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SPLITTING AND MODULARLY PERFECT FIELDS

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Let K be a field of characteristic $p \neq 0$. A field extension L/K is said to split when there exist intermediate fields J and D of L/K where J is purely inseparable over K, D is separable over K and $L = J \otimes_K D$. K is modularly perfect if $[K: K^p] \leqq p$. Every finitely generated extension of a modularly perfect field splits. This paper develops criteria for an arbitrary extension L/K to split and presents an example of an extension of a modularly perfect field which does not split. Necessary and/or sufficient conditions are also developed for the following to hold for an extension L/K: (a) L'/K splits for every intermediate field L'; (b) L'/K is modular for every intermediate field L'; (c) L/L' splits for every intermediate field L'; (d) L/L' is modular for every intermediate field L'.

Introduction. Let K be a field of characteristic $p \neq 0$. A field extension L/K is said to split when there exists intermediate fields J and D of L/K where J/K is purely inseparable, D/K is separable, and $L = J \otimes_K D$. It is a classic result that any normal algebraic field extension L/K must split. Recent papers have been concerned with nonalgebraic extensions L/K. Suppose there exists an intermediate field J of L/K such that L/J is separable and J/K is purely inseparable (hence $J = L \cap K^{p^{-\infty}}$). Under certain conditions, namely, if L has a separating transcendence basis over J [4], or if J is of bounded exponent over K [5], then L/K must split. That some conditions must be put on $L/J/K$ is illustrated by an example in [1].

A field extension L/K is called modular if L^{p^n} and K are linearly disjoint for all n. The importance of modular extensions was first observed by Sweedler [11] who used this property to characterize purely inseparable extensions of bounded exponent which were tensor products of simple extensions. In [4] it was shown that if L/K is an (arbitrary) modular extension then there must exist an intermediate field J such that L/J is separable and J/K is purely inseparable modular. It follows that any finitely generated modular extension must split. In [5], a field K such that $[K: K^p] \leqq p$ is called modularly perfect. Such fields are characterized by the fact that any extension L of such a field K must be modular over K. In view of the above results, a natural question is whether every extension of a modularly perfect field K must split. In part I we develop a number of criterion for a field extension to split. We construct an extension L of a modularly perfect field K which does not split. The field L also does not have a distinguished separable subfield [3], where D is distinguished in L/K if and only if D/K is separable and $L \subseteqq D(K^{p^{-\infty}})$. In the remainder of the paper we determine necessary and/or sufficient conditions for the following to hold for an extension L/K:

(a) L'/K splits for every intermediate field L';
(b) L'/K is modular for every intermediate field L';
(c) L/L' splits for every intermediate field L';
(d) L/L' is modular for every intermediate field L'.

I. Let $L \supseteqq K$ be fields of characteristic $p \neq 0$. An intermediate field D of L/K is called a distinguished separable intermediate field [3] if D is separable over K and $L \subseteqq D \otimes_K K^{p^{-\infty}}$.

REMARK 1.1. The following conditions are equivalent on K.

(1) L/K splits for every finitely generated field extension L/K.
(2) $[K: K^p] \leqq p$.
(3) L/K splits for every field extension L/K which has a distinguished separable intermediate field.

Proof. $1 \to 2$: Suppose $[K: K^p] > p$. Let x, y be p-independent in K and let z be transcendental over K. Then $L = K(z, zx^{p^{-1}} + y^{p^{-1}})$ is a finitely generated extension of K which does not split.

$2 \to 3$: By [5, Theorem 6, p. 1180], L/K is modular and whence splits [8, Corollary, p. 607].

$3 \to 1$: If L/K is finitely generated, then L/K has a distinguished separable intermediate field.

In [5, Theorem 6, p. 1180] it was shown that $[K: K^p] \leqq p$ if and only if for every field extension L/K there exists a separable field extension S of K (not necessarily in L) such that $L \subseteqq S \otimes_K (K^{p^{-\infty}} \cap L)$. Obviously S can be chosen as an intermediate field of L/K if and only if L/K splits. We now develop criterion for an extension L/K to split and present an example of an extension of a modularly perfect field which does not split.

LEMMA 1.2. *Let D be an intermediate field of L/K such that L/D is purely inseparable and D/K is separable. Then D is maximal separable if and only if $L^p \cap D \subseteqq K(D^p)$.*

Proof. Assume D is maximal and let $b \in L \backslash D$. If $b^p \in D \backslash K(D^p)$, then $D(b)/K$ is separable as follows: Let G be a p-basis for K. Then G is p-independent in D. Since $b^p \in D \backslash K(D^p)$, $G \cup \{b^p\}$ is p-independent in D. Hence there does not exist $c \in G$ such that $c \in D^p(b^p, G \backslash \{c\})$. Thus G is p-independent in $D(b)$ and $D(b)/K$ is separable. However this contradicts the maximality of D. Thus $b^p \in K(D^p)$ and $L^p \cap D \subseteqq K(D^p)$.

Conversely, assume $L^p \cap D \subseteqq K(D^p)$. If $L = D$, then D is maximal. Suppose $L \supset D$ and let $b \in L \backslash D$ be such that $b^p \in D$. Then $b^p \in L^p \cap D \subseteqq K(D^p)$ so $b \in D \otimes_K K^{p^{-1}}$. Thus $D(b)/K$ is not separable and hence D is maximal.

THEOREM 1.3. *Suppose $L \supseteqq K^{p^{-\infty}}$ and L/L^{p^∞} has a separating transcendence basis. Then L/K splits.*

Proof. Since $L \supseteqq K^{p^{-\infty}}$, $L^{p^\infty} \supseteqq K^{p^{-\infty}}$. Let D be a maximal separable extension of K in L^{p^∞}. We first show $D^{p^{-1}} \cap L^{p^\infty} \subseteqq D(K^{p^{-\infty}})$. If $b \in D^{p^{-1}} \cap L^{p^\infty}$, then $b^p \in K(D^p)$ by the previous lemma. Hence $b \in D(K^{p^{-1}}) \subset D(K^{p^{-\infty}})$, as desired. Now L^{p^∞}/D is purely inseparable, and hence $L^{p^\infty}/D(K^{p^{-\infty}})$ is also purely inseparable. We prove $L^{p^\infty} = D(K^{p^{-\infty}})$ by showing that each element $b \in L^{p^\infty}$ of exponent one over $D(K^{p^{-\infty}})$ is actually in $D(K^{p^{-\infty}})$. For such b, $b^p = \Sigma d_i e_i$ where $d_i \in D$ and $e_i \in K^{p^{-\infty}}$. Hence $b = \Sigma d_i^{p^{-1}} e_i^{p^{-1}}$ where each $d_i^{p^{-1}} \in L^{p^\infty}$ is of exponent one over D. As noted above, each $d_i^{p^{-1}} \in D(K^{p^{-\infty}})$ and thus $b \in D(K^{p^{-\infty}})$ and $L^{p^\infty} = D \otimes_K K^{p^{-\infty}}$. Now L/L^{p^∞} has a separating trancendence basis and L^{p^∞}/D is purely inseparable. Hence there exists an intermediate field D^* of L/D such that D^*/D is separable and $L = D^* \otimes_D L^{p^\infty}$ [4, Proposition 1, p. 2]. Thus $L = D^* \otimes_D (D \otimes_K K^{p^{-\infty}}) = D^* \otimes_K K^{p^{-\infty}}$. Since D^*/D and D/K are separable, D^*/K is separable.

COROLLARY 1.4. (1) *If $[K: K^p] \leqq p$, then L/K splits for every field extension L/K such that L/L^{p^∞} has a separating transcendence basis.*

(2) *Conversely, suppose K/K^{p^∞} has a separating transcendence basis. If L/K splits for every field extension L/K such that L/L^{p^∞} has a separating transcendence basis, then $[K: K^p] \leqq p$.*

Proof. (1) If $L \supseteqq K^{p^{-\infty}}$, then L/K splits by 1.3. If $L \not\supseteqq K^{p^{-\infty}}$, then $(L \cap K^{p^{-\infty}})/K$ has bounded exponent. Since $[K: K^p] \leqq p$, L/K is modular [5, Theorem 1, p. 1177] and hence splits [5, Theorem 3, p. 1178].

(2) Suppose $[K: K^p] > p$. Let T be a separating transcendence basis for K/K^{p^∞}. Then T is a p-basis for K and $|T| > 1$. Let $\{x, y\} \subseteqq T$ and set $L = K(z, zx^{p^{-1}} + y^{p^{-1}})$ where z is transcendental over K. If we show L/L^{p^∞} has a separating transcendence basis, we have a contradiction since L/K does not split. Now $T\backslash\{y\} \cup \{z, zx^{p^{-1}} + y^{p^{-1}}\}$ is a p-basis for L. $L^{p^\infty} \subseteqq \bigcap_{i=1}^{\infty} K(L^{p^i}) = K$, so $L^{p^\infty} = K^{p^\infty}$. $K/L^{p^\infty}(T)$ is separable algebraic so $L/L^{p^\infty}(T\backslash\{y\}, z, zx^{p^{-1}} + y^{p^{-1}})$ is separable algebraic

since $y \in L^{p^\infty}(T\backslash\{y\}, z, zx^{p^{-1}} + y^{p^{-1}})$. Thus L/L^{p^∞} has a separating transcendence basis.

PROPOSITION 1.5. *If L/K splits for every field extension L/K such that $L/L \cap K^{p^{-\infty}}$ is separable, then $[K: K^p] < \infty$.*

Proof. Suppose $[K: K^p] = \infty$ and let $\{x_1, \cdots, x_n, \cdots\}$ be a p-independent subset of K. Let $J = K(x_1^{p^{-1}}, x_2^{p^{-2}}, \cdots)$ and $L = J(z, z^{p^{-1}} + x_1^{p^{-2}}, \cdots, z^{p^{-n}} + x_1^{p^{-n-1}} + \cdots + x_n^{p^{-n-1}}, \cdots)$ where z is transcendental over J. Since L is the union of a chain of simple transcendental extensions of J, L/J is separable and $L \cap K^{p^{-\infty}} = J$. The proof that L/K does not split is completely analogous to the proof in [1].

Suppose K is a modularly perfect field, i.e., $[K: K^p] \leqq p$. Then for any field L which contains K, L is separable over $L \cap K^{p^{-\infty}}$. If $L \cap K^{p^{-\infty}} \neq K^{p^{-\infty}}$, then L/K must split. We now present an example where $L \supset K^{p^{-\infty}}$ and yet L/K does not split. This example indicates that the result presented in [5, Theorem 6, p. 1180] is in some sense the best possible.

EXAMPLE 1.6. Let P be a perfect field and let x, y, w_0 be algebraically independent indeterminates over P. Set $K = P(y)$ and $L = K^{p^{-\infty}}(x, w_0, w_1^{p^{-1}}, \cdots, w_n^{p^{-n}}, \cdots)$ where $w_1^{p^{-1}} = x^{p^{-1}} + y^{p^{-3}} w_0^{p^{-1}}$ and $w_n^{p^{-n}} = x^{p^{-1}} + y^{p^{-(2n+1)}} w_{n-1}^{p^{-n}}$. Then $[K: K^p] = p$ and $K^{p^{-\infty}} \subset L$. We show that L/K does not split. Assume $L = S \otimes_K K^{p^{-\infty}}$ where S is separable over K. Consider $S' = S(x, w_0)$. Since S' is finitely generated over S, S'/K has bounded exponent and hence splits since K is modularly perfect [5, Theorem 6, p. 1180]. Let $S' = S^* \otimes_K K(y^{p^{-t}})$, and hence $L = S' \otimes_{K(y^{p^{-t}})} K^{p^{-\infty}}$. Now by construction, $K(y^{p^{-t}}, x, w_0) \equiv K_t \subseteqq S'$ and $K_t \otimes_{K(y^{p^{-t}})} K^{p^{-\infty}} = K^{p^{-\infty}}(x, w_0)$. The fields which lie between $K^{p^{-\infty}}(x, w_0)$ and L are chained and each is a purely inseparable extension of exponent one of the previous one. Hence the same is true for the fields which lie between K_t and S'. Since $K^{p^{-\infty}}(x, w_0)/K_t$ is also modular, it follows that L/K_t is modular, and in fact $[K_t^{p^{-n}} \cap L: K_t] = p^{2n}$ for all n. Since any finitely generated extension of K_t in L is contained in $K_t^{p^{-n}} \cap L$ for some n, and $K_t^{p^{-n}} \cap L$ is modular over K_t with two elements in any subbase, we conclude that any finitely generated extension of K_t in L must be modular over K_t [9, Proposition 2.5, p. 76].

We now show that there is a field M which lies between L and K_t which is not modular over K_t. Since $w_n^{p^{-n}} = x^{p^{-1}} + y^{p^{-(2n+1)}} w_{n-1}^{p^{-n}}$, for large n, $w_n^{p^{-n}}$ will not be of exponent n over K_t. Thus assume $w_{n-1}^{p^{-(n-1)}}$ is of exponent $n-1$ over K_t and $w_n^{p^{-n}}$ is not of exponent n. It follows that $(y^{p^{-(2n+1)}})^{p^n} = y^{p^{-(n+1)}}$ is not an element of K_t. Let $M = K_t(y^{p^{-(2n+1)}}, w_n^{p^{-n}})$. If we show every higher derivation on M over K_t

maps $y^{p^{-(n+1)}}$ to 0, M/K_t is not modular [11, Theorem 1, p. 403]. Recall that a higher derivation of M over K_t is a sequence of K_t-linear maps $D^{(s)}=\{D_0=I, D_1, \cdots, D_s\}$ of M into itself such that $D_m(bc)=\sum_{i=0}^{m} D_i(b)D_{m-i}(c)$ for all $b, c \in M$, $m=0, \cdots, s$. We shall need the direct corollary to [13, p. 436] that $D_m(b^{p^r})=0$ if $p^r \nmid m$ and $D_m(b^{p^r})=(D_{m/p^r}(b))^{p^r}$ if $p^r \mid m$. We follow the method of Sweedler [11, Example 1.1, p. 405]. Assume there exists D_s such that $D_s(y^{p^{-(n+1)}}) \neq 0$. Then

$$\begin{aligned} w_{n-1}[D_{s/p^n}(y^{p^{-(2n+1)}})]^{p^n} &= D_s(w_{n-1}y^{p^{-(n+1)}}) \\ &= D_s(x^{p^{n-1}} + y^{p^{-(n+1)}}w_{n-1}) \\ &= [D_{s/p^n}(x^{p^{-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}})]^{p^n} . \end{aligned}$$

Hence

$$w_{n-1} = \frac{[D_{s/p^n}(x^{p^{-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}}]^{p^n}}{[D_{s/p^n}(y^{p^{-(2n+1)}})]^{p^n}}$$

which is an element of M^{p^n}.

Thus $w_{n-1}^{p^{-n}} \in M$ and hence $x^{p^{-1}} \in M$, a contradiction since $x^{p^{-1}}$ is not in L.

In the previous example, L is of transcendence degree two over K. The next result illustrates that this is the least degree possible for an extension which does not split.

PROPOSITION 1.7. *$[K: K^p] \leqq p$ if and only if L/K splits for every field extension L/K of transcendence degree one.*

Proof. Suppose $[K: K^p] \leqq p$. As usual it suffices to consider the case where $L \supseteqq K^{p^{-\infty}}$. If $L=L^p$, then $L=L^{p^\infty}$ and L/K splits by 1.4. If $L \supset L^p$, we show L has a separating transcendence basis over L^{p^∞}. Since $L \supset K^{p^{-\infty}}$, L/L^{p^∞} has transcendence degree 1. Let B be a p-basis for L. Since B is algebraically independent over L^{p^∞}, B consists of exactly one element and $L/L^{p^\infty}(B)$ is algebraic. Since $L/L^{p^\infty}(B)$ is separable, B is a separating transcendence basis for L/L^{p^∞} and 1.3 applies.

Conversely, suppose $[K: K^p] > p$. Let x, y be p-independent in K and let z be transcendental over K. Then L/K does not split where $L=K(z, zx^{p^{-1}}+y^{p^{-1}})$.

As Example 1.6 illustrates, not every extension L/K of a modularly perfect field need split. The following result gives several criteria for such an extension to split.

THEOREM 1.8. *Assume $[K: K^p]=p$ and $L \supseteqq K^{p^{-\infty}}$. The following are equivalent.*

(1) *L/K splits.*

(2) *There exists a maximal separable extension D of K in L such that L is modular over D.*

(3) *There exists a maximal separable extension D of K in L such that some relative p-basis for D over K remains p-independent in L.*

(4) *There exists a proper intermediate field D of L/K such that $L = D(K^{p^{-\infty}})$.*

Proof. Assume $L = D \otimes_K K^{p^{-\infty}}$. Then D satisfies properties (2), (3), and (4). Assume (2). Since D is a maximal separable extension of K in L, L/D is purely inseparable. Since $D((D(K^{p^{-\infty}}))^p) = D(K^{p^{-\infty}})$, $D(K^{p^{-\infty}})$ is pure in L/D [12, Definition, p. 41]. We claim $D^{p^{-1}} \cap L = D^{p^{-1}} \cap (D(K^{p^{-\infty}}))$. Let $b \in (D^{p^{-1}} \cap L) \backslash D$. Then $D(b)$ and $K^{p^{-1}}$ are not linearly disjoint over K since $D(b)$ is not separable over K. Since $[K^{p^{-1}}: K] = p$, $K^{p^{-1}} \subseteq D(b)$ and hence $D(b) = D(K^{p^{-1}})$. Thus $b \in D(K^{p^{-\infty}})$ and the claim is established. By [12, Proposition 2.7, p. 44], $D(K^{p^{-\infty}}) = D \otimes_K K^{p^{-\infty}} = L$ and (1) holds. Assume (3). A relative p-basis B for D over K is a p-basis for $D(K^{p^{-\infty}})$. Since this p-basis remains p-p-independent in L, $L/D(K^{p^{-\infty}})$ is separable. Since L/D, whence $L/D(K^{p^{-\infty}})$ is also purely inseparable, $L = D \otimes_K K^{p^{-\infty}}$ and (1) holds. Assume (4). Since D is proper, $D \not\supseteq K^{p^{-\infty}}$ and hence D/K splits, say $D = D' \otimes_K K^{p^{-n}}$. Then $L = D' \otimes_K K^{p^{-\infty}}$.

II. In this section we determine necessary and/or sufficient conditions for the following to hold for an arbitrary extension L/K;

(a) L'/K splits for any intermediate field L';

(b) L'/K is modular for any intermediate field L'.

We will need the following result.

LEMMA 2.1. *Suppose L/K splits and the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in a chain. If L' is an intermediate field of L/K, then $L'/(L' \cap K^{p^{-\infty}})$ is separable.*

Proof. We first note that $L \cap K^{p^{-\infty}}$ and L' are linearly disjoint over $L' \cap K^{p^{-\infty}}$. This follows since $L \cap K^{p^{-\infty}}/L' \cap K^{p^{-\infty}}$ is purely inseparable and the intermediate fields are chained. Now since L/K splits, $L/L \cap K^{p^{-\infty}}$ is separable and hence $(L \cap K^{p^{-\infty}})(L')$ is separable over $L \cap K^{p^{-\infty}}$. By [6, Corollary 6, p. 266], we conclude $L'/K^{p^{-\infty}} \cap L'$ is separable.

THEOREM 2.2. *Suppose L/K is inseparable but not purely inseparable. Then each condition in the following list implies the succeeding one.*

(1) *L/K splits and $(L \cap K^{p^{-\infty}})/K$ is simple.*

(2) *L'/K splits for every intermediate field L'.*

(3) *L/K splits and the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in a chain.*

Proof. (1) implies (2): Let L' be an intermediate field. By 2.1, $L'/(L' \cap K^{p^{-\infty}})$ is separable. Since $(L \cap K^{p^{-\infty}})/K$ is simple, $(L' \cap K^{p^{-\infty}})/K$ is of bounded exponent and so L'/K splits [5, Theorem 4, p. 1178].

(2) implies (3): Suppose the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ do not appear in a chain. Then there exist b, c in $L \cap K^{p^{-\infty}}$ such that $b \notin K(c)$, $c \notin K(b)$, and both b and c have some positive exponent i over K. Let $z \in L \backslash K$ be such that $K(z)/K$ is separable. Let $L' = K(z, zb + c)$. Now $K(z)$ is a distinguish maximal separable intermediate field of L'/K. Since L'/K splits, $L' = K(z) \otimes_K J$ by [8, Lemma, p. 607] where $J = L' \cap K^{p^{-\infty}}$. Let $\mathscr{X}$ be a linear basis of $K(z)/K$ with $1, z \in \mathscr{X}$. Now $zb + c = \sum x_j d_j$ where $x_j \in \mathscr{X}$ and $d_j \in J$. Thus

$$(*) \qquad z^{p^i} b^{p^i} + c^{p^i} = \sum x_j^{p^i} d_j^{p^i}, \ d_j^{p^i} \in J^{p^i} \cap K .$$

Since $K(z)/K$ is separable, $\mathscr{X}^{p^i}$ is linearly independent over K. Hence by equating coefficients in (*) we have $b^{p^i}, c^{p^i} \in J^{p^i}$. Thus $b, c \in J$. Hence $[J: K] > p^i$ and $[L': K(z)] > p^i$, a contradiction. Hence the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in a chain.

COROLLARY 2.3. *Suppose L/K has finite inseparability exponent. Then the following conditions are equivalent.*

(1) *L'/K is modular for every intermediate field L'.*

(2) *L/K is modular and $(L \cap K^{p^{-\infty}})/K$ is simple.*

(3) *L'/K splits for every intermediate field L'.*

(4) *L/K splits and $(L \cap K^{p^{-\infty}})/K$ is simple.*

Proof. A modular field extension with finite inseparability exponent splits. Also a field extension which splits and whose maximal purely inseparable subfield is simple is necessarily modular.

COROLLARY 2.4. *Suppose L/K is inseparable and $L/(L \cap K^{p^{-\infty}})$ has a finite separating transcendence basis. Then the following conditions are equivalent.*

(1) *L'/K is modular for every intermediate field L'.*

(2) *L'/K splits for every intermediate field L'.*

(3) *The intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in a chain.*

Proof. Assume (1). Let L' be an intermediate field of L/K. If $L' \supseteqq L \cap K^{p^{-\infty}}$, then $L'/L \cap K^{p^{-\infty}}$ has a finite separating transcendence basis [10, Theorem 1, p. 418] so L'/K splits by [4, Proposition 1, p. 2]. If $L' \not\supseteq L \cap K^{p^{-\infty}}$, then $L' \cap K^{p^{-\infty}}/K$ is a simple extension since the intermediate fields of $L \cap K^{p^{-\infty}}/K$ must be chained. For if they are not chained, choose b, c, z as in the proof of 2.2 and $K(z, zb + c)$ is not modular over K. Now we have $(L' \cap K^{p^{-\infty}})/K$ simple of bounded exponent, $L'/L' \cap K^{p^{-\infty}}$ separable since L'/K is assumed modular, and hence L'/K splits by [5, Theorem 4, p. 1178].

That (2) implies (3) is part of 2.2. Assume (3). Since $L/L \cap K^{p^{-\infty}}$ has a finite separating transcendence basis, L/K splits. By 2.1, if L' is an intermediate field of L/K, then $L'/L' \cap K^{p^{-\infty}}$ is separable. Since $L' \cap K^{p^{-\infty}}/K$ is modular, L'/K is modular by [11, Lemma 5(3), p. 407].

COROLLARY 2.5. *Suppose L/K is inseparable but not purely inseparable. Then each statement in the following list implies the succeeding one.*

(1) *L/K is modular and $(L \cap K^{p^{-\infty}})/K$ is simple.*

(2) *L'/K is modular for every intermediate field L'.*

(3) *L/K is modular and the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in a chain.*

Proof. Straightforward.

III. We now determine necessary and sufficient conditions for the following to hold;

(a) L/L' splits for any intermediate field L'.

(b) L/L' is modular for any intermediate field L'.

THEOREM 3.1. (1) *Suppose $L \not\supseteq K^{p^{-\infty}}$. Then L/L' splits for every intermediate field L' of L/K if and only if L/K is algebraic and L/K splits.*

(2) *Suppose $L \supseteq K^{p^{-\infty}}$. Then L/L' splits for every intermediate field L' of L/K if and only if $L = L^p$.*

Proof. (1) Suppose L/L' splits for every intermediate field L'. Let $J = L \cap K^{p^{-\infty}}$ and let $b \in J \backslash J^p$. Theh $b^{p^{-1}} \notin L$. Assume L/J is not algebraic and let $z \in T$, where T is a transcendence basis for L/J. Let $\hat{J} = J(T \backslash \{z\})$ and $w = -z^{2p}(z^p + b)^{-1}$. Then $z^{2p} + wz^p + wb = 0$. The polynomial $X^{2p} + wX^p + wb$ is irreducible over $\hat{J}(w)$ *by* Eisenstein's criterion. However $L/\hat{J}(w)$ does not split else $w^{p^{-1}}, w^{p^{-1}}b^{p^{-1}} \in L$ by [9, Lemma 3.7, p. 102] and so $b^{p^{-1}} \in L$. Thus L/K is algebraic. Clearly L/K splits. Conversely, suppose L/K is algebraic and splits, say $L = S \otimes_K J$ where S/K is separable algebraic. Then $L =$

$SL' \otimes_{L'} JL'$ for every intermediate field L' of L/K.

(2) Suppose L/L' splits for every L'. If L/K is algebraic, then L is separable algebraic over the perfect field $K^{p^{-\infty}}$, and is thus perfect. Suppose L/K is not algebraic and $L \neq L^p$. Let $b \in L \backslash L^p$. Then $L/\hat{J}(w)$ does not split, a contradiction, where $\hat{J}(w)$ is the field defined in (1) above. Hence $L = L^p$. Conversely, suppose $L = L^p$. Let L' be an intermediate field of L/K. Since $L = L^p$, $L'^{p^{-\infty}} \subseteqq L$. Hence L/L' splits by 1.3.

THEOREM 3.2. (1) *Suppose $L \not\supseteqq K^{p^{-\infty}}$. Then L/L' is modular for every intermediate field L' of L/K if and only if L/K is algebraic splits and L/L' is modular for every intermediate field L' of L/S where S is the maximal separable intermediate field of L/K.*

(2) *Suppose $L \supseteqq K^{p^{-\infty}}$. Then L/L' is modular for every intermediate field L' of L/K if and only if $L = L^p$.*

Proof. (1) Suppose L/L' is modular for every L'. If L/K is not algebraic, then we can construct the field $\hat{J}(w)$ of 3.1. Since $L/\hat{J}(w)$ is algebraic, $L/\hat{J}(w)$ cannot be modular else it would split. Thus L/K is algebraic. Since L/K is also modular, L/K splits. Conversely, suppose L/K is algebraic and splits and L/L' is modular for every intermediate field L' of L/S. Then L/L' is modular for every intermediate field L' of L/K by [7, Lemma 4, p. 340] since L/L' necessarily splits.

(2) Suppose L/L' is modular for every L'. If L/K is algebraic, then L is separable algebraic over the perfect field $K^{p^{-\infty}}$ and is thus perfect. Suppose L/K is not algebraic and $L \neq L^p$. Let $b \in L \backslash L^p$. Then $L/\hat{J}(w)$ is not modular, a contradiction, where $\hat{J}(w)$ is the field defined in 3.1. Hence $L = L^p$. Conversely, suppose $L = L^p$. Let L' be an intermediate of L/K. By 3.1(2), L/L' splits, say $L = D \otimes_{L'} L'^{p^{-\infty}}$ where D is separable over L'. Since $L'^{p^{-\infty}}/L'$ is modular, L/L' is modular.

COROLLARY 3.3. *Let L be a perfect field. Then L splits over every subfield.*

Proof. $Z_p^{p^{-\infty}} = Z_p \subseteqq L$ *and* $L = L^p$.

COROLLARY 3.4. *Consider the following statements:*

(a) *L/L' and L'/K split for every intermediate field L'.*

(b) *L/L' and L'/K are modular for every intermediate field L'.*

(c) *L/K is algebraic and splits and the intermediate fields of $(L \cap K^{p^{-\infty}})/K$ appear in chain.*

(d) *$L = L^p$ and $[K: K^p] \leqq p$.*

Then; (1) *Suppose* $L \not\supseteq K^{p^{-\infty}}$. *Then* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).
(2) *Suppose* $L \supseteq K^{p^{-\infty}}$. *Then* (a) $\Rightarrow$ (b) $\Leftrightarrow$ (d).

Proof. Straightforward.

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