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T AS AN $\mathscr{E}$ SUBMODULE OF G

W. J. WICKLESS

Let G be a mixed abelian group with torsion subgroup T. T is viewed as an $\mathscr{E}$ submodule of G, where $\mathscr{E} = \text{End}\, G$. It is shown that T is superfluous in G if and only if, $\forall_p$, either T_p is divisible or G/T_p is not p divisible. If G is not reduced, T is essential in G if and only if T contains a $Z(p^\infty)$. Let $I(G)$ $[I(T)]$ be the $\mathscr{E}$ injective hull of G $[T]$. Then $I(G) = I(T) \oplus X$ with X torsion free divisible and T is a pure subgroup of $I(G)$. This can be used to obtain several results; for example, if $Q \not\subseteq I(T)$, TFAE: 1. $T \operatorname{ess} G$, 2. $I(G) \cong I(T)$ as abelian groups, 3. $Q \not\subseteq I(G)$. The condition $T \operatorname{ess} G$ is characterized if T is a summand or if G is algebraically compact. If T is bounded or if T is a p-group, $T^1 = (0)$ and G is reduced cotorsion, T is not essential. In fact, for bounded T there is an $\mathscr{E}$ isomorphism $I(G) \cong I(T) \oplus I(G/T)$. Some information is obtained on the p-basic subgroups of $I(T)$ as a function of those of T. A condition is given for $I(T) \supseteq \oplus_c Q$. These last theorems specialize to $I({}_E T)$, where $E = \text{End}\, T$.

Preliminaries. In the last fifteen years several authors have written papers concerning an abelian group G viewed as a module over $\mathscr{E}$, its ring of endomorphisms.

Let G be a mixed abelian group with maximal torsion subgroup T. In this paper we consider T as an $\mathscr{E}$ submodule of G. We determine when T is superfluous in G and then study the more difficult question of determining when T is essential in G. (If $(0) \neq T \neq G$, it is easy to prove that T is neither essential nor superfluous as a Z submodule of G.)

The latter question leads to consideration of the injective hulls $I(T)$, $I(G)$—taken with respect to $\mathscr{E}$.

Our notation, with minor exceptions, is that of [1].

1. T as a superfluous submodule of G.

Henceforth, let G be a mixed abelian group, $T = t(G)$ its torsion subgroup and $\mathscr{E} = \text{End}\, G$. To avoid stating the trivial cases of our results we always assume $(0) \neq T \neq G$. We begin by characterizing those mixed G for which ${}_{\mathscr{E}}T$ is superfluous in ${}_{\mathscr{E}}G$ ($T \ll G$). In our context $T \ll G$ if and only if whenever K is a fully invariant subgroup of G with $K + T = G$, then $K = G$.

LEMMA 1. *Let $T = \bigoplus T_p$ be a decomposition of T into its p components. Then $T \ll G$ if and only if $T_p \ll G$, $\forall p$.*

Proof. The only if part of the implication is immediate since submodules of superfluous submodules are superfluous.

Suppose $T_p \ll G$, $\forall p$, and $T \not\ll G$. Then we must have $T + K = G$ for some fully invariant $K \neq G$. Clearly, $K \not\supseteq T_p$ for some p. Let $K' = K + \sum_{q \neq p} T_q$. Since K' is fully invariant with $K' + T_p = G$, $K' = G$.

Let $t \in T_p$ and suppose that t has order $o(t) = p^l$. Write $t = x + y$ with $x \in K$, $o(y) = n$, $(n, p) = 1$. If $a, b \in Z$ with $ap^l + bn = 1$, then $t = (ap^l + bn)t = bnt = bnx \in K$. Thus, $T_p \subseteq K$, a contradiction.

THEOREM 1. *$T \ll G$ if and only if, $\forall p$, either T_p is divisible or G/T_p is not p divisible.*

We prove the contrapositive in both directions.

Proof. Suppose $\exists p$ with T_p not divisible and G/T_p p divisible. Then $T_p \not\subseteq pG$ and $G = pG + T_p$. Thus, $T_p \not\ll G$ and, by Lemma 1, $T \not\ll G$.

Conversely, suppose $T \not\ll G$. Then $\exists p$ with $T_p \not\ll G$. Let K be a proper fully invariant subgroup with $K + T_p = G$. We cannot have T_p divisible, for then $K \supseteq \mathrm{Hom}(G, T_p)K = T_p$. (If $x \in K$, $o(x) = \infty$, and $t \in T_p$, the map $Zx \to Zt$ extends to G.)

G/T_p is p divisible if and only if $K \subseteq pG + T_p$. Assume that G/T_p is not p divisible. Then there is an $x \in K \backslash pG + T_p$. Therefore, $\forall t \in T_p$, the p-height of $x + t$ in G, $h_p^G(x + t)$, is zero.

Thus, for every positive integer l, $\bar{x} = x + p^l G$ must have order exactly p^l in $G/p^l G$. But then, $\forall t \in T_p$, we can construct an endomorphism of G mapping $x \to \bar{x} \to t$. This implies $K \supseteq T_p$, a contradiction. The theorem follows.

2. T as an essential submodule of G-basic results.

We next consider the more difficult problem of deciding when T is essential in $G (T \operatorname{ess} G)$. We first dispose of the nonreduced case.

THEOREM 2. *Let G be a nonreduced group. Then $T \operatorname{ess} G$ if and only if T contains a $Z(p^\infty)$.*

Proof. If $T \supseteq Z(p^\infty)$ then, $\forall x \in G$ with $o(x) = \infty$, $\exists \alpha \in \mathscr{E}$ with $0 \neq \alpha(x) \in Z(p^\infty)$. This, clearly, is enough to imply $T \operatorname{ess} G$.

Conversely, suppose T contains no $Z(p^\infty)$. Then, since G is not reduced, the maximum divisible subgroup D of G is nontrivial and torsion free. Hence $T \cap D = 0$, so T is not essential in G.

From now on we assume G is reduced.

To investigate the question of when $T \operatorname{ess} G$, it is natural to consider the $\mathscr{E}$ injective hulls. Let $I(G)$ be the injective hull of the module ${}_{\mathscr{E}}G$. Since ${}_{\mathscr{E}}T \leqq {}_{\mathscr{E}}G$ we can regard $I(T)$, the injective hull of ${}_{\mathscr{E}}T$, as a maximal $\mathscr{E}$ essential extension of T in $I(G)$. If $I(T)$ is constructed in this way we have an $\mathscr{E}$ decomposition: $I(G) = I(T) \oplus X$. Clearly, $T \operatorname{ess} G$ if and only if $X = (0)$.

THEOREM 3. *Let X be as above. Then X is torsion free divisible as an abelian group.*

Proof. If $t(X)$, the torsion subgroup of X, were nonzero, then $I(T) \oplus t(X)$ would be an $\mathscr{E}$ essential extension of T in $I(G)$ properly containing $I(T)$—a contradiction. Thus, X is torsion free. Since X is an injective module, X must also be divisible.

COROLLARY. *$T \operatorname{ess} G$ if and only if $I(T)$ and $I(G)$ are isomorphic $\mathscr{E}$ modules.*

Proof. Suppose $\theta: I(T) \to I(G)$ is an $\mathscr{E}$ isomorphism. Then $\theta(T) \operatorname{ess} I(G)$. By Theorem 3, $\theta(T) \cap X = (0)$. Thus, $X = (0)$ and $T \operatorname{ess} G$.

The next theorem is central for our results.

THEOREM 4. *T is a pure subgroup of $I(G)$ ($T \lhd I(G)$).*

Proof. Let $D(G)$ be the Z injective hull of G and let A be the injective left $\mathscr{E}$ module $\operatorname{Hom}_Z(\mathscr{E}, D(G))$. Regard $G \subseteqq A$ via $G \cong \operatorname{Hom}_{\mathscr{E}}(\mathscr{E}, G)$ and take $I(G)$ to be a maximal $\mathscr{E}$ essential extension of G in A. It suffices to show $T \lhd A$. Let $\delta \in T$ with $p\delta = 0$. Suppose $h_p^T(\delta) = m < \infty$, but $\delta = p^{m+1}\alpha$, $\alpha \in A$.

Write $\delta = p^m\delta'$, $\delta' \in T$. Then $T = \langle \delta' \rangle \oplus T'$ ([1], Corollary 27.2). Let $\pi \in \mathscr{E}$ be projection onto $\langle \delta' \rangle$. Then $\delta(\pi) = \pi(\delta) = \delta = p^{m+1}\alpha(\pi) = \alpha(p^{m+1}\pi) = 0$—a contradiction. Thus, we have proved: $\delta \in T[p] \to h_p^T(\delta) = h_p^A(\delta)$. This shows $T \lhd A$ ([1], (h), p. 114).

COROLLARY 1. *If T is a torsion group, $E = \operatorname{End} T$, then $T \lhd I({}_E T)$.*

This is proved by putting $G = T$ in the above.

COROLLARY 2. *Suppose $T \subset G$ with $T^1 = G^1$, G/T divisible. Then $T \operatorname{ess} G$. (Here T^1 $[G^1]$ denotes the first Ulm subgroup of T $[G]$.)*

Proof. Since $T \lhd I(G)$, G/T divisible, we have $G \lhd I(G)$. If

$G^1 = T^1$ and X is as in Theorem 3, $X \cap G = (0)$, so $X = (0)$. Thus, $T \operatorname{ess} G$.

COROLLARY 3. *Let $T \subset G$ with $T^1 = (0)$. Then $I(T)^1 = (0)$.*

Proof. $I(T)^1$ is an $\mathscr{E}$ submodule of $I(T)$. Since $T^1 = (0)$ and $T \lhd I(T)$, $I(T)^1 \cap T = (0)$. Thus, $I(T)^1 = (0)$.

THEOREM 5. *Let $T \subset G$ with $Q \not\subseteq I(T)$. Then TFAE*: 1. $T \operatorname{ess} G$; 2. $I(T) \cong I(G)$ *as abelian groups*; 3. $Q \not\subseteq I(G)$. *Moreover, if* 1—3 *hold, then* $T^1 = G^1$.

Proof. The implications $1 \to 2$, $2 \to 3$ are obvious. If $Q \not\subseteq I(G)$, then the X of Theorem 3 is zero, so $T \operatorname{ess} G$.

To prove the additional statement, note that $I(T)$ is an algebraically compact group ([1], p. 178) which, by assumption, contains no Q's. Thus, there can be no elements of infinite order in $I(T)^1$. If 1—3 hold, the same is true for $I(G)^1$. Thus, in this case, $G^1 = T^1$.

COROLLARY. *Let $T \subset G$ with $T^1 = (0)$. Then conditions* 1—3 *are equivalent. Moreover, if* 1—3 *hold, then* $G^1 = (0)$.

Proof. If $T^1 = (0)$, then $I(T)^1 = (0)$, so $Q \not\subseteq I(T)$.

Theorem 5 raises the questions: When are $I(T)$ and $I(G)$ isomorphic as abelian groups? Is this sufficient for $T \operatorname{ess} G$? Here is a partial result.

THEOREM 6. *Let $\bar{I}$ be the $\mathscr{E}$ injective hull of the factor module G/T. Write $I(T) = H \oplus K$, where H is the maximal torsion free divisible subgroup of $I(T)$. Let $r = \operatorname{rank} H$, $\bar{r} = \operatorname{rank} \bar{I}$. If r is infinite and $r \geqq \bar{r}$, then $I(G) \overset{+}{\cong} I(T)$.*

Proof. Embed $I(G)$ into $I(T) \oplus \bar{I}$ in the standard way (via $\alpha \oplus \beta$ where α and β are the extensions to $I(G)$ of $T \subset I(T)$ and $G \to G/T \subset \bar{I}$ respectively). Then, as $\mathscr{E}$ modules, $I(G) \oplus Y \cong I(T) \oplus \bar{I}$. Since $I(G) = I(T) \oplus X$, we have:

$$(*) \qquad I(T) \oplus X \oplus Y \cong I(T) \oplus \bar{I} .$$

The additive group of $\bar{I}$ is torsion free divisible, since $\bar{I}$ is the injective hull of a module whose additive group is torsion free. Thus, the number of Q's on the right-hand side of $(*)$ is $r + \bar{r} = r$, so $\operatorname{rank} X \leqq r$. But then, $I(G) = I(T) \oplus X \overset{+}{\cong} I(T)$.

EXAMPLE. For each prime p, let T_p be the group generated by $\{a_i \mid i = 0, 1, 2, 3, \cdots\}$ with relations $\{pa_0 = 0,\ p^n a_n = a_0,\ n = 1, 2, 3, \cdots\}$. Let $T = \bigoplus_p T_p$ and let $G = Q \oplus T$. Then $\bar{r} = 1$ and (as we will see in Theorem 13) $r \geqq c$. Thus, $I(G) \overset{+}{\cong} I(T)$. Since T is reduced, T is not essential in G.

3. *T* as an essential submodule of *G*—some special cases. In this section we consider the essentiality of T in G in some special cases. First we consider the situation for bounded T. The following theorem shows if T is bounded, then T is never essential in G.

THEOREM 7. *Let* $T \subset G$ *with* $nT = (0)$ *and let* $\bar{I} = I(G/T)$. *Then*:
1. $nI(T) = (0)$;
2. $I(G)$ *is* $\mathscr{E}$ *isomorphic to* $I(T) \oplus \bar{I}$.

Proof. Let $D(G)$, $D(T)$, $D(G/T)$ be the Z injective hulls of G, T, G/T and let A, B, C be the injective left $\mathscr{E}$ modules $\mathrm{Hom}_Z(\mathscr{E}, D(M))$ where $M = G, T, G/T$, respectively. As in Theorem 4, regard $T \subseteqq G \subseteqq I(G) \subseteqq A$. Suppressing the obvious isomorphism, write $A = B \oplus C$—an $\mathscr{E}$ direct sum. Under these identifications $T = B \cap G$.

To prove (1), recall $T \lhd A$, so in this case, $T \cap nA = nT = (0)$. Thus, if $x \in I(T)$ with $nx \neq 0$, then, for some $\lambda \in \mathscr{E}$, $0 \neq \lambda(nx) \in T \cap nA$—a contradiction.

To prove (2), first note that $B \cap I(G)$ is an essential extension of $T = B \cap G$. Choose $I(T) \subseteqq I(G)$ as before—with the additional requirement $I(T) \supseteqq B \cap I(G)$.

Let $x \in I(T)$, say $x = b + c$, $b \in B$, $c \in C$. Since C is torsion free and $nx = 0$, we must have $c = 0$. Thus, $I(T) \subseteqq B$. It follows that $I(T) = B \cap I(G)$.

Let $\pi \in \mathrm{Hom}_{\mathscr{E}}(A, C)$ be projection onto C and let $\pi' = \pi|_{I(G)}$. Clearly, $\mathrm{Ker}\,\pi' = B \cap I(G) = I(T)$, so write $I(G) = I(T) \oplus Y$ with π' a monomorphism on Y.

To finish the proof of (2), we claim $\pi'(Y)$ is an $\mathscr{E}$ injective hull of G/T. To see this, first note that if G/T is embedded in C via $e: g + T \to$ evaluation at $g + T$, we have $e(G/T) = \pi'(G) \subseteqq \pi'(Y)$, so $\pi'(Y)$ is an injective containing $e(G/T) \cong G/T$. Furthermore, if $0 \neq \pi'(y) \in \pi'(Y)$, then $\exists \lambda \in \mathscr{E}$ with $0 \neq \lambda(y) \in G \cap Y$. Thus, $0 \neq \pi'\lambda(y) = \lambda\pi'(y) \in \pi'(G) = e(G/T)$. This proves that $e(G/T)\ \mathrm{ess}\ \pi'(Y)$. The theorem follows.

EXAMPLE. Let $T = \bigoplus_{p \in P} Z(p)$, where P is an infinite set of primes, and let $G = Z \oplus T$. Then $T\ \mathrm{ess}\ G$, so $I(G) = I(T)$ and, in view of Theorem 4, $I(T)^1 = (0)$. Moreover, it is easy to see that $\bar{I} \cong_Z Q$. Thus, if T is an unbounded group direct summand of G, we need not have the decomposition of $I(G)$ given in (2).

The following gives one characterization of $T \operatorname{ess} G$ in the splitting case.

THEOREM 8. *Let* $T = \bigoplus T_p \subset G$. *Let* $k_p = \text{l.u.b.}\{l \mid G$ *has a* $Z(p^l)$ *summand*$\}$ *and let* $H = \{x \in G \mid o(x) = \infty, h_p^G(x) \geqq k_p \forall p\}$. *Then*:

(1) *If* $H = (0)$, $T \operatorname{ess} G$;

(2) *If* $G = T \bigoplus F$ *and* $T \operatorname{ess} G$, *then* $H = (0)$.

Proof. (1) is clear. To prove (2) suppose $G = T \bigoplus F$ and $0 \neq x \in H$. Then, for some positive integer n, $0 \neq nx \in H \cap F$. Clearly, nx cannot be mapped by an endomorphism of G onto any nonzero element of a bounded T_p.

If T_p is unbounded, then G has an unbounded p-basic subgroup, so $k_p = \infty$. Thus, $h_p^G(nx) = h_p^F(nx) = \infty$. If $\lambda \in \mathscr{E}$ with $0 \neq \lambda(nx) \in T_p$, then λ restricts to a nonzero map of the subgroup $\{m/p^k(nx) \mid m, k \in Z\} \subseteqq F$ into T_p. This is impossible since T_p is reduced. Thus, nx cannot be mapped by an endomorphism of G onto a nonzero element of any T_p. The result follows.

It is easy to describe when $T \operatorname{ess} G$ for algebraically compact G.

THEOREM 9. *Let* $T = \bigoplus T_p \subset G$ *with* G (*reduced*) *algebraically compact. Write* G *as a product of* p-*adic modules*, $G = \Pi G_p$. *Then* $T \operatorname{ess} G$ *if and only if*, $\forall p$, *either* $T_p = G_p$ *or* T_p *is unbounded*.

Proof. It is immediate that $T \operatorname{ess} G$ if and only if, $\forall p$, $T_p \operatorname{ess} G_p$. If $\exists p$ with $T_p \neq G_p$ and T_p bounded, then T_p is not essential in G_p.

Conversely, by considering projections onto summands of a p-adic basis for G_p, it is easy to see that T_p unbounded implies $T_p \operatorname{ess} G_p$.

We close this section with:

THEOREM 10. *Let* $T \subset G$ *with* G (*reduced*) *cotorsion*, T *a* p-*group*, $T^1 = (0)$. *Then* T *is not essential in* G.

Proof. If T is bounded, T is not essential. If T is an unbounded p-group, $(0) \neq P\operatorname{ext}(Q/Z, T) = [\operatorname{Ext}(Q/Z, T)]^1$. Since G is reduced cotorsion, $G \cong \operatorname{Ext}(Q/Z, G) \cong \operatorname{Ext}(Q/Z, T) \bigoplus \operatorname{Ext}(Q/Z, G/T)$ ([1] H, p. 234 and Lemma 55.2). Thus $G^1 \neq (0)$, $T^1 = (0)$ and T cannot be essential in G.

4. The structure of $I(T)$. In this section we prove three theorems concerning the structure of $I(T)$. With trivial modification, each of these theorems can be rewritten to give the same result for the injective hull of a torsion group over its own endomorphism ring.

Since $I(T)$ is algebraically compact, it is natural to try to find out what its p-basic subgroups look like as a function of the p-basic subgroups of T. In the case $T^1 = (0)$, this information would characterize $I(T)$ as an abelian group. The next result shows that $I(T)$ is generally large with respect to T.

THEOREM 11. *Let B $[B']$ be a p-basic subgroup of T $[I(T)]$. Let $f =$ final rank B. If $Z(p^k)$ occurs in B, then B' contains $\bigoplus_{\gamma \in \bar{\mathscr{A}}} \langle z_\gamma \rangle$ with $|\bar{\mathscr{A}}| = 2^{2^f}$, $o(z_\gamma) \geqq p^k$, $\forall \gamma$.*

Proof. Suppose B contains a $Z(p^k)$. Write $G = \langle b \rangle \oplus Y$, $o(b) = p^k$, and let $\bigoplus_{\alpha \in A} \langle b_\alpha \rangle \subseteq B$ with $|A| = f$, $o(b_\alpha) \geqq p^k \forall \alpha$.

Choose $\{A_\beta \mid \beta \in \mathscr{A}\}$ a collection of subsets of A such that: $|\mathscr{A}| = 2^f$, if F is any finite subset of $\mathscr{A}$ and $\beta_0 \in F$ then $[A_{\beta_0} \backslash \bigcup_{\beta \neq \beta_0, \beta \in F} A_\beta] \neq \varnothing$. (See [1[, Lemma 46.2.)

For $\beta \in \mathscr{A}$ define $\delta_\beta \in \mathrm{Hom}(\bigoplus \langle b_\alpha \rangle, \langle b \rangle)$ by $\delta_\beta(b_\alpha) = X_\beta(\alpha)b - X_\beta$ the characteristic function of A_β. Extend each δ_β to $\mathscr{E}$.

It is clear that the left ideals $\mathscr{E}\delta_\beta$ form a direct sum s in $\mathscr{E}$.

Let $\{C_\gamma \mid \gamma \in \bar{\mathscr{A}}\}$ be a family of subsets of $\mathscr{A}$ with the above independence property, $|\bar{\mathscr{A}}| = 2^{2^f}$. Consider:

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & \mathscr{E} \\ & & \big\downarrow{\scriptstyle \lambda_\gamma} & \swarrow{\scriptstyle \lambda'_\gamma} & \\ & & I(T) & & \end{array}$$

Here λ_γ is the $\mathscr{E}$ map defined by $\lambda_\gamma(\delta_\beta) = X_{C_\gamma}(\beta)b$, X_{C_γ} the characteristic function of the subset C_γ, and λ'_γ is the map obtained by injectively.

Let $z_\gamma = \lambda'_\gamma(1)$. We have $\delta_\beta(z_\gamma) = X_{C_\gamma}(\beta)b$. It is easy to see from this equation that $\{z_X \mid X \in \bar{\mathscr{A}}\}$ is a p independent set of elements of order $\geqq p^k$. This can be included as a summand of B'. The result follows.

Continuing with the same notation we have:

THEOREM 12. *If B' contains a $Z(p^k)$ so does B.*

Proof. If B' contains $Z(p^k)$ then $I(T)$ has a $Z(p^k)$ summand.

Therefore, so does Hom $(\mathscr{E}, D(T))$. ($I(T)$ can be regarded as a direct summand of Hom $(\mathscr{E}, D(T))$. Therefore, so does Hom $(\mathscr{E}, D(T)_p)$.

The pure exact sequence $0 \to t(\mathscr{E}) \to \mathscr{E} \to \mathscr{E}/t(\mathscr{E}) \to 0$ yields $0 \to [\mathscr{E}/t(\mathscr{E})]^* \to \mathscr{E}^* \to t(\mathscr{E})^* \to 0$, where $M^* = \mathrm{Hom}_Z(M, D(T)_p)$. This sequence is pure exact, so splits, since all its terms are algebraically compact. (In this proof "splits" means splits as an exact sequence of abelian groups.) Since $[\mathscr{E}/t(\mathscr{E})]^*$ is torsion free, $t(\mathscr{E})^*$ must have a $Z(p^k)$ summand.

Now $t(\mathscr{E})^* = [t(\mathscr{E})_p]^*$. Let B_0 be a basic subgroup for $t(\mathscr{E})_p$. Repeat the above procedure with $0 \to B_0 \to t(\mathscr{E})_p \to t(\mathscr{E})_p/B_0 \to 0$ to conclude that B_0^* must have a $Z(p^k)$ summand.

Since B_0 is a direct sum of cyclics, B_0 itself must have a $Z(p^k)$ summand. Thus, $\mathscr{E}$ and, therefore, Hom (G, T_p) have $Z(p^k)$ summands.

Let $\bar{B}$ be a p-basic subgroup for G. The p-pure exact sequence $0 \to \bar{B} \to G \to G/\bar{B} \to 0$ yields the p-pure exact sequence $0 \to (G/\bar{B})^\Delta \to G^\Delta \to (\bar{B})^\Delta$ where $M^\Delta = \mathrm{Hom}_Z(M, T_p)$. Since $(G/\bar{B})^\Delta \cong W \oplus \bigoplus_\gamma Q_\gamma$, where W is the p-adic completion of a direct sum of copies of the p-adic integers, this sequence also splits. It's not hard to show that $(\bar{B})^\Delta$ must have a $Z(p^k)$ summand.

Say $\bar{B} = \bar{B}_1 \oplus \bar{B}_2$, where $\bar{B}_1 = \bigoplus_\alpha Z(p^{l_\alpha})$ is a direct sum of finite p-power cyclics and $\bar{B}_2 = \bigoplus_\beta Z_\beta$ is free. Then $\bar{B}^\Delta = (\bar{B}_1)^\Delta \oplus (\bar{B}_2)^\Delta$, so one of these groups must contain a $Z(p^k)$ summand.

If $(\bar{B}_1)^\Delta \cong \prod_\alpha T_p[p^{l_\alpha}]$ has a $Z(p^k)$ summand, then $\bar{B}_1$ itself must, so T does.

If $(\bar{B}_2)^\Delta \cong \prod = \prod_\beta (T_p)_\beta$ has a $Z(p^k)$ summand, again T does. (If $\prod = \langle y \rangle \oplus Y$, $o(y) = p^k$, then $h_p^{\Pi}(p^{k-1}y) = k-1$. If $y = [y_\beta]$, $y_\beta \in (T_p)_\beta$, then, for some β_0, $h_p^{(T_p)\beta_0}(p^{k-1}y_{\beta_0}) = k-1$ and, therefore, $o(p^{k-1}y_{\beta_0}) = p$. Thus, y_{β_0} is contained in a $Z(p^k)$ summand of $(T_p)_{\beta_0}$.)

Thus, in either of the above cases, B contains a $Z(p^k)$.

In view of Theorem 5, it is of interest to discover when $Q \subseteq I(T)$. (Obviously, we must have $T^1 \neq (0)$.) We are unable to decide if $T^1 \neq (0)$ is also sufficient for $Q \subseteq I(T)$. We close the paper with a result in this direction. First, we need two lemmas.

LEMMA 2. *Let* $T = \oplus T_p \subset G$ *and suppose* $T_p^1 \neq (0)$ *whenever* $T_p \neq (0)$. *Then* ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Proof. If $t \in T \backslash T^1$, then $\Pi(t) \neq 0$, Π the projection onto $\langle a \rangle$, some $Z(p^k)$ summand of G. It is easy to construct $\theta \in \mathrm{Hom}_Z(\langle a \rangle, T_p^1)$ with $\theta\Pi(t) \neq 0$. Thus, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Let $\bar{\mathscr{E}} = \mathscr{E}/t(\mathscr{E})$. Since $t(\mathscr{E})T^1 = (0)$ we can regard T^1 as an $\bar{\mathscr{E}}$ module.

LEMMA 3. *Let* $\mathscr{I}$ *be the* $\bar{\mathscr{E}}$ *injective hull of* T^1 *and let* D *be* *the maximal divisible subgroup of* $I(T)$. *Then, under the assumption of Lemma* 2, $\mathscr{I} \cong D$.

Proof. By Lemma 2, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$, so $I_{\mathscr{E}}(T^1) = I(T)$.

Now $\mathscr{I}$ is an $\mathscr{E}$ essential extension of T^1, so we can regard $\mathscr{I} \subset I_{\mathscr{E}}(T^1) = I(T)$. Since $\mathscr{I}$ is an injective module over a ring with torsion free additive group, $\mathscr{I} \subseteqq D$. But D is an $\bar{\mathscr{E}}$ essential extension of T^1. Thus, $\mathscr{I} = D$.

THEOREM 13. *Let* $E = \operatorname{End} T$, $\bar{E} = E/t(E)$ *and suppose* $R\colon \bar{\mathscr{E}} \to \bar{E}$ *is onto, where* R *is the restriction map. Then, if* T^1 *is unbounded,* $I(T) \supseteqq \bigoplus_c Q$.

Proof. Let $T_1 = \{\bigoplus T_p \mid T_p^1 \neq 0\}$, $T_2 = \{\bigoplus T_p \mid T_p^1 = (0)\}$. Clearly, T_1 and T_2 are $\mathscr{E}$ submodules and $I(T) \cong I(T_1) \bigoplus I(T_2)$. It suffices to show $I(T_1) \supseteqq \bigoplus_c Q$, so, without loss of generality, assume $T = T_1$. Then Lemma 3 applies, so it is enough to construct c independent elements of infinite order in $\mathscr{I} \cong D$.

Choose $\{x_i \mid i = 1, 2, 3, \cdots\} \subseteqq T^1$ with $\{o(x_i) = p_i^{s_i}\}$ unbounded. For each fixed i, choose distinct $\bigoplus_{j=1}^{\infty} \langle b_{ij} \rangle$ part of a p_i-basic subgroup of T such that $\sum_{i,j} \langle b_{ij} \rangle$ is direct and such that $o(b_{ij}) \geqq p_i^{j^2}$. (Each T_p is reduced with $T_p^1 \neq (0)$, thus has an unbounded basic.) Finally, choose $\{x_{ij}\} \subseteqq T$ with $p_i^j x_{ij} = x_i$.

Now define $\delta_i \in \operatorname{Hom}_Z(\bigoplus_j \langle b_{ij} \rangle, T_{p_i})$ by $\delta_i(b_{ij}) = x_{ij}$. Each δ_i is a small homomorphism (see [1], Lemma 46.3) so each δ_i extends to an endomorphism of T_{p_i} and, thus, to an endomorphism of T. Still call this extension δ_i.

LEMMA 4. $\sum_i \bar{\mathscr{E}} \bar{\delta}_i$ *is an* $\bar{\mathscr{E}}$ *direct sum in* $\bar{E}$. *Here* $\bar{\delta}_i = \delta_i + t(E)$ *and* $\bar{E}$ *is regarded as a left* $\bar{\mathscr{E}}$ *module in the natural way.*

The proof of Lemma 4 is not difficult and is left to the reader.

Let $\{N_\alpha \mid \alpha \in A\}$ be a family of subsets of the natural numbers with $|A| = c$ such that if $F \subseteqq A$ is finite and $\alpha_0 \in F$ then $[N_{\alpha_0} \backslash \bigcup_{\alpha \in F, \alpha \neq \alpha_0} N_\alpha]$ is countable.

For all $\alpha \in A$, consider the diagram of $\bar{E}$ modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_i \bar{\mathscr{E}} \bar{\delta}_i & \longrightarrow & \bar{E} \\ & & \Big\downarrow \lambda_\alpha & \swarrow \lambda'_\alpha & \\ & & \mathscr{I} & & \end{array}$$

Here λ_α is the $\bar{\mathscr{E}}$ map defined by $\lambda_\alpha(\bar{\delta}_i) = X_{N_\alpha}(i) x_i$, X_{N_α} the characteristic function of N_α, and λ'_α the $\bar{\mathscr{E}}$ map obtained by injectivity.

Set $z_\alpha = \lambda'_\alpha(\bar{1})$, $\bar{1}$ the identity of the ring $\bar{E}$. Since $R\colon \bar{\mathscr{E}} \to \bar{E}$

is onto, choose $\bar{\sigma}_i \in \bar{\mathscr{E}}$ with $R(\bar{\sigma}_i) = \bar{\bar{\delta}}_i$.

Then $\bar{\sigma}_i(z_\alpha) = \lambda'_\alpha(\bar{\sigma}_i\bar{1}) = \lambda'_\alpha(\bar{\bar{\delta}}_i) = X_{N_\alpha}(i)x_i$. This equation, together with $\{o(x_i)\}$ unbounded, easily implies that $\{z_\alpha \mid \alpha \in A\}$ is an independent set of elements of infinite order. Thus, $I(T) \supseteqq \bigoplus_c Q$.

COROLLARY. *Let T be a torsion group with T^1 unbounded and $E = \operatorname{End} T$. Then $I_E(T) \supseteqq \bigoplus_c Q$.*

Added in proof. The proof of Theorem 13 can be modified, using a procedure similar to that of Theorem 11, to construct $\bigoplus_{2^c} Q \subseteqq I(T)$.

REFERENCES

1. L. Fuchs, *Infinite Abelian Groups*, Vol. 1, Academic Press, New York, (1970).

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THE UNIVERSITY OF CONNECTICUT
STORRS, CT 06268