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LEVEL SETS OF DERIVATIVES

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We consider real-valued functions defined on intervals on the real line R, and we denote the extended real line by $\bar{R}$.

The theme of this paper is the idea that, when a function has a derivative that is equal to some $A \in \bar{R}$ on a dense set, the derivative can take other (finite) values only on a rather thin set. Our most general result shows that, in particular, the hypothesis "the derivative is equal to A on a dense set" can be replaced by "at each point of a dense set, at least one Dini derivate equals A." As corollaries we obtain unified and rather simple proofs of some more special known results, which we now state.

A function can be discontinuous at each point of a dense set and yet be continuous at each point of a co-meager (residual) subset of its domain. However, the following theorem of Fort [4] shows that such a function cannot be differentiable at each point of a nonmeager set.

THEOREM F. *If $f: I \rightarrow R$ where I is an open interval and if f is discontinuous at each point of a dense subset of I, then the set of points where f has a (finite) derivative is meager in I.*

(For a different proof, see [1], p. 131; two rediscoveries are in [3] and [10].)

Recently, Cargo [2] used harmonic analysis to prove

THEOREM C. *If f is a real-valued function of finite variation defined on a compact interval I, and if, for some $A \in R$, $f'(x) = A$ on a dense subset of I, then the set of those points at which f has a (finite) derivative different from A is meager in I.*

In 1903 W. H. Young [11] proved

THEOREM Y. *If $f: I \rightarrow R$ where I is an open interval, then the set of all points at which at least one of the Dini derivates of f is infinite is a G_δ subset of I.*

In this paper we use real-variable methods to establish a result (Theorem 2) that includes Theorems F and C (without the hypothesis of finite variation) as corollaries. We also give a short, elementary proof of Theorem Y, observe that Theorem F is an easy consequence of Theorem Y, and then prove a theorem (Theorem 3) that has Theorems 2, Y, F, and C as corollaries.

2. The main theorems.

THEOREM 1. *Let $f: I \to R$ where I is an interval, and let $A \in \bar{R}$. If $f'(x) = A$ on a dense subset of I, then the set of those points at which f has a (finite) derivative different from A is meager in I.*

Note that Theorem C is an immediate consequence of Theorem 1. Since each interval is a Baire space with respect to the inherited metric, we have

COROLLARY 1. *If $f: I \to R$ has a (finite) derivative at each point of the interval I, if $A \in R$, and if $f'(x) = A$ on a dense subset of I, then the set of points at which $f'(x) = A$ is nonmeager and co-meager in I; and, hence, each subinterval of I contains uncountably many points at which $f'(x) = A$.*

Theorem 1 is a special case of, but easier to prove than, the following result.

THEOREM 2. *Let $f: I \to R$ where I is an interval, and let $A \in \bar{R}$. If at each point of a dense subset of I at least one of the Dini derivates of f has the value A, then the set of those points at which f has a (finite) derivative different from A is meager in I.*

Clearly, Theorem C is a corollary of Theorem 2.

To prove that Theorem F is a corollary of Theorem 2, suppose that a function f is discontinuous at each point of a dense subset of an open interval I. Let F denote the set of points in I at which f has a (finite) derivative. We want to prove that F is meager in I. Let $D_{+\infty}(D_{-\infty})$ denote the set of points in I at which at least one Dini derivate of f is equal to $+\infty(-\infty)$. Then $D_{+\infty} \cup D_{-\infty}$ is dense in I, since f is clearly continuous at any point at which all Dini derivates are finite. Hence, each open subinterval of I contains an open interval in which either $D_{+\infty}$ or $D_{-\infty}$ is dense. Call an open subinterval of I distinguished if either $D_{+\infty}$ or $D_{-\infty}$ is dense in the subinterval, and let G denote the union of all distinguished intervals. Our previous observation shows that $I \backslash G$ is nowhere dense in I. Clearly, G is separable since R is separable. According to Lindelöf's covering theorem, $G = \bigcup_n G_n$ where $\{G_1, G_2, \cdots\}$ is a countable set of (not necessarily disjoint) distinguished intervals. According to Theorem 2, each $F \cap G_n$ is meager in G_n and, hence, in I. Finally, $F = \{F \cap (I \backslash G)\} \cup \bigcup_n (F \cap G_n)$ is meager in I, as desired.

Proofs of Theorems 1 *and* 2. In each theorem, it is enough to consider the set S where $f'(x) < A$, since the set where $f'(x) > A$ is the set where $(-f)'(x) < -A$. If A is finite, S is contained in $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{n,m}$ where $E_{n,m}$ consists of all points x in I such that $y \in I$ and $0 < |y - x| < 1/n$ imply that $(f(y) - f(x))/(y - x) < A - 1/m$; if $A = +\infty$, replace $A - 1/m$ by m. To show that S is meager in I, we have only to show that each $E_{n,m}$ is nowhere dense.

Suppose that some $E_{N,M}$ is dense in some open interval J. In Theorem 1, there is a dense set of points x at which $f'(x) = A$; let x_0 be such a point in J. Since $E_{N,M}$ is also dense in J, for each positive k, there exists $x_k \in E_{N,M} \backslash \{x_0\}$ such that $x_k \to x_0$ as $k \to \infty$. Thus, if k is so large that $|x_0 - x_k| < 1/N$, then $(f(x_k) - f(x_0))/(x_k - x_0) < A - 1/M$ (or $< M$ if $A = +\infty$). Letting $k \to \infty$, we get $f'(x_0) \leqq A - 1/M$ (or $\leqq M$), contradicting $f'(x_0) = A$. Therefore, each $E_{n,m}$ is nowhere dense.

In Theorem 2, at each point of a dense set least one of the Dini derivates has the value A; let x_0 be a point of the dense set that is also in J. Then there exists, for each positive integer k, a point $z_k \in J \backslash \{x_0\}$ such that, as $k \to \infty$, $z_k \to x_0$ and $(f(z_k) - f(x_0))/(z_k - x_0) \to A$. As for Theorem 1, for each positive integer k, there exists a point $x_k \in E_{N,M} \backslash \{x_0\}$ between x_0 and z_k. For all sufficiently large k, we have $0 < |x_0 - x_k| < 1/N$ and $0 < |z_k - x_k| < 1/N$. Hence, since $x_k \in E_{N,M}$, for all sufficiently large k, we have $(f(x_0) - f(x_k))/(x_0 - x_k) < A - 1/M$ (or M) and $(f(z_k) - f(x_k))/(z_k - x_k) < A - 1/M$ (or M). Clearly,

$$\frac{f(z_k) - f(x_0)}{z_k - x_0} = \frac{f(z_k) - f(x_k)}{z_k - x_k}\frac{z_k - x_k}{z_k - x_0} + \frac{f(x_k) - f(x_0)}{x_k - x_0}\frac{x_k - x_0}{z_k - x_0};$$

and the right-hand side of the last equation is a convex combination of the two difference quotients, each of which is less than $A - 1/M$ (or M) for all sufficiently large k. Letting $k \to \infty$, we obtain $A \leqq A - 1/M$ (or M), which is a contradiction; and, again, each $E_{n,m}$ is nowhere dense in I.

The original proof of Theorem Y is quite complicated (see [11] or [9], pp. 402–404). We now give a simple, elementary proof.

Proof of Theorem Y. For each positive integer n, let F_n denote the set of all $x \in I$ such that $|(f(y) - f(x))/(y - x)| \leqq n$ whenever $y \in I$ and $0 < |y - x| < 1/n$. Also, let F denote the set of all points at which each Dini derivate of f is finite. Then it is geometrically clear (and not difficult to prove analytically) that $F = \bigcup_{n=1}^{\infty} F_n$. Once we prove that each F_n is closed in I we shall be done. Suppose that n is a positive integer and that x is a limit point of F_n in I. We want to prove that $x \in F_n$. Let y be a point of I such that $0 < |y - x| < 1/n$. We want to prove that

$$(1) \qquad \left| \frac{f(y) - f(x)}{y - x} \right| \leqq n \, .$$

Since x is a limit point of F_n, there exists a sequence $z_1, z_2, z_3, \cdots$ of points of $F_n \backslash \{x, y\}$ such that $z_k \to x$ as $k \to \infty$. Next, note that, for each positive integer k,

$$(2) \quad \frac{f(y) - f(z_k)}{y - z_k} = \frac{f(y) - f(x)}{y - x} \frac{y - x}{y - z_k} + \frac{f(x) - f(z_k)}{x - z_k} \frac{x - z_k}{y - z_k} \, .$$

Since $z_k \to x$ as $k \to \infty$ and $z_k \in F_n$ for each k, it follows that

$$\left| \frac{f(x) - f(z_k)}{x - z_k} \right| \leqq n$$

for all sufficiently large k. From $\lim_{k \to \infty}(x - z_k)/(y - z_k) = 0$, we conclude that

$$\lim_{k \to \infty} \frac{f(x) - f(z_k)}{x - z_k} \frac{x - z_k}{y - z_k} = 0 \, .$$

Finally, since $\lim_{k \to \infty}(y - x)/(y - z_k) = 1$, we see from (2) that

$$(3) \qquad \lim_{k \to \infty} \frac{f(y) - f(z_k)}{y - z_k} = \frac{f(y) - f(x)}{y - x} \, .$$

Since $z_k \in F_n$ for each k and $\lim_{k \to \infty} |y - z_k| = |y - x| < 1/n$, it follows that

$$(4) \qquad \left| \frac{f(y) - f(z_k)}{y - z_k} \right| \leqq n \text{ for all sufficiently large } k \, .$$

From (3) we obtain

$$(5) \qquad \lim_{k \to \infty} \left| \frac{f(y) - f(z_k)}{y - z_k} \right| = \left| \frac{f(y) - f(x)}{y - x} \right| \, .$$

We conclude from (4) and (5) that (1) holds, as desired.

Thus, $F = \bigcup_{n=1}^{\infty} F_n$ is an F_σ subset of I, and $I \backslash F$ is a G_δ subset of I, that is, the set of all points at which at least one of the Dini derivates of f is infinite is a G_δ subset of I. This completes the proof of Theorem Y.

Next, we shall prove that Theorem F is a simple consequence of Theorem Y. As we noted above, the set of discontinuities of f is a subset of the set of all points at which at least one of the Dini derivates of f is infinite. Since the former set is dense in I, so is the latter. By Theorem Y, the latter set is a G_δ subset of I. Since a dense G_δ subset is co-meager (see [8], p. 135), it follows that the set of points at which all four Dini derivates are finite is meager in I. Finally, the set of points at which f has a (finite) derivative is meager in I because it is a subset of the latter set.

3. An extension. Next, we shall prove a theorem that has Theorem 2 as a direct corollary. If the domain of a real-valued function f contains an open interval containing a real number x, we define the set $D(f; x)$ of derivates of f at x to consist of all $A \in \bar{R}$ for which there exists a sequence $x_1, x_2, x_3, \cdots$ of real numbers distinct from x and converging to x such that $\lim_{n\to\infty}(f(x_n) - f(x))/(x_n - x) = A$ (see [7], pp. 115–116). The set $D^+(f; x)$ of right derivates of f at x and the set $D_-(f; x)$ of left derivates of f at x are defined in the obvious way. Clearly, $D(f; x) = D^+(f; x) \cup D_-(f; x)$. One can prove that $D(f; x)$ is a closed subset of $\bar{R}$ and, if f is continuous in a neighborhood of x, that $D(f; x)$ is an interval. The usual Dini derivates are extreme unilateral derivates (see [7], p. 116). For example, the upper right (Dini) derivate of f at x is just the largest element of $D^+(f; x)$, that is

$$f^+(x) = \limsup_{u\to x+} \frac{f(u) - f(x)}{u - x} = \max D^+(f; x) .$$

Of course, f has a derivative at x in the extended sense if and only if $D(f; x)$ consists of just one point of $\bar{R}$.

THEOREM 3. *Let $f\colon I \to R$ where I is an open interval, and let $A \in \bar{R}$. Then the set of x such that $D(f; x)$ contains at least one element of $\{A, +\infty, -\infty\}$ is a G_δ subset of I.*

Proof. If $A = +\infty$ or $A = -\infty$, the desired conclusion follows from Theorem Y, which we just proved.

Suppose that $A \in R$. Let F denote the set of all points at which each derivate of f is finite; let D_A denote the set of all $x \in I$ such that $A \in D(f; x)$; and, for each positive integer n, let E_n denote the set of all $x \in I$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - A \right| \geqq \frac{1}{n}$$

whenever $y \in I$ and $0 < |y - x| < 1/n$.

First, let us prove that $I\backslash D_A = \bigcup_{n=1}^{\infty} E_n$. Suppose that $x \in \bigcup_{n=1}^{\infty} E_n$. Then $x \in E_n$ for some positive integer n. If $x_k \to x$ as $k \to \infty$ where $x_k \in I\backslash\{x\}$ for each k, then $0 < |x_k - x| < 1/n$ for all sufficiently large k; hence, since $x \in E_n$,

$$\left| \frac{f(x_k) - f(x)}{x_k - x} - A \right| \geqq \frac{1}{n}$$

for all sufficiently large k. Thus, $(f(x_k) - f(x))/(x_k - x)$ cannot converge to A as $k \to \infty$, that is, $x \in I\backslash D_A$. Next, suppose that $x \in I\backslash\bigcup_{n=1}^{\infty} E_n$. Then, for each positive integer n, $x \in I\backslash E_n$; and, hence, there exists $y_n \in I$ such that $0 < |y_n - x| < 1/n$ and

$$\left| \frac{f(y_n) - f(x)}{y_n - x} - A \right| < \frac{1}{n} .$$

Then $y_n \to x$ as $n \to \infty$, $y_n \in I\backslash\{x\}$ for each n, and $(f(y_n) - f(x))/(y_n - x) \to A$ as $n \to \infty$; consequently, $x \in D_A$, that is, $x \notin I\backslash D_A$, as desired.

Next, let us prove that, for each positive integer n, $F \cap E_n$ is closed in F. Let $x_0 \in F$ be a limit point of $F \cap E_n$. We want to prove that $x_0 \in E_n$. Given $y \in I$ such that $0 < |y - x_0| < 1/n$, it will suffice to prove that

$$(6) \qquad \left| \frac{f(y) - f(x_0)}{y - x_0} - A \right| \geqq \frac{1}{n} .$$

Since x_0 is a limit point of $F \cap E_n$, there exists a sequence $x_1, x_2, x_3, \cdots$ of points of $E_n\backslash\{x_0, y\}$ such that $x_k \to x_0$ as $k \to \infty$. Now, clearly, f is continuous at x_0 since $x_0 \in F$. Hence,

$$(7) \qquad f(x_k) \longrightarrow f(x_0) \quad \text{as} \quad k \longrightarrow \infty .$$

Since $x_k \to x_0$ as $k \to \infty$, it follows that $0 < \lim_{k\to\infty} |y - x_k| = |y - x_0| < 1/n$. Thus, there exists a positive integer k_1, such that $0 < |y - x_k| < 1/n$ if $k > k_1$. Since $x_k \in E_n$ for each k, it follows that

$$(8) \qquad \left| \frac{f(y) - f(x_k)}{y - x_k} - A \right| \geqq \frac{1}{n} \text{whenever } k > k_1 .$$

From (7) we obtain

$$(9) \qquad \lim_{k\to\infty} \left| \frac{f(y) - f(x_k)}{y - x_k} - A \right| = \left| \frac{f(y) - f(x_0)}{y - x_0} - A \right| ;$$

and (9) combined with (8) yields (6). Thus, each $F \cap E_n$ is closed in F.

Since $F \cap (I\backslash D_A) = F \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (F \cap E_n)$, it follows that $F \cap (I\backslash D_A)$ is an F_σ subset of F. By Theorem Y, F is an F_σ subset of I. Moreover, if U is an F_σ subset of V, and V is an F_σ subset of W, then U is an F_σ subset of W (see [8], p. 63). Hence, $F \cap (I \backslash D_A)$ is an F_σ subset of I. Finally, by De Morgan's law, $I \backslash \{F \cap (I \backslash D_A)\} = \{I \backslash F\} \cup D_A$ is a G_δ subset of I, that is, the set of x such that $D(f; x)$ contains at least one element of $\{A, +\infty, -\infty\}$ is a G_δ subset of I. This completes the proof of the theorem.

Next, let us prove a corollary of Theorem 3 that, in turn, has Theorem 2 as a direct corollary.

COROLLARY 2. *Let $f: I \to R$ where I is an interval, and let $A \in \bar{R}$. If, at each point of a dense subset of I, A is a derivate of f, then the set of those points at which f has a (finite) derivative different from A is meager in I.*

Proof. Without loss of generality we may, and do, assume that I is open.

Since $D_A = \{x \in I: A \in D(f; x)\}$ is, by hypothesis, dense in I and $D_A \subset D_A \cup (I \backslash F)$ where F is the set of all points at which each Dini derivate of f is finite, it follows that $D_A \cup (I \backslash F)$ is dense in I. According to Theorem 3, $D_A \cup (I \backslash F)$ is a G_δ subset of I. Since $D_A \cup (I \backslash F)$ is a dense G_δ subset of I, it is co-meager in I, that is, $I \backslash \{D_A \cup (I \backslash F)\} = \{I \backslash D_A\} \cap F$ is meager in I. Since the subset of I where $f'(x)$ exists (finite) and $f'(x) \neq A$ is a subset of $\{I \backslash D_A\} \cap F$, it, too, must be meager in I.

4. Conclusion. We note that a trivial modification of the proof of Theorem 2 yields Corollary 2 directly. Also, "finite" may be deleted in the statements of Theorems 1 and 2.

When this investigation was in the final stages, we discovered that it overlaps some recent research of Garg [5]. In particular, our Theorem 1 follows from Garg's Proposition 3.9 and also from his Corollary 5.2.

While this paper was in press, we learned of Filipczak's paper [3a]. Our Theorem 2 is a corollary of his lemma (p. 74). However, our Theorem 3 is in some sense stronger than that lemma since it asserts that a potentially smaller set is residual.

Finally, it should be pointed out that our observation that Fort's theorem is an easy consequence of Young's theorem was anticipated by Garg [6] in 1962.

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