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FIX-FINITE HOMOTOPIES

HELGA SCHIRMER

A well-known result by H. Hopf states that every selfmap f of a polyhedron $|K|$ can be deformed into a selfmap f' which has only a finite number of fixed points and is arbitrarily close to the given one. In addition one can locate all fixed points of f' in maximal simplexes. A map which has a finite fixed point set is here called a fix-finite map, and a homotopy $F: |K| \times I \to |K|$ is called a fix-finite homotopy if the map $f_t = F(\cdot, t)$ is fix-finite for every $t \in I$. We extend Hopf's result to homotopies, and show that two homotopic selfmaps f_0 and f_1 of a polyhedron $|K|$ which are fix-finite and have all their fixed points located in maximal simplexes can be related by a homotopy which is fix-finite and arbitrarily close to the given one. All fixed points of F can again be located in as high-dimensional simplexes as possible. Some simple properties are derived from the fact that the fix-finite homotopy is constructed in such a way that its fixed point set is a one-dimensional polyhedron in $|K| \times I$.

A. Introduction. In 1929 H. Hopf [2], Satz V, proved a well-known theorem which states that every selfmap f of a polyhedron can be deformed into a selfmap f' which is arbitrarily close to f and has only a finite number of fixed points. The construction of f' can be carried out so that all fixed points of f' are, in Hopf's terminology, "regular", i.e., they are located in maximal simplexes. We call a map which has only a finite number of fixed points a *fix-finite map*, and formulate Hopf's result accordingly.

THEOREM 1 (*Hopf*). *Let f be a selfmap of a polyhedron $|K|$. Given $\varepsilon > 0$, there exists a selfmap f' of $|K|$ such that*

(1) *f' is fix-finite,*

(2) *all fixed points of f' are contained in maximal simplexes of $|K|$,*

(3) *the distance $d(f, f') < \varepsilon$.*

We ask in this paper whether a similar result can be obtained for homotopies. We call a map $F: |K| \times I \to |K|$ (where I is the unit interval) a *fix-finite homotopy* if the map $f_t: |K| \to |K|$ defined by $f_t(x) = F(x, t)$ is a fix-finite map for every $t \in I$, and ask therefore whether two selfmaps f_0 and f_1 of a polyhedron $|K|$ which are fix-finite and homotopic can be related by a homotopy which is fix-finite and arbitrarily close to the given one. We shall show that this is possible if all fixed points of f_0 and f_1 are contained in maximal simplexes, and we shall construct the fix-finite homotopy so that its fixed points are again located as nicely as possible. They clearly cannot all be located in maximal simplexes of $|K|$, but they can be located in simplexes which are either maximal, or faces of maximal dimension. Let us make these notions precise.

We denote by $|K|$ a polyhedron which is the realization of a finite simplicial complex K, by σ an open simplex of K, by $\bar{\sigma}$ its closure, and by $\dim \sigma$ its dimension. $\sigma < \tau$ means that σ is a face of the simplex τ. The (open) *star* st σ of σ consists of all simplexes τ of $|K|$ with $\sigma < \tau$. A simplex σ is called *maximal* if $\sigma = \operatorname{st} \sigma$, and we call it a *hyperface* if $\dim \operatorname{st} \sigma = \dim \sigma + 1$. A *fixed point of a homotopy* $F\colon |K| \times I \to |K|$ is defined as a point $x \in |K|$ with $F(x, t) = x$ for some $t \in I$. If f, f' are maps and d is the metric of $|K|$, then the sup metric is given by

$$d(f, f') = \sup \{d(f(x), f'(x)) \mid x \in X\} .$$

We use this terminology to state our main result.

THEOREM 2. *Let F be a homotopy between two selfmaps f_0 and f_1 of a polyhedron $|K|$, let f_0 and f_1 be fix-finite, and let all their fixed points be contained in maximal simplexes. Given $\varepsilon > 0$, there exists a homotopy F' from f_0 to f_1 such that*

(1) *F' is fix-finite,*

(2) *all fixed points of F' are contained in maximal simplexes or hyperfaces of $|K|$,*

(3) $d(F, F') < \varepsilon$.

Special cases of Theorem 2 are known. Weier [6] constructed a fix-finite homotopy satisfying (1) and a condition related to (2) if $|K|$ is a 2-dimensional pseudomanifold satisfying a certain connectedness condition, and in [4], Satz III we constructed a fix-finite homotopy satisfying (1) and (3) if $|K|$ is an orientable and triangulable finite dimensional manifold without boundary.

The proof of Theorem 2 given below is related to Hopf's proof of Theorem 1. Hopf started with a simplicial approximation of the given map, and then carried out a succession of changes on simplexes of increasing dimension which freed the simplicial approximation of fixed points on all but maximal simplexes. The final result is a map which is again simplicial and satisfies Theorem 1. Hopf's proof is readily available in [1], pp. 117-118, where the successive changes are called "*Hopf constructions*".

In our proof of Theorem 2 a homotopy is altered successively on simplexes of increasing dimension by a "*Hopf construction for homotopies*" which is described in §B. As this construction can only be applied to simplicial homotopies, it is first necessary to approximate the given homotopy by a simplicial one. This leads to a proof of Theorem 2 in three steps. In the first, the given maps f_0 and f_1 are, with the help of the Hopf construction, approximated by fix-finite simplicial maps g_0 and g_1, and fix-finite homotopies H_i from f_i to g_i (where $i = 0, 1$) are obtained in a manner reminiscent of [4]. A homotopy between the simplicial maps g_0 and g_1 has a simplicial approximation relative to $|K| \times \{0\} \cup |K| \times \{1\}$, on which a succession of Hopf constructions for homotopies is carried out in Step 2, leading to a fix-finite homotopy G' from g_0 to g_1. Finally, in Step 3, the desired homotopy F' is obtained by constructing a homotopy from g_0 to g_1 as the composite of H_0^{-1}, F, and H_1, changing it to a homotopy G' as in Step 2, and forming the composite of H_0, G', and H_1^{-1}, where all compositions are made with suitable scale changes to ensure closeness between F and F'.

The homotopy F' is constructed in such a way that the set

$$\operatorname{Fix} F' = \{(x, t) \in |K| \times I \mid F'(x, t) = x\}$$

is a finite one-dimensional polyhedron. Some simple consequences of this fact are given in §D. One of them is the existence of an upper bound M so that the number of fixed points of f'_t is $\leqq M$ for every $t \in I$.

B. A Hopf construction for homotopies. Let G be the realization of a simplicial function $P \to K$, where P is a suitable complex with $|P| = |K| \times I$, and let τ be a given simplex of $|P|$. The Hopf construction for homotopies, which frees G of all fixed points on τ as long as $G(\tau)$ is not maximal in $|K|$, will be the basic tool in the second step of the proof of Theorem 2 and we shall embody its results in the rather technical Lemma 1 below. We write $G: |P| \to |K|$ to indicate that G is the realization of a simplicial function from P to K. The construction of K_L, the barycentric subdivision of K modulo the subcomplex L, can e.g. be found in [3], p. 49. If $L = \phi$, then it is the ordinary barycentric subdivision of K. A *refinement* of K is a complex obtained from K by means of a finite number of subdivisions modulo subcomplexes. $\mu(K)$ denotes the *mesh* of $|K|$, i.e., the maximum of the diameters of its simplexes.

LEMMA 1. *Let P be a complex with $|P| = |K| \times I$, let $G: |P| \to |K|$ be simplicial and $\pi: |P| \to |K|$ be the first projection. If τ is a simplex of $|P|$ for which $\pi(\tau)$ is contained in a simplex ρ of $|K'|$,* *where* K' *is a refinement of* K, *if* $\tau \cap \operatorname{Fix} G \neq \phi$ *where* $\operatorname{Fix} G = \{(x, t) \in |P| \mid G(x, t) = \pi(x, t)\}$, *and if* $G(\tau)$ *is not maximal in* $|K|$, *then there exists a simplicial map* $G': |P_Q| \to |K|$, *with* $Q = P \backslash \operatorname{st} \tau$, *so that*

(1) $\tau \cap \operatorname{Fix} G' = \phi$,

(2) $G = G'$ *on* $|Q|$,

(3) $d(G, G') \leqq 2\mu(K)$.

Proof. Let ρ^* be a maximal simplex of K' with $\rho < \rho^*$, and σ^* be a maximal simplex of K with $\rho^* \subset \sigma^*$. Then

$$\pi(\tau) \subset \rho \subset \bar{\rho}^* \subset \bar{\sigma}^* .$$

If $\sigma = G(\tau)$, then $\pi(\tau) \cap \sigma \neq \phi$ implies $\sigma < \sigma^*$.

Define $G: |P_Q| \to |K|$ on the vertices of P_Q as follows: If $v \in Q$, let $G'(v) = G(v)$. If $\tau_j \in \operatorname{st} \tau \backslash \tau$ and v is the vertex of P_Q contained in τ_j, let $G'(v)$ be any vertex of σ, and if v is the vertex of P_Q contained in τ, let $G'(v)$ be any vertex of σ^* which is not a vertex of σ. (As σ is not maximal, such a vertex exists.) It can be checked that G' extends to a simplicial map $G': |P'_Q| \to |K|$. The proof that G' satisfies the conditions (1), (2), and (3) closely parallels arguments in [1], p. 117–118, and is omitted.

C. The proof.

Step 1. Construction of fix-finite simplicial maps g_i which are fix-finitely homotopic to the given maps f_i.

We begin with a simple lemma.

LEMMA 2. *Let* $|K|$ *be a connected polyhedron,* $x \in |K|$, *and the carrier* σ *of* x *in* $|K|$ *maximal. Given* $\delta > 0$, *there exists a* $y \in \sigma$ *with* $d(x, y) < \delta$ *whose carrier in any refinement of* K *is maximal.*

Proof. $|K|$ is connected, therefore σ is of dimension $p > 0$. As the number of refinements of $\bar{\sigma}$ is countable, the dimension of the union A of the $(p-1)$-skeletons of all refinements is $p - 1$, and $y \in \sigma \backslash A$ with $d(x, y) < \delta$ exists and satisfies the lemma.

The result of Step 1 is given as the next lemma, where

$$\operatorname{diam} H = \sup \{d(H(x, t), H(x, t')) \mid x \in |K|, t, t' \in I\}$$

denotes the diameter of a homotopy $H: |K| \times I \to |K|$.

LEMMA 3. *Let* $f_i: |K| \to |K|$, $i = 0, 1$, *be two selfmaps of a polyhedron* $|K|$ *which are fix-finite and have all their fixed points located in maximal simplexes of* $|K|$. *Given* $\varepsilon > 0$, *there exist a* *refinement K' of K, refinements K_i'' of the first barycentric subdivision of K', simplicial maps $g_i: |K_i''| \to |K'|$, and homotopies H_i from f_i to g_i so that*

(1) *H_i is fix-finite and has all its fixed points located in the maximal simplexes of $|K|$,*

(2) *the fixed points of g_i are located in distinct maximal simplexes of $|K_i''|$,*

(3) $\operatorname{diam} H_i < \varepsilon/4$,

(4) $\mu(K') < \varepsilon/8(n+1)$, *where $n = \dim |K|$.*

Proof. We can assume that $|K|$ is connected, otherwise the construction is made on each component.

(i) We first construct two maps $f_i': |K| \to |K|$ and homotopies H_i' from f_i to f_i' such that all carriers of fixed points of f_i' are maximal in every refinement of K, all carriers of fixed points of H_i' are maximal in $|K|$, and $\operatorname{diam} H_i' < \varepsilon/2$.

Consider f_0, and let $\operatorname{Fix} f_0 = \{c_j\}$ be its fixed point set. As f_0 is uniformly continuous, we can select β with $0 < \beta < \varepsilon/16$ so that, for all $c_j \in \operatorname{Fix} f_0$, the open β-balls $U(c_j, \beta)$ are pairwise disjoint and each $U(c_j, \beta)$ is contained in the carrier of c_j in $|K|$. Now select γ with $0 < \gamma < \beta/2$ such that $d(x, f_0(x)) < \beta/2$ for all $x \in \cup\{U(c_j, \gamma) \mid c_j \in \operatorname{Fix} f_0\}$. According to Lemma 2 each $U(c_j, \gamma)$ contains a point c_j' whose carrier in all refinements of $|K|$ is maximal. If $x \in \bar{U}(c_j, \gamma)\backslash\{c_j'\}$, let y be the point in which the ray from c_j' to x intersects the boundary $\operatorname{Bd} U(c_j, \gamma)$, and z the point on the segment from c_j to y for which

$$d(c_j, z) = \frac{d(c_j, y)}{d(c_j', y)} \cdot d(c_j', x) .$$

To define a map f_{0j}' from $\bar{U}(c_j, \gamma)$ to $U(c_j, \beta)$, denote by $\overrightarrow{ab}$ the (free) vector from a to b in $U(c_j, \beta)$, and determine $f_{0j}'(x)$ for $x \neq c_j'$ by

$$\overrightarrow{c_j' f_{0j}'(x)} = \overrightarrow{c_j' x} + \overrightarrow{z f_0(z)} ;$$

also let $f_{0j}' = c_j'$.

As we have for all $x \in \bar{U}(c_j, \gamma)$

$$\begin{aligned} d(f_{0j}'(x), c_j) &\leqq d(f_{0j}'(x), x) + d(x, c_j) \\ &= d(f_0(z), z) + d(x, c_j) < \beta/2 + \gamma < \beta , \end{aligned}$$

this construction is well defined.

Now define $f_0': |K| \to |K|$ by

$$f_0'(x) = \begin{cases} f_{0j}'(x) & \text{if } \; x \in \cup\{U(c_j, \gamma) \mid c_j \in \operatorname{Fix} f_0\} , \\ f_0 & \text{otherwise} . \end{cases}$$

f_0' is continuous, and its fixed point set is $\operatorname{Fix} f_0' = \{c_j'\}$. Hence all carriers of its fixed points are maximal in every refinement of $|K|$.

If $f'_0(x) \neq f_0(x)$, then $x \in U(c_j, \gamma)$ for some $c_j \in \operatorname{Fix} f_0$. Denote, for $0 < t \leqq 1$, by $c_j(t)$ the point which divides the segment from c_j to c'_j in the ratio $t: (1-t)$, and define $H'_{0j}(x, t)$ as the point in $U(c_j, \beta)$ which is obtained in a manner analogous to $f'_{0j}(x)$ but with the use of $c_j(t)$ instead of c'_j. Also put $H'_{0j}(x, 0) = f_0(x)$. Then a homotopy H'_0 from f_0 to f'_0 can be constructed from the H'_{0j} in the same way in which f'_0 was constructed from the f'_{0j}. If $f'_0(x) = f_0(x)$, then H'_0 is the constant homotopy, if $f'_0(x) \neq f_0(x)$, then the set $\{H'_0(x, t) \mid 0 \leqq t \leqq 1\}$ lies in some $U(c_j, \beta)$. Hence $\operatorname{diam} H'_0 < 2\beta < \varepsilon/8$. The construction of H'_0 shows that all carriers of its fixed points are maximal in K.

The map f'_1 and the homotopy H'_1 from f_1 to f'_1 are obtained analogously.

(ii) We now describe the construction of the maps g_i and the homotopies H''_i from f'_i to g_i.

Choose ρ_0 with $0 < \rho_0 < \varepsilon/32$ so that for each $c'_j \in \operatorname{Fix} f'_0$ with carrier κ_j in $|K|$ the set $\bar{U}(c'_j, 4\rho_0) \subset \kappa_j$, and so that the $\bar{U}(c'_j, 4\rho_0)$ are pairwise distinct. As f'_0 is uniformly continuous, there exists a δ_0 with $0 < \delta_0 \leqq \rho_0$ so that

$$f'_0(\bar{U}(c'_j, \delta_0)) \subset \bar{U}(c'_j, \rho_0) \quad \text{for all} \quad c'_j \in \operatorname{Fix} f'_0 .$$

Furthermore choose η_0 with $0 < \eta_0 \leqq \rho_0$ so that

$$d(x, f'_0(x)) \geqq \eta_0 \quad \text{if} \quad d(x, \operatorname{Fix} f'_0) \geqq \delta_0 .$$

Determine ρ_1, δ_1, η_1 analogously for f'_1, and select a refinement K' of K so that $\mu(K') < \min\{\delta_0, \delta_1, \eta_0/(2n+1), \eta_1/(2n+1)\}$, where n is the dimension of K.

Let ψ_0 be a simplicial approximation of f'_0 which maps a refinement of the first barycentric subdivision of K' into K', and choose g_0 as a map which is obtained from $|\psi_0|$ by a succession of Hopf constructions in the same way in which f' is obtained from $|\psi|$ in the proof of Theorem 2 on p. 118 of [1]. Then g_0 is a simplicial map $|K''_0| \to |K'|$, where K''_0 again refines the first barycentric subdivision of K'. It is fix-finite, has all its fixed points located in distinct maximal simplexes of $|K''_0|$, and $d(|\psi_0|, g_0) \leqq 2n\mu(K')$. As $d(f'_0, |\psi_0|) \leqq \mu(K')$, we have $d(f'_0, g_0) \leqq (2n+1)\mu(K') < \eta$.

Next, let us construct a homotopy H''_0 from f'_0 to g_0. If $x \notin \cup\{U(c'_j, \delta_0) \mid c'_j \in \operatorname{Fix} f'_0\}$, then it follows from [1], p. 118 that $g_0(x) = |\psi_0|(x)$. As ψ_0 is a simplicial approximation of f'_0, it is possible to define $H''_0(x, t)$ by

$$H''_0(x, t) = tf'_0(x) + (1-t)g_0(x) .$$

From $d(x, f'_0(x)) \geqq \eta$ and $d(f'_0, g_0) < \eta$ follows $H''_0(x, t) \neq x$ for all $0 \leqq t \leqq 1$.

Now consider one of the sets $\bar{U}(c'_j, \delta_0)$ contained in a maximal simplex κ_j of $|K|$. H''_0 has already been defined on Bd $\bar{U}(c'_j, \delta_0) \times I$ such that

$$d(c'_j, H''_0(x, t)) \leqq d(c'_j, f'_0(x)) + d(f'_0(x), g_0(x)) \leqq 2\rho_0 .$$

Let further $H''_0(x, 0) = f'_0(x)$ and $H''_0(x, 1) = g_0(x)$ for all $x \in \bar{U}(c'_j, \delta_0)$.

Then H''_0 is defined on Bd $(\bar{U}(c'_j, \delta_0) \times I)$, has values in $\bar{U}(c'_j, 2\rho_0)$, and its fixed point set consists of $c'_j \times \{0\}$ and finitely many points in $U(c'_j, \delta_0) \times \{1\}$. To extend H''_0 over all of $\bar{U}(c'_j, \delta_0) \times I$, let $\tilde{c}_j = (c'_j, 1/2)$, and determine for every point $\tilde{x} = (x, t) \in (\bar{U}(c'_j, \delta_0) \times I)\backslash\{c_j\}$ the point $\tilde{y} = (y, s)$ as the one in which the ray from $\tilde{c}_j$ to $\tilde{x}$ intersects Bd $(\bar{U}(c'_j, \delta_0) \times I)$. Let $\tilde{d}$ denote the product metric in $|K| \times I$, and define $H''_0(x, t)$ by

$$\overrightarrow{c'_j H''_0(x, t)} = \overrightarrow{c'_j x} + \overrightarrow{\lambda y H''_0(y, s)} ,$$

where

$$\lambda = \tilde{d}(\tilde{c}_j, \tilde{x})/\tilde{d}(\tilde{c}_j, \tilde{y}) .$$

As $d(c'_j, x) \leqq \delta_0$, $0 < \lambda \leqq 1$, and $d(y, H''_0(y, s)) \leqq \delta_0 + 2\rho_0 \leqq 4\rho_0$, we obtain in this way a point $H''_0(x, t) \in \bar{U}(c'_j, 4\rho_0)$. Finally, let $H''_0(c'_j, 1/2) = c'_j$.

In this way H''_0 is extended over $\cup\{\bar{U}(c'_j, \delta_0) \times I \mid c'_j \in \operatorname{Fix} f'_0\}$, yielding a homotopy $H''_0\colon |K| \times I \to |K|$ from f'_0 to g_0 which is fix-finite and has all its fixed points located in the maximal simples κ_j of $|K|$. If $x \in \cup\{\bar{U}(c'_j, \delta_0) \mid c'_j \in \operatorname{Fix} f'_0\}$, then $\sup\{H''_0(x, t), H''_0(x, t') \mid t, t' \in I\} \leqq d(f'_0, g_0) < \eta$, and if $x \in \bar{U}(c'_j, \delta_0)$ for some $c'_j \in \operatorname{Fix} f'_0$, then $\{H''_0(x, t) \mid t \in I\} \subset \bar{U}(c'_j, 4\rho_0)$, so $\sup\{H''_0(x, t), H''_0(x, t') \mid t, t' \in I\} \leqq 8\rho_0$. Hence $\operatorname{diam} H''_0 < \varepsilon/4$. The construction of $H''_1\colon |K| \times I \to |K|$ is analogous.

(iii) Define finally a homotopy H_i from f_i to g_i by

$$H_i(x, t) = \begin{cases} H'_i(x, 2t) & \text{for } 0 \leqq t \leqq 1/2 , \\ H''_i(x, 2t-1) & \text{for } 1/2 \leqq t \leqq 1 . \end{cases}$$

Then $\operatorname{diam} H_i \leqq \operatorname{diam} H'_i + \operatorname{diam} H''_i < \varepsilon/4$, and H_0 and H_1 satisfy Lemma 3.

Step 2. Construction of a fix-finite homotopy between two fix-finite simplicial maps.

The aim of Step 2 is the construction of a fix-finite homotopy between the fix-finite and simplicial maps g_i of Lemma 3. It will be achieved with the help of a succession of Hopf constructions for homotopies. For this purpose, we need to realise $|K| \times I$ as a suitable simplicial complex P. If K', K_0'' and K_1'' are the complexes obtained in Lemma 3, then we require that P is a simplicial complex with $|P| = |K| \times I$ and satisfies the following two conditions:

(P1) $K_0'' \times \{0\}$ and $K_1'' \times \{1\}$ are subcomplexes of P,

(P2) if $\tau \in |P|$ is a simplex and $\pi\colon |P| \to |K|$ the first projection, then $\pi(\tau) \subset \rho$, where ρ is a simplex of K'.

P can easily be obtained by starting with the complex usually associated with the polyhedron $|K'| \times I$ and then refining it modulo the complements of the simplicial neighborhoods of those simplexes in $K' \times \{0\}$ and $K' \times \{1\}$ which are subdivided in K_0'' resp. K_1''.

We state one more technical detail as a lemma.

LEMMA 4. *Let P' be a refinement of P, let $G_s\colon |P'| \to |K'|$ be a simplicial map, and $\tau \in |P'|$ so that $\tau \cap \operatorname{Fix} G_s \neq \phi$. If τ is neither maximal nor a hyperface in $|P'|$, thn $G_s(\tau)$ is not maximal in $|K'|$.*

Proof. Let $G_s(\tau) = \sigma$, where σ is a simplex of $|K'|$, and $\pi(\tau) \subset \rho$, where $\rho \in |K'|$. As $\tau \cap \operatorname{Fix} G_s \neq \phi$ implies $\pi(\tau) \cap \sigma \neq \phi$, we have $\rho = \sigma$, and $\dim \rho \leqq \dim \tau$. By assumption there exists a simplex $\tau^* \in |P'|$ with $\tau < \tau^*$ and $\dim \tau \leqq \dim \tau^* - 2$, therefore

$$\dim \rho + 1 \leqq \dim \tau^* - 1 \leqq \dim \pi(\tau^*) ,$$

so $\pi(\tau^*) \not\subset \rho$. But $\pi(\tau) \subset \rho$ implies $\pi(\bar{\tau}^*) \cap \rho \neq \phi$, hence ρ cannot be maximal in $|K'|$. As $\rho = \sigma$, $G_s(\tau)$ cannot be maximal either.

The next lemma contains the result of Step 2.

LEMMA 5. *Let K', K_i'' and $g_i\colon |K_i''| \to |K'|$ be as in Lemma 3. If g_0 and g_1 are related by a homotopy G, then there exists a homotopy G' relating them such that*

(i) G' is fix-finite and has all its fixed points located in maximal simplexes or hyperfaces of $|K|$,

(ii) $d(G, G') < \varepsilon/4$.

Proof. Again we can assume that $|K|$ is connected. Let P satisfy (P1) and (P2). We first select as a simplicial approximation of G a simplicial map $G_s\colon |P'| \to |K'|$, where P' is a refinement of P obtained by a finite number of subdivisions modulo $(K_0'' \times \{0\}) \cup (K_1'' \times \{1\})$, so that G_s satisfies $G_s = G$ on $(|K_0''| \times \{0\}) \cup (|K_1''| \times \{1\})$ and $d(G, G_s) < \mu(K')$. The existence of G_s follows from [3], p. 55.

If $\tilde{x}_0 = (x_0, t_0)$ is a vertex of $|P'|$ with $G_s(x_0, t_0) = x_0$, then x_0 is a vertex of $|K'|$ and hence not maximal. Lemma 1 allows us to make a Hopf construction which results in a simplicial map G'_s: $|P''| \to |K'|$, where P'' refines P', for which $G'_s(x_0, t_0) \neq x_0$ and $G'_s = G_s$ on $|P' \backslash \mathrm{st}\{\tilde{x}_0\}|$. Hence any vertex $\tilde{x} \in |P''| \cap \mathrm{Fix}\, G'_s$ must also be a vertex of $|P' \backslash \{\tilde{x}_0\}|$. We can therefore make further Hopf constructions for all such vertices until we arrive at a simplicial map, denoted again by $G'_s: |P''| \to |K'|$, where P'' refines P', which is fixed point free on all vertices of $|P''|$. As G_s is fixed point free on the vertices of $(|K''_0| \times \{0\}) \cup (|K''_1| \times \{1\})$, we have $G'_s = G_s$ on this subcomplex.

Next we carry out a succession of Hopf constructions for all one-dimensional simplexes $\tau \in |P''|$ for which $\tau \cap \mathrm{Fix}\, G'_s \neq \phi$ and $G'_s(\tau)$ is not maximal in $|K'|$, then for all two-dimensional simplexes with the same property, and so on. According to (P2) and Lemmas 1 and 4 we can continue until we arrive at a simplicial map $G'_s: |P''| \to |K'|$, which equals G_s on the subpolyhedron $(|K''_0| \times \{0\}) \cup (|K''_1| \times \{1\})$ of $|P''|$ and is fixed point free on all simplexes of $|P''|$ which are neither maximal nor hyperfaces.

If τ is a hyperface of $|P''|$ for which $\tau \cap \mathrm{Fix}\, G'_s \neq \phi$, then it follows (as in [1], pp. 118–119) from the fact that G'_s is linear on $\bar{\tau}$ and that $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G'_s = \phi$ that G'_s has at most one fixed point on τ. Now consider a maximal simplex $\tau \in |P''|$ with $\tau \cap \mathrm{Fix}\, G'_s \neq \phi$. Then $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G'_s$ is empty or a finite set $\{\tilde{x}_j\}$. Let $\tilde{x}_j = (x_j, t_j)$, and select $\tilde{x}_0 = (x_0, t_0) \in \tau$ so that $t_0 \neq t_j$ for all t_j. For any $\tilde{x} = (x, t) \in \bar{\tau} \backslash \{\tilde{x}_0\}$, let $\tilde{y} = (y, u)$ be the point in which the ray from $\tilde{x}_0$ to $\tilde{x}$ intersects $\mathrm{Bd}\, \tau$, and modify G'_s on $\bar{\tau}$ to G' by defining $G'(x, t)$ as the point in $\bar{\sigma} = G'_s(\bar{\tau})$ with

$$\overrightarrow{x_0 G'(x, t)} = \overrightarrow{x_0 x} + \lambda \overrightarrow{y G'_s(y, u)}\,, \quad \text{where} \quad \lambda = \tilde{d}(\tilde{x}_0, \tilde{x}) / \tilde{d}(\tilde{x}_0, \tilde{y})\,.$$

As $\pi(\bar{\tau}) \subset \bar{\sigma}$ and $\bar{\sigma}$ is convex, this yields a point $G'(x, t) \in \bar{\sigma}$. Also let $G'(x_0, t_0) = x_0$. Then $\bar{\tau} \cap \mathrm{Fix}\, G'$ consists of the union of the segments from $\tilde{x}_0$ to all the $\tilde{x}_j$ if $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G' \neq \phi$, and otherwise of the point $\tilde{x}_0$ alone. If we carry out this construction on all maximal simplexes of $|P''|$ on which G'_s has fixed points, we obtain a fix-finite homotopy $G': |P''| \to |K'|$, where P'' refines P' and hence P. By construction $G'(x, 0) = g_0(x)$ and $G'(x, 1) = g_1(x)$ for all $x \in |K|$. If $\tilde{x} = (x, t) \in \mathrm{Fix}\, G'$, then $\tilde{x}$ is contained in a maximal simplex or hyperface of $|P''|$ and hence of $|P|$. It follows from (P2) that x is contained in a maximal simplex or hyperface of $|K'|$ and hence of $|K|$.

Each point $\tilde{x} \in |P|$ is moved during the succession of Hopf constructions at most n times, where again n is the dimension of $|K|$, and by a distance of at most $2\mu(K')$ on each move. During the last change of G'_s to G' it is moved by a distance of at most $\mu(K')$. So we have

$$d(G_s, G') \leqq (2n + 1)\mu(K') ,$$

and hence, according to (4) of Lemma 3,

$$d(G, G') \leqq 2(n + 1)\mu(K') < \varepsilon/4 .$$

We see that G' satisfies Lemma 5.

Step 3. Construction of a fix-finite homotopy between the given maps.

It remains to paste the constructed homotopies together in a suitable way to find a homotopy F' satisfying Theorem 2. Given $F: |K| \times I \rightarrow |K|$ as in Theorem 2 and $\varepsilon > 0$, we can choose δ with $0 < \delta < 1$ so that $d(F(x, t), F(x, t')) < \varepsilon/4$ for all $x \in |K|$ and $t, t' \in I$ with $|t - t'| < \delta$. Use the homotopies H_0, H_1 obtained in Lemma 3 and define $F'': |K| \times I \rightarrow |K|$ as a homotopy which equals $H_0H_0^{-1}FH_1H_1^{-1}$ apart from a scale change by

$$F''(x, t) = \begin{cases} H_0(x, 2t/\delta) & \text{if } 0 \leqq t \leqq \delta/2 , \\ H_0(x, 2(1 - t/\delta)) & \text{if } \delta/2 \leqq t \leqq \delta , \\ F(x, (t - \delta)/(1 - 2\delta)) & \text{if } \delta \leqq t \leqq 1 - \delta , \\ H_1(x, \delta(t + \delta - 1)/2) & \text{if } 1 - \delta \leqq t \leqq 1 - \delta/2 , \\ H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leqq t \leqq 1 . \end{cases}$$

Then $d(F, F'') < \varepsilon/2$.

The homotopy $G: |K| \times I \rightarrow |K|$ defined by $G(x, t) = F''(x, t(1 - \delta) + \delta/2)$ for all $(x, t) \in |K| \times I$ equals $H_0^{-1}FH_1$ apart from a scale change and is hence a homotopy from g_0 to g_1. Replace it by a homotopy G' according to Lemma 5, and define $F': |K| \times I \rightarrow |K|$ by

$$F'(x, t) = \begin{cases} H_0(x, 2t/\delta) & \text{if } 0 \leqq t \leqq \delta/2 , \\ G'(x, (t - \delta/2)/(1 - \delta)) & \text{if } \delta/2 \leqq t \leqq 1 - \delta/2 , \\ H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leqq t \leqq 1 . \end{cases}$$

It is easy to check that F' is a homotopy satisfying Theorem 2.

D. Some properties of the fix-finite homotopy. The proof of Theorem 2 allows an easy description of Fix F'.

PROPOSITION 1. *The homotopy F' in Theorem 2 can be chosen* *so that* $\mathrm{Fix}\, F'$ *is a one-dimensional finite polyhedron in* $|K| \times I$ *without horizontal edges.*

Here a horizontal edge means an edge contained in a section $|K| \times \{t\}$, for some $t \in I$. Note that $\mathrm{Fix}\, F'$, though constructed as a polyhedron, was not constructed as a subpolyhedron of $|P|$, and its projection $\pi(\mathrm{Fix}\, F')$ is not a subpolyhedron of $|K|$.

As $\mathrm{Fix}\, F'$ has a simple structure, it has simple properties. We collect a few. The first two are immediate consequences of the homotopy and additivity axioms of the fixed point index $i(f, x)$ of the selfmap f of a polyhedron at the isolated fixed point x.

PROPOSITION 2. *Let e be an edge of* $\mathrm{Fix}\, F'$. *Then the index of f'_t along e is constant,* i.e.,

$$i(f'_t, x) = i(f'_s, y) \quad \textit{if} \quad (x, t) \in e \quad \textit{and} \quad (y, s) \in e\,.$$

PROPOSITION 3. *Let $v = (x, t)$ be a vertex of* $\mathrm{Fix}\, F'$. *Then the index of f_t at x is the sum of the indices of fixed points chosen on all edges of* $\mathrm{Fix}\, F'$ *either leading towards v or away from v,* i.e.,

$$i(f'_t, x) = \sum_k i(f'_{t_k}, x_k)\,,$$

where all (x_k, t_k) lie on edges $e_k \in \mathrm{st}\, v$, with e_k distinct, and the sum taken over all edges in $\mathrm{st}\, v \cap \{|K| \times [0, t)\}$ *(resp. in* $\mathrm{st}\, v \cap \{|K| \times (t, 1]\}$*).*

Finally we note that F' is "uniformly" fix-finite.

PROPOSITION 4. *There exists a positive integer M so that the number of fixed points of f'_t is $\leqq M$ for all $t \in I$.*

Proof. It suffices to choose M as the number of edges in $\mathrm{Fix}\, F'$, as no section $|K| \times \{t\}$ can intersect the closure of an edge of $\mathrm{Fix}\, F'$ more than once.

E. Conclusion. For a single selfmap f of a polyhedron $|K|$ the construction of a fix-finite map which is arbitrarily close to f and has all its fixed points contained in maximal simplexes is only a first step on the road to the construction of a map homotopic to f which has a minimal number of fixed points. It is, in fact, possible to obtain a map g homotopic to f which has exactly $N(f)$ fixed points, where $N(f)$ is the Nielsen number of f, as long as $|K|$ satisfies the Shi condition, which is a somewhat stronger connectedness condition. (See [5] or [1], p. 140.) Hence a similar question arises for homotopies.

Problem. If f_0 and f_1 are two selfmaps of a polyhedron $|K|$ which satisfies the Shi condition, if f_0 and f_1 are homotopic and have each exactly $N(f_0)$ fixed points, does there exist a homotopy F from f_0 to f_1 so that, for every $t \in I$, the map $f_t = F(\cdot, t)$ has exactly $N(f_0)$ fixed points?

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CARLETON UNIVERSITY
OTTAWA, CANADA KIS 5B6