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FANS AND EMBEDDINGS IN THE PLANE

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We prove that every fan which is locally connected at its vertex can be embedded in the plane. This gives a solution to a problem raised by J. J. Charatonik and Z. Rudy.

1. Introduction and definitions. In 1963, K. Borsuk [4] constructed a fan which is not embeddable in the plane. Hence, the question arises to characterize those fans which are embeddable in the plane. In particular, in [5] it was asked whether each contractible fan is embeddable in the plane. In an attempt to solve this problem in the negative, J. J. Charatonik and Z. Rudy constructed a contractible fan which is locally connected at its vertex. They conjectured ([6], p. 215) that this fan is not embeddable in the plane. We show in this paper that each fan, which is locally connected at its vertex, is embeddable in the plane (see Theorem 5.2). We will also establish, for fans, several equivalences between the local connectedness at the vertex and other conditions. In a forthcoming paper [11] the author has shown that each contractible fan is locally connected at its vertex, and hence embeddable in the plane.

By a *continuum* we mean a compact connected metric space. A *dendroid* is an arc-wise connected and hereditarily unicoherent continuum. By a *fan* we understand a dendroid which has exactly one branch-point, and we call this branch-point the *vertex* of the fan. If x, y are points in a dendroid X, then we denote by $[x, y]$ the unique arc in X having x and y as end-points. The weak-cut order $\leqq$, with respect to a point p, in a dendroid X is given by

$$x \leqq y \text{ if and only if } [p, x] \subset [p, y] .$$

We denote by I the unit closed interval $[0, 1]$ of reals, and the symbol $B(x, \varepsilon)$ denotes the open ball of radius ε about the point x. We use the symbol $\cong$ to denote that two spaces are homeomorphic. The symbol R, as used in Lemma 3.1, denotes a set of indices.

2. Embeddings in the plane. A cover $U = \{U_1, U_2, \cdots, U_n\}$ of a space is called an *ε-chain* if the nerve (see [8], p. 318) of U is an arc and $\text{diam}(U_i) < \varepsilon$ for $i = 1, 2, \cdots, n$. A continuum X is said to be *arc-like* if for each $\varepsilon > 0$ there exists an ε-chain covering X. A point e of an arc-like continuum X is called an *end-point* provided for each $\varepsilon > 0$ there exists an ε-chain $U_1, U_2, \cdots, U_n$ covering X such that

$$(1) \qquad e \in U_1 \Big\backslash \bigcup_{i=2}^{n} U_i \ .$$

It is known (see [9], p. 148) that every 0-dimensional compact metric space K is homeomorphic to a subset of the Cantor ternary set $C \subset [0, 1]$, and hence K possesses a natural order $\leqq$. We will call this ordering the *induced ordering* on K. The main result of this section is Theorem 2.2. We start with the following lemma.

LEMMA 2.1. *Let X be a compact metric space and let $\{J_\alpha\}$, $\alpha \in A$, be the decomposition of X into components. Let $\varepsilon > 0$ and let K be a 0-dimensional compact set in X, with induced ordering $\leqq$ such that:*

(2) *J_α is an arc-like continuum for each $\alpha \in A$,*

(3) *$J_\alpha \cap K = \{e_\alpha\}$, where e_α is an end-point of J_α for each $\alpha \in A$.*
Then there exists an open cover U of X such that U is a finite union of disjoint ε-chain $V_i (i = 1, 2, \cdots, t)$, where $V_i = \{U(i, j)\} (j = 1, 2, \cdots, k(i))$ such that:

(4) $K \subset \bigcup_{i=1}^{t} U(i, 1) \backslash \bigcup_{i=1}^{t} \bigcup_{j=2}^{k(i)} U(i, j)$,

(5) *all nonadjacent elements of U have positive distance,*

(6) *for each i, $1 \leqq i \leqq t$, there exist $a_i, b_i \in K$ such that:*

$$K \cap U(i, 1) = \{x \in K \mid a_i \leqq x \leqq b_i\} \ .$$

Proof. Denote by 0 the minimal and by 1 the maximum element of K. Let $g: X \to K$ be defined by $g(x) = e_\alpha$ if $x \in J_\alpha$, then g is a monotone retraction. Let

(7) $x_0 = \sup \{e \in K \mid$ for each $e' \leqq e$ there exists an open cover of $g^{-1}([0, e'])$ satisfying the conclusion of Lemma 2.1$\}$,

then $x_0 \geqq 0$. By (2) and (3) there exists an ε-chain $U_1, U_2, U_3, \cdots, U_k$ in X covering $g^{-1}(x_0)$ such that

$$K \cap \bigcup_{j=2}^{k} U_j = \emptyset \ .$$

Since $g^{-1}(x_0) \subset \bigcup_{j=1}^{k} U_j$ and K is 0-dimensional there exists a closed and open set $H \subset K$ such that $g^{-1}(H) \subset \bigcup_{j=1}^{k} U_j$. Moreover, we can choose H such that

$$H \cap K = \{x \in K \mid a \leqq x \leqq b\}$$

for some a and b in K. If $a > 0$, define $x_1 = \sup \{x \in K \mid x < a\}$, then $x_1 \notin U_1$ and $x_1 < a$. By (7) there exists a cover U of $g^{-1}([0, x_1])$ satisfying the conclusions of the lemma (if $a = 0$, take $U = \emptyset$). Since $g^{-1}([0, x_1])$ is open in X we may assume that $\cup U \subset g^{-1}([0, x_1])$. Hence

$$U \cup \{U_j \cap g^{-1}(H) \mid j = 1, 2, \cdots, k\}$$

is a cover of $g^{-1}([0, b])$ satisfying the conclusion of the lemma. It follows the definition of x_0 that $x_0 = b$.

If $x_0 = b = 1$, we are done, whence suppose $x_0 < 1$ and let $x_2 = \inf\{x \in K \mid x > x_0\}$. By repeating the argument above, replacing x_0 by x_2, one can show that $g^{-1}([0, x_2])$ can be covered with a cover satisfying the conclusion of the lemma, contrary to (7), since $x_2 > x_0$.

We will call a cover U that satisfies the conclusion of Lemma 2.1 an *ε-cover* of X.

THEOREM 2.2. *Let X be a compact metric space and K a closed subset of X. Let $\{J_\alpha\}$, $\alpha \in A$, be the decomposition of X into components such that:*

(8) *$J_\alpha \cap K = \{e\}$, where e is an end-point of J_α for each $\alpha \in A$,*

(9) *J_α is an arc-like continuum for each $\alpha \in A$. Then there exists an embedding $h: X \to I^2$ such that $h(K) = h(X) \cap l$, where $l = \{(x, y) \in I^2 \mid y = 0\}$.*

Proof. Notice that by (8) K is 0-dimensional. By Lemma 2.1, there exists for each $\varepsilon > 0$ an ε-cover of X. Let U_1 be a 1/2-cover of X and $\eta > 0$ such that η is the minimum distance between two nonintersecting elements of U_1. By induction we construct a sequence of covers $U_1, U_2, \cdots$ of X such that U_n refines U_{n-1}, U_n is a $(1/2)^n$-cover, no sub-chain of less than nine links of U_n connects two nonintersecting elements of U_{n-1}.

Given a cover U of X, satisfying the conclusion of Lemma 2.1, we label the chains $V_1, V_2, \cdots, V_t$ of U such that $\inf\{x \mid x \in K \cap V_i\} < \inf\{x \mid x \in K \cap V_j\}$ if $i < j$, and the links of the chain $V_i = \{U(i, 1), U(i, 2), \cdots, U(i, k(i))\}$ such that $K \cap V_i \subset U(i, 1)$. If U and U^* are both covers of X, satisfying the conclusion of Lemma 2.1, then we say that U follows the pattern $\{(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{k(1)}), \cdots, (a_t, b_{k(t)})\}$ in U^* if the jth link of the ith chain of U is contained in the b_jth link of the a_ith chain of U^*(i.e., $U(i, j) \subset U^*(a_i, b_j)$).

There exist in I^2 a sequence of open sets $D_1, D_2, \cdots$ such that D_n is a finite union of $(1/2)^n$-chains whose elements are interiors of rectangles, and such that D_n follows a pattern in D_{n-1} that U_n follows in U_{n-1}, each element of D_{n-1} contains the closure of an element of D_n, while the closure of each element of D_n lies in an element of D_{n-1} and the first link of each chain of D_n intersects l in a nondegenerate interval, while the closure of all other elements of D_n are contained in $I^2 \backslash l (n = 1, 2 \cdots)$.

The existence of the open sets D_n satisfying the above follows from an argument similar to one used by R. H. Bing (see [3], p. 654), the only difference being that in each cover D_{n-1} we insert, in the next step, finitely many, instead of one, new chains and we require the first link of each chain of D_n to intersect l in a nondegenerate interval, while the closures of all other elements of D_n are contained in $I^2\backslash l$.

The latter facts can be established by dividing each chain of D_{n-1} into finitely many "strips" in each of which we insert, in the next step, a new chain in such a way that we always insert new links on a predescribed "side" of already chosen previous links.

It follows from Theorem 11 of [2] that X is homeomorphic with the continuum $Y = D_1^* \cap D_2^* \cap \cdots$, where D_n^* denotes the union of the elements of D_n and moreover it follows from the choice of D_n that Y satisfies the conclusion of Theorem 2.2, and the proof is complete.

3. Fans locally connected at the vertex.

A fan X has *property P*[1], if for each sequence of points $\{x_i\}$ in X ($i = 1, 2, \cdots$) converging to the vertex v of X we have

$$Ls[v, x_i] = \{v\} . \tag{1}$$

THEOREM 3.1. *Let X be a fan with vertex v and*

(2) $X = \bigcup_{r\in R}\{J_r \mid J_r \cong [0, 1]$ *for each* $r \in R$ *and* $J_{r_1} \cap J_{r_2} = \{v\}$ *if* $r_1 \neq r_2 \in R\}$,

then the following are equivalent:

(3) *X has property P,*

(4) *for each $\varepsilon > 0$, there exists a connected open neighborhood U of v such that* $\operatorname{diam}(U) \leqq \varepsilon$ *and* $\operatorname{Bd}(U) \cap J_r$ *is connected for every* $r \in R$,

(5) *X is locally connected at v.*

Proof. (3) $\to$ (4). Let $\varepsilon > 0$ be given and let $\leqq$ be the weak-cut order of X with respect to v. Define $V = B(v, e)$,

$$x(r) = \inf\{x \in X \mid x \in J_r \cap \operatorname{Bd}(V)\} \text{ if } J_r \cap \operatorname{Bd}(V) \neq \varnothing ,$$

$$Q_r = \begin{cases} \{y \in J_r \mid y \geqq x(r)\} & \text{if } J_r \cap \operatorname{Bd}(V) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases} \tag{6}$$

and $Q = \bigcup_{r\in R} Q_r$. It follows that $v \notin \bar{Q}$, since if $\{v_i\}$ is a sequence in Q converging to v, then $v_i \geqq x(r_i)$ for some $r_i \in R$, and hence $Ls[v, v_i] \cap \operatorname{Bd}(V) \neq \varnothing$, contrary to (3).

Let $U = X\backslash\bar{Q}$, then U is an open neighborhood of v and $\operatorname{diam}(U) \leqq$

[1] It follows from the definition that property P is related to the notion of a Q-point or a P-point (cf. [1] and [7], respectively).

$\operatorname{diam}(V) = \varepsilon$. We will show that U satisfies all conditions of (4). We claim that

(7) if $z \in U$ and $x < z$, then $x \in U$, or, equivalently, if $x \in \bar{Q}$ and $z \geqq x$, then $z \in \bar{Q}$.

To this end, suppose that (7) is false. Hence $x \in \bar{Q}$, let $\{x_i\}$ be a sequence in Q converging to x. Then $x_i \geqq x(r_i) \in \operatorname{Bd}(V)$ for some $r_i \in R (i = 1, 2, \cdots)$. We may assume that the sequence $\{x(r_i)\}$ converges to a point $x_0 \in J_{r_0} \cap \operatorname{Bd}(V)$ for some $r_0 \in R$.

By ([9], p. 171), $Ls[x_i, x(r_i)]$ is a continuum and since $[x_i, x(r_i)] \subset Q (i = 1, 2, \cdots)$ we have $Ls[x_i, x(r_i)] \subset \bar{Q} \subset X \backslash \{v\}$. Moreover, since X is hereditarily unicoherent, it follows that $[x, x_0] \subset Ls[x_i, x(r_i)] \subset \bar{Q} \subset X \backslash \{v\}$ and we consider two cases as follows:

Case 1. $z \in [x, x_0]$. Then $z \in \bar{Q}$.

Case 2. $z \notin [x, x_0]$. Then, since $z > x$, $z > \max\{x, x_0\}$ and consequently $z > x_0 \geqq x(r_0)$. Hence $z \in Q$ by (6) and the definition of Q.

In both case we conclude that $z \in \bar{Q}$, contrary to the assumptions in (7) and the proof of (7) is complete. It follows from (7) that U is connected. In order to show that $J_r \cap \operatorname{Bd}(U)$ is connected for each $r \in R$, we will show that if $x, y \in J_r \cap \operatorname{Bd}(U)$, say $x < y$, and $z \in [x, y]$, then $z \in J_r \cap \operatorname{Bd}(U)$.

Since $x \in J_r \cap \operatorname{Bd}(U) = J_r \cap \bar{U} \cap \bar{Q}$ and $z > x$, it follows from (7) that $z \in \bar{Q}$. Moreover, since $y \in \bar{U}$, there exists a sequence $\{y_i\}$ in U converging to y. Since $Ls[v, y_i]$ is a continuum ([9], *p.* 171), containing both y and v and X is hereditarily unicoherent, it follows that $[v, y] \subset Ls[v, y_i]$. As $z \in [v, y]$, we may assume that there exists a sequence $\{z_i\}$, where $z_i \in [v, y_i]$, converging to z. By (7), $z_i \in U$ and whence $z \in \bar{U}$. Obviously $z \in J_r$ and we conclude $z \in J_r \cap \bar{U} \cap \bar{Q} = J_r \cap \operatorname{Bd}(U)$.

$(4) \rightarrow (5)$: Trivial.

$(5) \rightarrow (3)$: Suppose X does not have property P. Let $\{x_i\}$ be a sequence of points in X converging to v such that $Ls[v, x_i] = K \neq \{v\}$.

Let $\varepsilon > 0$ be such that $\operatorname{diam}(K) > 3\varepsilon$ and let U be a connected neighborhood of v such that $\operatorname{diam}(U) < \varepsilon$. Then there exists an index $i > 0$ such that $x_i \in U$ and $[v, x_i] \cap [X \backslash B(v, 2\varepsilon)] \neq \emptyset$. But then $\bar{U}$ and $[v, x_i]$ are two continua in X whose intersection is not connected, contradicting the fact that X is hereditarily unicoherent, and the proof is complete.

4. Decompositions of fans. We say that a space X is a $(q = c)$-*space* if, in X, every quasi-component is connected. In other words, for $(q = c)$-spaces the quasi-components and the components coincide. We will show that if a fan is locally connected at the vertex v of

X, then $X\backslash\{v\}$ is a $(q=c)$-space.

THEOREM 4.1. *Let X be a fan which is locally connected at the vertex v of X and*

$$X=\bigcup_{r\in R}\{J_r \mid J_r\cong[0,1] \text{ for each } r\in R \text{ and } J_{r_1}\cap J_{r_2}=\{v\} \text{ if } r_1\neq r_2\in R\}.$$

Then $X\backslash\{v\}$ is a $(q=c)$-space and $\{J_r\backslash\{v\}\}$, $r\in R$, is the decomposition of $X\backslash\{v\}$ into quasi-components.

Proof. It is sufficient to show that if $r_0\neq r_1\in R$, then there exists a closed and open set $G\subset X\backslash\{v\}$ such that

$$(1)\qquad J_{r_0}\backslash\{v\}\subset G\subset X\backslash J_{r_1}.$$

By Theorem 3.1 there exists for each $n(n=1,2,\cdots)$ a neighborhood U_n of v such that $\operatorname{diam}(U_n)<1/n$, $\bar{U}_{n+1}\subset U_n$ and $\operatorname{Bd}(U_n)\cap J_r$ is connected for each $r\in R$. We may assume that $J_{r_0}\cap\operatorname{Bd}(U_1)\neq\emptyset\neq J_{r_1}\cap\operatorname{Bd}(U_1)$. Let $R_n=\{r\in R\mid\operatorname{Bd}(U_n)\cap J_r\neq\emptyset\}(n=1,2,\cdots)$, then $R_n\subset R_{n+1}$ and $\bigcup_{n=1}^{\infty}R_n=R$.

Let Y be the space obtained from $\operatorname{Bd}(U_1)$ by identifying all components of $\operatorname{Bd}(U_1)$ to a point and let $f\colon\operatorname{Bd}(U_1)\to Y$ be the natural projection. It follows ([9], p. 148) that $\dim Y=0$. Since

$$f(J_{r_0}\cap\operatorname{Bd}(U_1))\neq f(J_{r_1}\cap\operatorname{Bd}(U_1)),$$

there exists a closed and open set H_1^* in Y such that

$$f(J_{r_0}\cap\operatorname{Bd}(U_1))\subset H_1^*\subset Y\backslash f(J_{r_1}\cap\operatorname{Bd}(U_1)).$$

Let $H_1=f^{-1}(H_1^*)$, then H_1 is a closed and open set in $\operatorname{Bd}(U_1)$. Define $A_1=\{r\in R_1\mid J_r\cap H_1\neq\emptyset\}$ and $B_1=\{r\in R_1\mid J_r\cap H_1=\emptyset\}$, then $A_1\cap B_1=\emptyset$ and $A_1\cup B_1=R_1$. Moreover, since H_1 is closed and open in $\operatorname{Bd}(U_1)$, we have that

$$P_1=\bigcup_{r\in A_1}\{J_r\backslash\{v\}\}\quad\text{and}\quad Q_1=\bigcup_{r\in B_1}\{J_r\backslash\{v\}\}$$

are disjoint and closed subsets of $X\backslash\{v\}$.

By induction we will construct sets A_n and B_n such that

$$(2)\qquad A_{n-1}\subset A_n,\ B_{n-1}\subset B_n,\ A_n\cap B_n=\emptyset \text{ and } A_n\cup B_n=R_n$$

and if $P_n=\bigcup_{r\in A_n}\{J_r\}$ and $Q_n=\bigcup_{r\in B_n}\{J_r\}$ then P_n and Q_n are disjoint and closed subsets of $X\backslash\{v\}(n=1,2,\cdots)$.

Suppose A_{n-1} and B_{n-1} have been constructed. Since $P_{n-1}\cap\operatorname{Bd}(U_n)$ and $Q_{n-1}\cap\operatorname{Bd}(U_n)$ are disjoint closed subsets of $\operatorname{Bd}(U_n)$ and

$J_r \cap \mathrm{Bd}(U_n)$ is connected for each $r \in R$, it follows as above, replacing U_1, $J_{r_0} \cap \mathrm{Bd}(U_1)$ and $J_{r_1} \cap \mathrm{Bd}(U_1)$ by U_n, $P_{n-1} \cap \mathrm{Bd}(U_n)$ and $Q_{n-1} \cap \mathrm{Bd}(U_n)$ respectively, that there exists a closed and open subset H_n of $\mathrm{Bd}(U_n)$ such that

$$P_{n-1} \cap \mathrm{Bd}(U_n) \subset H_n \subset \mathrm{Bd}(U_n) \backslash Q_{n-1} .$$

Let $A_n = \{r \in R_n \,|\, J_r \cap H_n \neq \varnothing\}$ and $B_n = \{r \in R_n \,|\, J_r \cap H_n = \varnothing\}$, then A_n and B_n satisfy (2).

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, then $A \cup B = R$ and $A \cap B = \varnothing$. Let $G = \bigcup_{r \in A} \{J_r \backslash \{v\}\}$ and $G_n = \bigcup_{r \in A_n} \{J_r \backslash \bar{U}_n\}$. Since G_n is open in X and $G = \bigcup_{n=1} G_n$, it follows that G is open in X. Similarly $X \backslash (G \cup \{v\}) = \bigcup_{r \in B} \{J_r \backslash \{v\}\}$ is open in X. Hence G is both open and closed in $X \backslash \{v\}$ and, since $r_0 \in A_1$ and $r_1 \in B_1$, (1) is proved.

5. **Property P and embeddings in the plane.** The main result of this section is Theorem 5.2 where we prove that if a fan is locally connected at its vertex, then it can be embedded in the plane. This result gives a solution to problem 1015 of [6].

Since every fan is hereditarily decomposable and hence 1-dimensional ([9], p. 206), we can consider every fan as a subspace of I^3. We start with the following lemma.

LEMMA 5.1. *Let X be a fan, with vertex v and*

$$X = \bigcup_{r \in R} \{J_r \,|\, J_r \cong [0, 1] \text{ for each } r \in R \text{ and } J_{r_1} \cap J_{r_2} = \{v\} \text{ if } r_1 \neq r_2 \in R\}$$

such that $\{J_r \backslash \{v\}\}$, $r \in R$, is the decomposition of $X \backslash \{v\}$ into quasi-components, then there exists an embedding $f: X \backslash \{v\} \to C \times I^3$ such that each quasi-component of $X \backslash \{v\}$ is contained in $\{c\} \times I^3$ for some $c \in C$, and

$$\overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) \subset C \times \{v\} , \tag{1}$$

where $C \subset [0, 1]$ denotes the Cantor ternary set.

Proof. We may assume that $X \subset I^3$. By ([9], p. 148), there exists a continuous function $g: X \backslash \{v\} \to C$ such that the quasi-components of $X \backslash \{v\}$ coincide with the point-inverses of g. Then the function $f: X \backslash \{v\} \to C \times I^3$ defined by $f(x) = (g(x), x)$ is an embedding. Only (1) remains to be shown. Let

$$(c_0, x_0) \in \overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) , \tag{2}$$

and let $\{(c_i, x_i)\} (i = 1, 2, \cdots)$ be a sequence of points in $f(X \backslash \{v\})$ converging to (c_0, x_0). We may assume that the sequence $\{x_i\}$ in X, where $x_i = f^{-1}((c_i, x_i))$, converges to a point $y \in X$. We consider two cases as follows:

Case 1. $y \neq v$. Then the sequence $\{f(x_i)\}$, where $f(x_i) = (c_i, x_i)$, converges to $f(y)$. Hence $f(y) = (c_0, x_0)$, contrary to (2).

Case 2. $y = v$. Then $x_0 = v$ and whence (1) holds.

These two cases complete the proof of the lemma.

THEOREM 5.2. *Let X be a fan which is locally connected at the vertex v of X, then X is embeddable in the plane.*

Proof. Let

$$X = \bigcup_{r \in R} \{J_r \mid J_r \cong [0, 1] \text{ for each } r \in R \text{ and } J_{r_1} \cap J_{r_2} = \{v\} \text{ if } r_1 \neq r_2 \in R\}.$$

It follows from 4.1 that $\{J_r \backslash \{v\}\}$, $r \in R$, is the decomposition of $X \backslash \{v\}$ into quasi-components. Hence by Lemma 5.1 there exists an embedding $f: X \backslash \{v\} \to C \times I^3$ such that each quasi-component of $X \backslash \{v\}$ is contained in $\{c\} \times I^3$ for some $c \in C$ and

$$\overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) \subset C \times \{v\} .$$

It follows that $\overline{f(X \backslash \{v\})}$ satisfies all conditions of Theorem 2.2, where $K = \overline{f(X \backslash \{v\})} \cap (C \times \{v\})$. Hence there exists an embedding $h: \overline{f(X \backslash \{v\})} \to I^2$ such that $h(K) = h(\overline{f(X \backslash \{v\})}) \cap l$, where $l = \{(x, y) \in I^2 \mid y = 0\}$. Let $\pi: I^2 \to I^2/l$ be the natural projection. It follows (see [9], p. 533) that $I^2 \cong I^2/l$ and whence the mapping $g: X \to I^2/l$ defined by

$$g(x) = \begin{cases} \pi \circ h \circ f(x) & \text{if } x \neq v , \\ \pi(l) & \text{if } x = v \end{cases}$$

is the required embedding.

REMARK. J. J. Charatonik and Z. Rudy constructed a fan X which is locally connected at its vertex (see [6], p. 215). They conjectured that this fan is not embeddable in the plane. The above theorem disproves their conjecture and gives a solution to problem 1015 of [6].

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