not centralize B, a contradiction.

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