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# ON THE LOW DIMENSIONAL COHOMOLOGY OF SOME INFINITE DIMENSIONAL SIMPLE LIE ALGEBRAS

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**The two dimensional cohomology, with values in the base field $K$ of characteristic 0, of a simple Lie algebra attached to any non-Euclidean indecomposable Cartan matrix is computed. We find that $\dim(H^2(\mathscr{L}, K))$ equals the nullity of the Cartan matrix which defines $\mathscr{L}$. We also show that there is an invariant 3-cocycle of $\mathscr{L}$ if and only if the matrix defining $\mathscr{L}$ is symmetrizable. This yields cohomological interpretations for all the known isomorphism class invariants of these algebras.**

Introduction. About ten years ago V.G. Kac [7] and R. Moody [12] independently discovered a class of infinite dimensional simple Lie algebras defined over fields of zero characteristic which are natural generalizations of the finite-dimensional split simple Lie algebras, and which possess many of the same structural features. More recently, these algebras have attracted wide interest, due to the fact that one has a formula, the Macdonald-Kac formula, similar to Weyl's character formula, for certain modules of these algebras. This formula has surprising connections with additive number theory and sheds new light on Dedekind's $\eta$-function. See [6], [8], [9], and [11] for these developments and also the references therein.

A classification theory describing the isomorphism classes of these algebras is lacking, due to the fact that there are no versions of the well known conjugacy theorems which hold in the finite dimensional case. However, in [3], it is shown that the nullity of the Cartan matrix which defines the algebra is indeed an isomorphism class invariant, since this is the dimension of the space of outer derivations. If the Cartan matrix defining the algebra has the property that it is symmetrizable (see § 1 for definitions) then the algebra possesses a nondegenerate symmetric associative bilinear form. This turns out to be an isomorphism class invariant as well, see [3]. These two invariants are the only known ones to date.

The purpose of the present paper is to give new interpretations to these invariants, via cohomology. We will show that if $\mathscr{L}$ is one of the algebras under consideration, arising from the Cartan matrix $(A_{ij})$, then $H^2(\mathscr{L}, K)$ has dimension equal to the nullity of $(A_{ij})$. Here $H^2(\mathscr{L}, K)$ is the 2-cohomology of $\mathscr{L}$ with values in the base field $K$. We go on to show that there is an invariant

3-cocycle if and only if $(A_{ij})$ is symmetrizable, and hence, if this is the case, then $H^3(\mathscr{L}, K)$ is nonzero. We thus obtain a cohomological interpretation of all the known isomorphism class invariants for these algebras. Of course, these results are well known and easy to prove in the finite dimensional case, (see [5]), because of Weyl's theorem on complete reducibility. However, in the infinite dimensional case we do not have such techniques available and so our methods are more computational.

In § 1 we will briefly recall the structural properties necessary for our investigation and also fix the cohomology notation we will use. Section 2 contains our main result on $H^2(\mathscr{L}, K)$, and in the final section we investigate invariant 3-cocycles. Thanks go to R. Moody for numerous suggestions concerning these matters, and to J. Lepowsky for suggesting this type of investigation to me.

**1. Basic facts and notation.** We will use the notation in [3] but for the convenience of the reader we recall this. For more information the reader may consult [2], [6], [7] and [12].

An $l \times l$ integral matrix, $(A_{ij})$, is a Cartan matrix if $A_{ii} = 2$, $A_{ij} \leqq 0$, and $A_{ij} = 0 \Leftrightarrow A_{ij} = 0$, for $1 \leqq i, j \leqq l$, $i \neq j$. The Cartan matrix is symmetrizable if and only if there are positive rational numbers $\varepsilon_1, \cdots, \varepsilon_l$ such that $A_{ij}\varepsilon_j = A_{ji}\varepsilon_i$ for $1 \leqq 1, j \leqq l$. We always assume that our Cartan matrix is indecomposable, which is the same as requiring that the associated Dynkin diagram is connected. This assumption is a matter of convenience and the results we obtain can easily be extended to the case where the matrix is decomposable. Also, we assume that $(A_{ij})$ is not one of the sixteen types of Euclidean Cartan matrices, (see [1] for a description of these). The reason for this assumption is that the algebras attached to the Euclidean Cartan matrices have null roots and are not simple and hence our methods do not directly apply to them.

To any $l \times l$ indecomposable non-Euclidean Cartan matrix $(A_{ij})$ and any field $K$ of characteristic 0 there is associated a Lie algebra, $\hat{\mathscr{L}}$, over $K$ which is called the universal Cartan matrix Lie algebra of type $(A_{ij})$, or universal C.M. algebra, for short. $\hat{\mathscr{L}}$ is generated by $3l$ elements $e_i, f_i, h_i$, $1 \leqq i \leqq l$, subject to the four relations $[e_i, f_j] = \delta_{ij}h_i$, $[e_i, h_j] = A_{ji}e_i$, $[f_i, h_j] = -A_{ji}f_i$, and $[h_i, h_j] = 0$, for $1 \leqq i, j \leqq l$. It turns out $\hat{\mathscr{L}}$ has a unique maximal ideal $\mathscr{R}$ (see [2] and [10] for a description) and we let $\mathscr{L}$ denote the corresponding simple factor algebra, and again let $e_i, f_i, h_i$ denote the images, in $\mathscr{L}$, of the generators. $\mathscr{L}$ is called the standard C.M. algebra of type $(A_{ij})$ over $K$.

Let $V$ be the free $\boldsymbol{Z}$-module of rank $l$ with basis elements

$\alpha_1, \cdots, \alpha_l$ so that $V = \boldsymbol{Z}\alpha_1 \oplus \cdots \oplus \boldsymbol{Z}\alpha_l$. If $\mathscr{H}$ denotes the linear span of $h_1, \cdots, h_l$ in $\mathscr{L}$ then $\dim \mathscr{H}$ equals the rank of the Cartan matrix $(A_{ij})$, (see [3]), and we let $V$ act on $\mathscr{H}$ via $\alpha_i(h_j) = A_{ji}$. There is a subset, $\Delta$, of $V$ such that $\mathscr{L} = \mathscr{H} + \sum_{\alpha\in\Delta}\mathscr{L}_\alpha$, (all sums direct), where $\mathscr{L}_\alpha$ is a subspace of $\mathscr{L}$ and $[\mathscr{L}_\alpha, \mathscr{L}_\beta] \subseteq \mathscr{L}_{\alpha+\beta}$. If $x \in \mathscr{L}_\alpha$, $h \in \mathscr{H}$ then $[x, h] = \alpha(h)x$. Also, $\pm\alpha_i \in \Delta$ and $\mathscr{L}_{\alpha_i} = Ke_i$, $\mathscr{L}_{-\alpha_i} = Kf_i$ for $1 \leqq i \leqq l$. If $\alpha = \sum_{i=1}^l d_i\alpha_i \in \Delta$ then either $d_i \geqq 0$ for all $i$, or $d_i \leqq 0$ for all $i$. Also, $0 \notin \Delta$. One has $\alpha \in \Delta \Leftrightarrow -\alpha \in \Delta$, and the elements of $\Delta$ are called roots of $\mathscr{L}$. By the above we can speak of positive and negative roots, and let $\Delta^+$ denote the set of positive roots and $\Delta^- = -\Delta^+$ denote the set of negative roots. We let $\mathscr{L}^+ = \sum_{\alpha\in\Delta^+}\mathscr{L}_\alpha$ and $\mathscr{L}^- = \sum_{\alpha\in\Delta^-}\mathscr{L}_\alpha$ so that $\mathscr{L} = \mathscr{L}^- \oplus \mathscr{H} \oplus \mathscr{L}^+$. $\mathscr{L}$ possesses an automorphism $\eta$ of period 2 satisfying $\eta(e_i) = f_i$ for $1 \leqq i \leqq l$, so that $\eta(\mathscr{L}^+) = \mathscr{L}^-$. Since we assume that $(A_{ij})$ is non-Euclidean we have that if $\alpha \in \Delta$ then there is some $h \in \mathscr{H}$ for which $\alpha(h) \neq 0$. If $\alpha = \sum_{i=1}^l d_i\alpha_i \in \Delta$ we let $\mathfrak{l}(\alpha) = \sum_{i=1}^l d_i$. Notice that if $\alpha \in \Delta^+$ and $\mathfrak{l}(\alpha) \geqq 2$ then each element of $\mathscr{L}_\alpha$ is a linear combination of elements of the form $[x_\beta, x_\gamma]$ where $\beta, \gamma \in \Delta^+$, $x_\beta \in \mathscr{L}_\beta$, $x_\gamma \in \mathscr{L}_\gamma$ and $\beta + \gamma = \alpha$. A similar statement holds for $\alpha \in \Delta^-$. $V_K$ denotes the space $K \otimes_Z V$ and $\langle\cdot, \cdot\rangle\colon V_K \times V_K \to K$ denotes the nondegenerate symmetric bilinear form on $V_K$ for which the basis $\alpha_1, \cdots, \alpha_l$ is orthonormal.

Let $\mathscr{L}_0 = \mathscr{H}$, and for any integer $n \geqq 1$ let

$$\mathscr{L}^n = \sum_{\substack{\alpha\in\Delta \\ |\mathfrak{l}(\alpha)|=n}} \mathscr{L}_\alpha \quad \text{and} \quad \mathscr{L}_n = \mathscr{H} + \sum_{\substack{\alpha\in\Delta \\ |\mathfrak{l}(\alpha)|\leqq n}} \mathscr{L}_\alpha .$$

Then we have $\mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots \subseteq \mathscr{L}_n \subseteq \cdots$, and $\mathscr{L}_{n+1} = \mathscr{L}_n \oplus \mathscr{L}^{n+1}$ for all $n \geqq 0$.

It is known that $\mathscr{L}$ possesses a nondegenerate symmetric associative bilinear form if and only if $(A_{ij})$ is symmetrizable (see [3]). Moreover, any two such forms on $\mathscr{L}$ are scalar related, since the radical of such a form is an ideal of $\mathscr{L}$, and $\mathscr{L}$ is simple. From this it easily follows that if $(\cdot, \cdot)\colon \mathscr{L} \times \mathscr{L} \to K$ is such a form and if $(e_i, f_i) = 0$ for some $i \in \{1, \cdots, l\}$ then the form is identically 0.

At one point in our argument we will need a slightly bigger algebra than $\mathscr{L}$ which is described as follows. The radical $\mathscr{R}$ of $\hat{\mathscr{L}}$ can be written as $\mathscr{R} = \mathscr{R}^- \oplus \mathscr{H}_{\mathscr{R}} \oplus \mathscr{R}^+$ where $\mathscr{R}^+$ (resp. $\mathscr{R}^-$) is just $\mathscr{R} \cap \hat{\mathscr{L}}^+$ (resp. $\mathscr{R} \cap \hat{\mathscr{L}}^-$), and $\mathscr{H}_{\mathscr{R}}$ is the center of $\hat{\mathscr{L}}$, and is in $\hat{\mathscr{H}} = \bigoplus_{i=1}^l Kh_i \subseteq \hat{\mathscr{L}}$. Now $\mathscr{R}^+$ and $\mathscr{R}^-$ are ideals of $\hat{\mathscr{L}}$ and we let $\bar{\mathscr{L}}$ denote the factor algebra of $\hat{\mathscr{L}}$ by $\mathscr{R}^- \oplus \mathscr{R}^+$, and we let $\bar{\mathscr{H}}$ denote the linear span of $h_1, \cdots, h_l$ in $\bar{\mathscr{L}}$ so that $\dim \bar{\mathscr{H}} = l$. Then $\bar{\mathscr{L}} = \bar{\mathscr{H}} \oplus \sum_{\alpha\in\Delta}\bar{\mathscr{L}}_\alpha$, and $\mathscr{L}$ is a factor of $\bar{\mathscr{L}}$ by its center $\bar{\mathscr{H}}_C = \{h \in \bar{\mathscr{H}} \mid \alpha(h) = 0 \text{ for all } \alpha \in \Delta\}$. Clearly the dimension of $\bar{\mathscr{H}}_C$

is the nullity of $(A_{ij})$. When convenient we will identify $V_K$ and $(\bar{\mathscr{H}})^*$.

When dealing with cohomology we will use the notation in [4, Problem 12 of Chapter 1, § 3, p. 88-91], which we now recall. We readjust this notation to the particular case of the trivial one dimensional $\mathscr{L}$-module $K$. For any integer $n \geqq 1$, $C^n(\mathscr{L}, K)$ denotes the vector space of alternating linear maps of $\mathscr{L} \times \cdots \times \mathscr{L}$ ($n$ copies) into $K$, and $C^0(\mathscr{L}, K) = K$. $C^*(\mathscr{L}, K) = \bigoplus_{n=0}^{\infty} C^n(\mathscr{L}, K)$. For any $f \in C^n(\mathscr{L}, K)$ and $x, x_1, \cdots, x_{n+1} \in \mathscr{L}$ we let

$$(i(x)f)x_1, \cdots, x_{n-1}) = f(x, x_1, \cdots, x_{n-1}) ,$$

$$(\theta(x)f)(x_1, \cdots, x_n) = -\sum_{j=1}^{n} f(x_1, \cdots, x_{j-1}[x, x_j], x_{j+1}, \cdots, x_n) ,$$

and

$$(df)(x_1, \cdots, x_{n+1}) = \sum_{1 \leqq i < j \leqq n+1} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}) ,$$

where, as usual, a circumflex over a symbol denotes its omission. The maps $\theta(x)$, $i(x)$, and $d$ extend to linear maps on $C^*(\mathscr{L}, K)$ and $\theta$ is a representation of $\mathscr{L}$. We have that

( i ) $\theta(x)i(y) - i(y)\theta(x) = i([x, y])$ for all $x, y \in \mathscr{L}$,

( ii ) $di(y) + i(y)d = \theta(y)$ for all $y \in \mathscr{L}$,

(iii) $d\theta(y) = \theta(y)d$ for all $y \in \mathscr{L}$,

and (iv) $d^2 = 0$.

Let $Z^n(\mathscr{L}, K)$ be the kernel of $d$ restricted to $C^n(\mathscr{L}, K)$, and let $B^n(\mathscr{L}, K) = d(C^{n-1}(\mathscr{L}, K))$, $Z^*(\mathscr{L}, K) = \bigoplus_{n=0}^{\infty} Z^n(\mathscr{L}, K)$ and $B^*(\mathscr{L}, K) = \bigoplus_{n=0}^{\infty} B^n(\mathscr{L}, K)$. From (iv) we have that $B^n(\mathscr{L}, K) \subseteqq Z^n(\mathscr{L}, K)$ and so let $H^n(\mathscr{L}, K) = Z^n(\mathscr{L}, K)/B^n(\mathscr{L}, K)$. $H^n(\mathscr{L}, K)$ is called the $n$th cohomology space of $\mathscr{L}$ with values in $K$, and $H^*(\mathscr{L}, K) = \bigoplus_{n=0}^{\infty} H^n(\mathscr{L}, K)$ is called the cohomology space of $\mathscr{L}$ with values in $K$. It is clear that $H^0(\mathscr{L}, K) \cong K$ and that $H^1(\mathscr{L}, K)$ is naturally isomorphic to $(\mathscr{L}/[\mathscr{L}, \mathscr{L}])^*$ so that for the simple C.M. algebras $\mathscr{L}$ we have $H^1(\mathscr{L}, K) = (0)$. Elements in $Z^*(\mathscr{L}, K)$ (resp. $B^*(\mathscr{L}, K)$) are called cocycles (resp. coboundaries). An element $f \in C^n(\mathscr{L}, K)$ is invariant if and only if $\theta(x)f = 0$ for all $x \in \mathscr{L}$. It is easy to see that if $f \in C^n(\mathscr{L}, K)$ is invariant then $f \in Z^n(\mathscr{L}, K)$. Moreover, if $f \in Z^n(\mathscr{L}, K)$ then $\theta(x)f \in B^n(\mathscr{L}, K)$ for all $n \geqq 0$, since then, by (ii), $\theta(x)f = d(i(x)f)$ for any $x \in \mathscr{L}$. If $f \in C^n(\mathscr{L}, K)$ then $\mathrm{Rad}(f) = \{x \in \mathscr{L} \mid f(x, x_1, \cdots, x_{n-1}) = 0$ for all $x_i \in \mathscr{L}, 1 \leqq i \leqq n-1\}$. If $f \in Z^n(\mathscr{L}, K)$ is invariant we have that $\mathrm{Rad}(f)$ is an ideal of $\mathscr{L}$.

It is obvious that $\dim H^n(\mathscr{L}, K)$ is an isomorphism class invariant of $\mathscr{L}$, and so is the existence or nonexistence of nontrivial invariant elements in $Z^n(\mathscr{L}, K)$.

2. $H^2(\mathscr{L}, K)$. Throughout this and the final section we let $\mathscr{L}$ denote a simple C.M. algebra over the field $K$ of characteristic 0 which is attached to the indecomposable $\mathfrak{l} \times \mathfrak{l}$ Cartan matrix $(A_{ij})$ which is non-Euclidean. We are going to show that the dimension of $H^2(\mathscr{L}, K)$ equals the nullity of $(A_{ij})$. To do this we let $Z_0^2(\mathscr{L}, K) = \{f \in Z^2(\mathscr{L}, K) | \mathscr{H} \subseteqq \mathrm{Rad}(f)\}$ and $B_0^2(\mathscr{L}, K) = Z_0^2(\mathscr{L}, K) \cap B^2(\mathscr{L}, K)$. Our first step is to show that $Z_0^2(\mathscr{L}, K) \cong V_K$ and this is accomplished via a construction which associates to any $\gamma \in V_K$ an elemect $f_\gamma \in Z_0^2(\mathscr{L}, K)$. Next, we show that $B_0^2(\mathscr{L}, K) \cong \mathscr{H}^*$, and finally that $Z^2(\mathscr{L}, K) = Z_0^2(\mathscr{L}, K) + B^2(\mathscr{L}, K)$. It then follows that $H^2(\mathscr{L}, K) \cong Z_0^2(\mathscr{L}, K)/B_0^2(\mathscr{L}, K)$ and hence, $\dim H^2(\mathscr{L}, K)$ equals $\mathfrak{l}$-rank $(A_{ij})$, which is the nullity of $(A_{ij})$.

PROPOSITION 2.1. *For any $\gamma \in V_K$ there is an element $f_\gamma \in Z_0^2(\mathscr{L}, K)$ such that $f_\gamma(e_i, f_j) = \delta_{ij}\langle\gamma, \alpha_i\rangle$ for $1 \leqq i, j \leqq \mathfrak{l}$. Moreover, $Z_0^2(\mathscr{L}, K)$ is isomorphic to $V_K$ and hence* $\dim Z_0^2(\mathscr{L}, K) = \mathfrak{l}$.

*Proof.* Fix $\gamma \in V_K$ and let $\tau \in \bar{\mathscr{H}}^*$ be the corresponding element. Thus, $\langle\gamma, \alpha_i\rangle = \tau(h_i)$ for $h_i \in \bar{\mathscr{H}}$, $1 \leqq i \leqq \mathfrak{l}$. Let $p$ be the projection of $\bar{\mathscr{L}}$ onto $\bar{\mathscr{H}}$ given by the decomposition $\bar{\mathscr{L}} = \bar{\mathscr{H}} \oplus \sum_{\alpha \in \Delta} \bar{\mathscr{L}}_\alpha$ and define $\bar{f}_\gamma: \bar{\mathscr{L}} \times \bar{\mathscr{L}} \to K$ by $\bar{f}_\gamma(x, y) = \tau(P([x, y]))$ for all $x, y \in \bar{\mathscr{L}}$. Clearly, $\bar{f}_\gamma$ is an alternating form on $\bar{\mathscr{L}}$, and also, $\bar{\mathscr{H}} \subseteqq \mathrm{Rad}(\bar{f}_\gamma)$, since $[\bar{\mathscr{H}}, \bar{\mathscr{H}}] = (0)$ and $[\bar{\mathscr{H}}, \bar{\mathscr{L}}_\alpha] \subseteqq \mathscr{L}_\alpha \subseteqq \mathrm{Ker}\ P$ for all $\alpha \in \Delta$. It follows that we can define $f_\gamma: \mathscr{L} \times \mathscr{L} \to K$ by $f_\gamma(x + \bar{\mathscr{H}}_C, y + \bar{\mathscr{H}}_C) = \bar{f}_\gamma(x, y)$ for any $x, y \in \bar{\mathscr{L}}$. $f_\gamma$ is alternating and $\mathscr{H} \subseteqq \mathrm{Rad}(f_\gamma)$. We have that $f_\gamma(e_i, f_j) = \tau(\delta_{ij}h_i) = \delta_{ij}\langle\gamma, \alpha_i\rangle$ for $1 \leqq i, j \leqq \mathfrak{l}$. Finally, using the Jacobi identity of $\bar{\mathscr{L}}$ we easily obtain that $f_\gamma \in Z_0^2(\mathscr{L}, K)$. This yields the existence part of our result.

Clearly the mapping which assigns to any $\gamma \in V_K$ the element $f_\gamma \in Z_0^2(\mathscr{L}, K)$ is a linear injection. Next, assume that $f \in Z_0^2(\mathscr{L}, K)$, and let $\alpha, \beta \in \Delta$, $x_\alpha \in \mathscr{L}_\alpha$, $x_\beta \in \mathscr{L}_\beta$. We have that for any $h \in \mathscr{H}$, $0 = f([x_\alpha, x_\beta], h) = f([x_\alpha, h], x_\beta) - f([x_\beta, h], x_\alpha) = (\alpha + \beta)(h)f(x_\alpha, x_\beta)$. It follows that $f(\mathscr{L}_\alpha, \mathscr{L}_\beta) = (0)$ unless $(\alpha + \beta)(h) = 0$ for all $h \in \mathscr{H}$. In particular, since $(A_{ij})$ is non-Euclidean, we have that $f(\mathscr{L}_{\alpha_i}, \mathscr{L}_\beta) = (0)$ for all $\beta \in \Delta$, $\beta \neq -\alpha_i$. Say $f(e_i, f_i) = \gamma_i$ for $1 \leqq i \leqq \mathfrak{l}$, and let $\gamma = \sum_{i=1}^{\mathfrak{l}} \gamma_i \alpha_i$. Then $f(e_i, f_j) = f_\gamma(e_i, f_j)$ for $1 \leqq i, j \leqq \mathfrak{l}$. Letting $g = f - f_\gamma$ we find that $g \in Z_0^2(\mathscr{L}, K)$ and that $e_i, f_i \in \mathrm{Rad}(g)$ for $1 \leqq i \leqq \mathfrak{l}$. But it is easy to see that the radical of any element in $Z^2(\mathscr{L}, K)$ is a subalgebra of $\mathscr{L}$, and it follows from this that $\mathrm{Rad}(g) = \mathscr{L}$, and hence, that $f = f_\gamma$. Thus, $V_K$ is isomorphic to $Z_0^2(\mathscr{L}, K)$.

LEMMA 2.2. $B_0^2(\mathscr{L}, K) \cong \mathscr{H}^*$.

*Proof.* If $\phi \in C^1(\mathscr{L}, K) = \mathscr{L}*$ then $d\phi \in B^2(\mathscr{L}, K)$ is given by $d\phi(\mathfrak{l}_1, \mathfrak{l}_2) = -\phi([\mathfrak{l}_1, \mathfrak{l}_2])$, so since $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$, we have that $d\phi = 0$ if and only if $\phi = 0$. Now $d\phi \in B_0^2(\mathscr{L}, K)$ if and only if $\mathscr{H} \subseteqq \mathrm{Rad}(d\phi)$. Clearly, if $\mathscr{L}_\alpha \subseteqq \mathrm{Ker}\,\phi$ for all $\alpha \in \Delta$ then $\mathscr{H} \subseteqq \mathrm{Rad}(d\phi)$. Conversely, if $\mathscr{H} \subseteqq \mathrm{Rad}(d\phi)$ then let $\alpha \in \Delta$, $x_\alpha \in \mathscr{L}_\alpha$, and choose $h \in \mathscr{H}$ such that $\alpha(h) \neq 0$. Then $0 = d\phi(x_\alpha, h) = -\alpha(h)\phi(x_\alpha)$, so that $\phi(x_\alpha) = 0$, and hence $\mathscr{L}_\alpha \subseteqq \mathrm{Ker}\,\phi$ for all $\alpha \in \Delta$. Thus, $B_0^2(\mathscr{L}, K)$ is isomorphic to $\{\phi \in \mathscr{L}^* | \mathscr{L}_\alpha \subseteqq \mathrm{Ker}\,\phi$ for all $\alpha \in \Delta\}$, and it is clear that this last space is isomorphic to $\mathscr{H}^*$.

We are now in a position to obtain the main result of this section.

THEOREM 2.3. $Z^2(\mathscr{L}, K) = Z_0^2(\mathscr{L}, K) + B^2(\mathscr{L}, K)$. *In particular, the dimension of* $H^2(\mathscr{L}, K)$ *equals the nullity of the Cartan matrix* $(A_{ij})$ *which defines the C.M. algebra* $\mathscr{L}$.

*Proof.* As in [3] we let $\{e_{\beta_i}\}_{i=1}^\infty$ be a basis of $\mathscr{L}^+$ such that $e_{\beta_i} \in \mathscr{L}_{\beta_i}$ for all $i \geqq 1$ and $e_{\beta_j} = e_j$ for $1 \leqq j \leqq \mathfrak{l}$. Let $e_{-\beta_j} = \eta(e_{\beta_j})$ so that $\{e_{-\beta_j}\}_{j=1}^\infty$ is a basis of $\mathscr{L}^-$ and $e_{-\beta_j} = f_j$ for $1 \leqq j \leqq \mathfrak{l}$. For each $i \geqq 1$ we choose $h_i \in \mathscr{H}$ such that $\beta_i(h_i) = 2$ for all $i \geqq 1$ and $h_j = [e_j, f_j]$ for $1 \leqq j \leqq \mathfrak{l}$. This choice of an infinite collection of $h_i$'s is possible because $(A_{ij})$ is non-Euclidean.

If $f \in Z^2(\mathscr{L}, K)$ we define $\phi \in \mathscr{L}^*$ by the equations $\phi(e_{\beta_i}) = (1/2) f(e_{\beta_i}, h_i)$, $\phi(e_{-\beta_i}) = -1/2\, f(e_{-\beta_i}, h_i)$ for $i \geqq 1$ and $\phi(h) = 0$ for all $h \in \mathscr{H}$. We certainly have that $f + d\phi \in Z^2(\mathscr{L}, K)$. If $i, j, k \geqq 1$ then $0 = df(e_{\beta_i}, h_j, h_k)$ implies that $\beta_i(h_j)f(e_{\beta_i}, h_k) = \beta_i(h_k)f(e_{\beta_i}, h_j)$. Taking $k = i$ in this yields $2f(e_{\beta_i}, h_j) = \beta_i(h_j)f(e_{\beta_i}, h_i) = 2\beta_i(h_j)\phi(e_{\beta_i})$, so that $f(e_{\beta_i}, h_j) = \beta_i(h_j)\phi(e_{\beta_i}) = \phi([e_{\beta_i}, h_j]) = -d\phi(e_{\beta_i}, h_j)$. Thus, $(f + d\phi)(e_{\beta_i}, h_j) = 0$ for all $i, j \geqq 1$. Similarly, $(f + d\phi)(e_{-\beta_i}, h_j) = 0$ for all $i, j \geqq 1$ so if $g = f + d\phi$ then we have that $g(\mathscr{L}^+, \mathscr{H}) = (0) = g(\mathscr{L}^-, \mathscr{H})$, and $g \in Z^2(\mathscr{L}, K)$. Next, note that $0 = dg(e_i, f_i, h_j) = -g([e_i, f_i], h_j) + g([e_i, h_j], f_i) - g([f_i, h_j], e_i)$, which implies $g(h_i, h_j) = \alpha_i(h_j)g(e_i, f_i) - \alpha_i(h_j)g(e_i, f_i) = 0$ for $1 \leqq i, j \leqq \mathfrak{l}$. It now follows that $\mathscr{H} \subseteqq \mathrm{Rad}(g)$ so that $f + d\phi = g \in Z_0^2(\mathscr{L}, K)$, and hence $f \in Z_0^2(\mathscr{L}, K) + B^2(\mathscr{L}, K)$. Combining this with Proposition 2.1 and Lemma 2.2 now yields our result.

REMARK 2.4. (a) Since in [3] it is shown that the dimension of the outer derivation algebra of $\mathscr{L}$ is the nullity of $(A_{ij})$ and since this space is isomorphic to $H^1(\mathscr{L}, \mathscr{L})$ we have that $H^2(\mathscr{L}, K) \cong H^1(\mathscr{L}, \mathscr{L})$. Moreover, it is shown in [3] that for any $n \geqq 2$ there exists a symmetrizable, and also a nonsymmetrizable, Cartan matrix with nullity $n$.

(b) There are no nonzero invariant elements in $C^2(\mathscr{L}, K)$. Indeed, if $f \in C^2(\mathscr{L}, K)$ and $\theta(x)f = 0$ for all $x \in \mathscr{L}$ then we must have that $f \in Z^2(\mathscr{L}, K)$. But then if $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \in \mathscr{L}$ we have that $0 = df(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) = -f([\mathfrak{l}_1, \mathfrak{l}_2], \mathfrak{l}_3) + f([\mathfrak{l}_1, \mathfrak{l}_3], \mathfrak{l}_2) - f([\mathfrak{l}_2, \mathfrak{l}_3], \mathfrak{l}_1)$. Since $0 = \theta(\mathfrak{l}_1)f(\mathfrak{l}_2, \mathfrak{l}_3) = -f([\mathfrak{l}_1, \mathfrak{l}_2], \mathfrak{l}_3) - f(\mathfrak{l}_2, [\mathfrak{l}_1, \mathfrak{l}_3]) = -f([\mathfrak{l}_1, \mathfrak{l}_2], \mathfrak{l}_3) + f([\mathfrak{l}_1, \mathfrak{l}_3], \mathfrak{l}_2)$, we obtain $f([\mathfrak{l}_2, \mathfrak{l}_3], \mathfrak{l}_1) = 0$. Thus, $f([\mathscr{L}, \mathscr{L}], \mathscr{L}) = 0$, and this implies that $f = 0$.

**3. Invariants in $C^3(\mathscr{L}, K)$.** Our main goal here is to show that there is a nonzero invariant element in $C^3(\mathscr{L}, K)$ if and only if the matrix $(A_{ij})$ which defines $\mathscr{L}$ is symmetrizable. Moreover, we go on to show that if $(A_{ij})$ is symmetrizable then the vector space of invariant elements in $C^3(\mathscr{L}, K)$ is of dimension one and does not lie in $B^3(\mathscr{L}, K)$. It follows that when $(A_{ij})$ is symmetrizable we have $H^3(\mathscr{L}, K) \neq (0)$. Of course if $f \in C^3(\mathscr{L}, K)$ is invariant then it is in $Z^3(\mathscr{L}, K)$. We begin by recalling the following result of [3].

THEOREM 3.1. *$\mathscr{L}$ possesses a nondegenerate symmetric associative bilinear form if and only if the matrix $(A_{ij})$ which defines $\mathscr{L}$ is symmetrizable.*

Using this result we have

LEMMA 3.2. *If the matrix $(A_{ij})$ which defines $\mathscr{L}$ is symmetrizable then $Z^3(\mathscr{L}, K)$ has a nonzero invariant element.*

*Proof.* Define $f \in C^3(\mathscr{L}, K)$ by $f(x, y, z) = ([x, y], z)$ for all $x, y, z \in \mathscr{L}$ where $(\cdot, \cdot)\colon \mathscr{L} \times \mathscr{L} \to K$ is any nondegenerate symmetric associative bilinear form on $\mathscr{L}$. $f$ is nonzero since $(\cdot, \cdot)$ is nondegenerate and $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$. An easy computation, [see 5], shows that $f$ is invariant and hence is in $Z^3(\mathscr{L}, K)$.

We now assume that $f \in Z^3(\mathscr{L}, K)$ is a nonzero invariant element. Thus, if $\mathfrak{l}_i \in \mathscr{L}$ for $1 \leqq i \leqq 4$, we have

$$f([\mathfrak{l}_1, \mathfrak{l}_2], \mathfrak{l}_3, \mathfrak{l}_4) = -f(\mathfrak{l}_2, [\mathfrak{l}_1, \mathfrak{l}_3], \mathfrak{l}_4) - f(\mathfrak{l}_2, \mathfrak{l}_3, [\mathfrak{l}_1, \mathfrak{l}_4]) . \tag{3.3}$$

Setting $\mathfrak{l}_1 = h, \mathfrak{l}_2 = h' \in \mathscr{H}$ and letting $\mathfrak{l}_3 = x_\alpha \in \mathscr{L}_\alpha, \mathfrak{l}_4 = x_\beta \in \mathscr{L}_\beta$ for $\alpha, \beta \in \Delta$, we obtain, from (3.3), that

$$(\alpha + \beta)(h)f(h', x_\alpha, x_\beta) = 0 . \tag{3.4}$$

Setting $\mathfrak{l}_2 = h', \mathfrak{l}_3 = h'' \in \mathscr{H}$ and letting $\mathfrak{l}_4 = x_\alpha \in \mathscr{L}_\alpha$ in (3.3) we obtain, taking $\mathfrak{l}_1 = h \in \mathscr{H}$ such that $\alpha(h) \neq 0$,

$$f(h', h'', x_\alpha) = 0 . \tag{3.5}$$

LEMMA 3.6. *Let $f \in Z^3(\mathscr{L}, K)$ be a nonzere invariant element. Then there is a symmetric bilinear form $(\cdot, \cdot)\colon \mathscr{H} \times \mathscr{H} \to K$ such that $(h_k, h_i) = f(h_k, e_i, f_i)$ for $1 \leqq k, i \geqq l$.*

*Proof.* For $1 \leqq k, i \leqq l$ define $B_{ki} = f(h_k, e_i, f_i)$. Then if $k \neq i$ we have, using (3.3), that

$$B_{ki} = f([e_k, f_k], e_i, f_i) = -f(f_k, [e_k, e_i], f_i) - f(f_k, e_i, [e_k, f_i]) .$$

Since $i \neq k$, $[e_k, f_i] = 0$, and hence, using (3.3) once again, we obtain $B_{ki} = f(e_k, [e_i, f_k], f_i) + f(e_k, f_k, h_i) = B_{ik}$. Thus, the $l \times l$ matrix $(B_{ki})$ is symmetric. Clearly, if $d_1, \cdots, d_l \in K$ and $\sum_{k=1}^{l} d_k h_k = 0$, then $\sum_{k=1}^{l} d_k f(h_k, e_i, f_i) = \sum_{k=1}^{l} d_k B_{ki} = 0$, so that we can define $(h_k, h_i) = B_{ki}$ for $1 \leqq k, i \leqq l$, and extend by linearity to obtain the desired pairing.

Our goal now is to show that this symmetric pairing on $\mathscr{H}$ is nondegenerate. To do this the following lemma is crucial.

LEMMA 3.7. *If $f \in Z^3(\mathscr{L}, K)$ is invariant then*

$$f(l_1, [l_2, l_3], l_4) = f(l_1, [l_2, l_4], l_3) + f(l_1, l_2, [l_3, l_4]) ,$$

*for any $l_i \in \mathscr{L}$, $1 \leqq i \leqq 4$.*

*Proof.* $0 = \theta(l_1)f = di(l_1)f + i(l_1)df = di(l_1)f$ by (ii) of § 1. Thus, $i(l_1)f \in Z^2(\mathscr{L}, K)$ so that

$$i(l_1)f([l_2, l_3], l_4) = i(l_1)f([l_2, l_4], l_3) + i(l_1)f(l_2, [l_3, l_4]) .$$

This is the desired equality.

LEMMA 3.8. *Let $f \in Z^3(\mathscr{L}, K)$ be a nonzero invariant element and let $(\cdot, \cdot)\colon \mathscr{H} \times \mathscr{H} \to K$ denote the corresponding symmetric bilinear form defined by $(h_k, h_i) = f(h_k, e_i, f_i)$ for $1 \leqq k, i \leqq l$. Then the form $(\cdot, \cdot)$ is nondegenerate.*

*Proof.* Assume $h \in \mathscr{H}$ and $(h, h') = 0$ for all $h' \in \mathscr{H}$. We will show that $h = 0$. We begin by noting that if $1 \leqq i, j, k \leqq l$, then $f(h_i, h_j, h_k) = f(h_i, [e_j, f_j], h_k) = f(h_i, [e_j, h_k], f_j) + f(h_i, e_j, [f_j, h_k])$, by Lemma 3.7. Thus, $f(h_i, h_j, h_k) = 0$, so we get $f(h, h', h'') = 0$ for any $h', h'' \in \mathscr{H}$. Since $(h, h') = 0$ for all $h' \in \mathscr{H}$ we have that $f(h, e_i, f_i) = 0$ for $1 \leqq i \leqq l$. These facts, together with (3.4) and (3.5) imply that $f(h, l, l') = 0$ for any $l, l' \in \mathscr{L}_1$. Now we use induction and assume that $f(h, l, l') = 0$ for any $l, l' \in \mathscr{L}_n$ where $n \geqq 1$. Let $\alpha, \beta \in \Delta$, $|l(\alpha)| = |l(\beta)| = n + 1$, and let $x_\alpha \in \mathscr{L}_\alpha$, $x_\beta \in \mathscr{L}_\beta$. Then if $\alpha + \beta \neq 0$, (3.4) implies $f(h, x_\alpha, x_\beta) = 0$. Assume $\alpha \in \Delta^+$ and that $\beta = -\alpha$. Since $n+1 \geqq 2$ we can assume that $x_\alpha = [x_\gamma, e_i]$ for some $\gamma \in \Delta^+$, $x_\gamma \in \mathscr{L}_\gamma$, and $1 \leqq i \leqq \mathfrak{l}$. Then

$$f(h, x_\alpha, x_\beta) = f(h, [x_\gamma, e_i], x_\beta) = f(h, [x_\gamma, x_\beta], e_i) - f(h, x_\gamma, [x_\beta, e_i]) ,$$

by Lemma 3.7. By induction each of these terms is 0.

We now have that $h \in \mathrm{Rad}(f)$, so since $\mathscr{L}$ is simple, $f \neq 0$, and $\mathrm{Rad}(f)$ is an ideal of $\mathscr{L}$, it follows that $h = 0$ as desired.

THEOREM 3.9. *$C^3(\mathscr{L}, K)$ has a nonzero invariant element if and only if the Cartan martrix defining $\mathscr{L}$ is symmetrizable. Moreover, the space of invariant elements in $C^3(\mathscr{L}, K)$ is at most one dimensional.*

*Proof.* Let $f \in C^3(\mathscr{L}, K)$ be a nonzero invariant element, and let $(\cdot, \cdot)\colon \mathscr{H} \times \mathscr{H} \to K$ be the nondegenerate symmetric form associated to $f$. Recall that $f(h_k, e_i, f_i) = B_{ki} = (h_k, h_i)$ for $1 \leqq k, i \leqq \mathfrak{l}$. We have that

$$\begin{aligned} B_{ki} = f(h_k, e_i, f_i) &= -\frac{1}{2} f(f_i, [e_i, h_i], h_k) \\ &= -\frac{1}{2} f(f_i, [e_i, h_k], h_i) - \frac{1}{2}(f_i, e_i, [h_i, h_k]) , \end{aligned}$$

by Lemma 3.7. Thus, since $[h_i, h_k] = 0$, we obtain that

$$B_{ik} = B_{ki} = \frac{1}{2} f(h_i, A_{ki}e_i, f_i) = \frac{1}{2} A_{ki}B_{ii} , \quad \text{for} \quad 1 \leqq i, k \leqq \mathfrak{l} .$$

Since $(\cdot, \cdot)\colon \mathscr{H} \times \mathscr{H} \to K$ is nondegenerate we obtain that $B_{ii} \neq 0$ for $1 \leqq i \leqq \mathfrak{l}$. Thus, we can replace $f$ by $(1/B_{11})f$, if necessary, to assume $B_{11} = 1$. Let $\varepsilon_i = 1/2B_{ii}$ for $1 \leqq i \leqq \mathfrak{l}$. Then we have that

$$A_{ji}\varepsilon_i = \frac{1}{2} A_{ji}B_{ii} = B_{ij} = B_{ji} = \frac{1}{2} A_{ij}B_{jj} = A_{ij}\varepsilon_j , \quad \text{for} \quad 1 \leqq i, j \leqq \mathfrak{l} .$$

Since $(A_{ij})$ is indecomposable and integral, and since $\varepsilon_1 = 1/2$, we see that $\varepsilon_j \in \boldsymbol{Q}$ and $\varepsilon_j > 0$ for $1 \leqq j \leqq \mathfrak{l}$. Thus, $(A_{ij})$ is symmetrizable. Moreover, all the $\varepsilon_i$'s are completely determined by the condition $\varepsilon_i$ equals 1/2, and so it follows that the space of invariant elements in $C^3(\mathscr{L}, K)$ is at most one dimensional.

We close by noting the following:

COROLLARY 3.10. *If the Cartan matrix defining $\mathscr{L}$ is symmetrizable then $H^3(\mathscr{L}, K) \neq (0)$.*

*Proof.* Let $(\cdot, \cdot)\colon \mathscr{L} \times \mathscr{L} \to K$ be any nondegenerate symmetric associative bilinear form and let $f \in Z^3(\mathscr{L}, K)$ be the invariant element defined by $f(x, y, z) = (x, [y, z])$, for all $x, y, z \in \mathscr{L}$. It is enough to show that $f \notin B^3(\mathscr{L}, K)$.

Assume $f \in B^3(\mathscr{L}, K)$ and choose $g \in C^2(\mathscr{L}, K)$ such that $dg = f$. Let $\mathscr{S}$ be the subalgebra of type $A_1$ in $\mathscr{L}$ generated by the elements $e_1, h_1, f_1$, and let $\bar{g}$ denote the restriction of $g$ to $\mathscr{S} \times \mathscr{S}$. Then $\bar{g} \in C^2(\mathscr{S}, K)$ and if $s \in \mathscr{S}$ then $0 = \theta(s)f = \theta(s)dg = d\theta(s)g$, and this implies that $\theta(s)g \in Z^2(\mathscr{L}, K)$. Thus, $\theta(s)\bar{g} \in Z^2(\mathscr{S}, K)$ for all $s \in \mathscr{S}$. Since $C^2(\mathscr{S}, K)$ is a finite dimensional $\mathscr{S}$-module, it is completely reducible, and so it follows, see [5], that $\bar{g} \in Z^2(\mathscr{S}, K)$, and hence $d\bar{g} = 0$. We have $0 = d\bar{g}(h_1, e_1, f_1) = dg(h_1, e_1, f_1) = f(h_1, e_1, f_1) = (h_1, [e_1, f_1]) = ([h_1, e_1], f_1) = -2(e_1, f_1)$, so that $(e_1, f_1) = 0$. From this it easily follows that $(\cdot, \cdot)\colon \mathscr{L} \times \mathscr{L} \to K$ is degenerate. This is the desired contradiction.

REMARK 3.11. It would be of great interest to have more information on the cohomology of $\mathscr{L}$, such as the dimension of $H^n(\mathscr{L}, K)$ for $n \geqq 3$, or the existence or nonexistence of invariant elements in $C^n(\mathscr{L}, K)$ for $n \geqq 4$. This information would possibly yield new isomorphism class invariants for standard C.M. algebras.

## REFERENCES

1. S. Berman, R. Moody, and M. Wonenburger, *Cartan matrices with null roots and finite Cartan matrices*, Indiana Univ. Math. J., **21** (1971), 1091–1099.

2. S. Berman, *On the construction of simple Lie algebras*, J. Algebra, **27** (1973), 158–183.

3. ————, *On derivations of Lie algebras*, Canad. J. Math., **XXVII** (1976), 174–180.

4. N. Bourbaki, *Lie Groups and Lie Algebras, Part I*: Chapters 1–3, Addison-Wesley Publishing Co., Reading, Massachusetts, 1975.

5. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc., **63** (1948), 85–124.

6. H. Garland and J. Lepowsky, *Lie algebra homology and the Macdonald-Kac formulas*, Invent. Math., **34** (1976), 37–76.

7. V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Math. U.S.S.R.-Izv., **2** (1968), 1271–1311.

8. ————, *Infinite-dimensional Lie algebras and Dedekind's η-function*, (English translation) Functional Anal. Appl., **8** (1974), 68–70.

9. I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Invent. Math., **15** (1972), 91–143.

10. R. Marcuson, *Representations and radicals of a class of infinite dimensional Lie algebras*, Indiana Univ. Math. J., **23** (1974), 883–887.

11. R. Moody, *Macdonald identities and Euclidean Lie algebras*, Proc. Amer. Math. Soc., **48** (1975), 43–52.

12. R. Moody, *A new class of Lie algebras*, J. Algebra, **10** (1968), 211–230.

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